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On the character table of 2-groups

Shousaku Abe

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Abstract. We shall show that there are infinite pairs of non-direct product 2-groups with the same character. They are not pairs of the generalized quaternion group and dihedral group.

Key words: p-groups, characters.

Let Q_m and D_m denote the generalized quaternion group and the dihedral group of order 2^{m+1} ($m \geq 2$), respectively. For each prime p, there exists a pair of p-groups which aren't isomorphic but have the same character. If $p = 2$, they are D_m and Q_m. In this paper we study pairs of 2-groups which satisfy such a property.

We use the following notation throught this paper.

1. The dihedral group

$$D_m = \langle a, b \mid a^{2^m} = 1, b^2 = 1, b^{-1}ab = a^{-1} \rangle \ (m \geq 2).$$

2. The generalized quaternion group

$$Q_m = \langle a, b \mid a^{2^m} = 1, b^2 = a^{2^{m-1}}, b^{-1}ab = a^{-1} \rangle \ (m \geq 2).$$

To state our results, we have to introduce the following groups:

3. $\mathrm{GF}(2^n) \otimes \log_2 D_2 = (\mathrm{GF}(2^n)^3, *_D)$ with

$$x *_D y = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + x_2 y_2 + x_2 y_1)$$

   for $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. D stands for this group when n is obvious under discussion.
4. $\mathrm{GF}(2^n) \otimes \log_2 Q_2 = (\mathrm{GF}(2^n)^3, *_Q)$ with

$$x *_Q y = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + x_1 y_1 + x_2 y_2 + x_2 y_1)$$

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for $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. Q stands for this group when n is obvious under discussion.

The logarithm log is a continuous differentiable map and it's a group homomorphism of the multiplicative group $\mathbb{R}^{\times}$ of positive numbers to the additive group $\mathbb{R}^{+}$ of real line. Caluculating finite p-groups with some generators and relations, it follows that each element is represented by a unique vector in $\mathrm{GF}(p)^t$ with $t \in \mathbb{N}$ and the set of thier vectors is $\mathrm{GF}(p)^t$ itself and that the operator constructs of polynomial functions. Hence we employ log as a group homomorphism from a p-group with generators and relations into the associated vector space with polynomial functions. $\otimes$ is short for $\otimes_{\mathrm{GF}(p)}$ and is the same as the usual tensor product of vector spaces.

It is easy to see that $\mathrm{GF}(2) \otimes \log_2 D_2 \cong D_2$ and $\mathrm{GF}(2) \otimes \log_2 Q_2 \cong Q_2$. If $n \mid e$ then we have

$$\mathrm{GF}(2^e) \otimes \log_2 D_2 \supseteq \mathrm{GF}(2^n) \otimes \log_2 D_2 \quad \text{and}$$
$$\mathrm{GF}(2^e) \otimes \log_2 Q_2 \supseteq \mathrm{GF}(2^n) \otimes \log_2 Q_2.$$

Our main theorem is the following:

Theorem *Let $n \in \mathbb{N}$. If n is odd, then $\mathrm{GF}(2^n) \otimes \log_2 D_2$ and $\mathrm{GF}(2^n) \otimes \log_2 Q_2$ are not isomorphic but have the same character table. If n is even, then two groups are isomorphic. They are not direct product groups of some D_m and Q_m.*

The organization of the paper is as follows. First, we find conjugacy classes of two groups in Proposition 1 and 2. Next, we make an isomorphic decision in Proposition 3. Last, we prove that they are not direct product.

We shall find conjugacy classes of two groups.

Proposition 1 *Let $D = \mathrm{GF}(2^n) \otimes \log_2 D_2$. For $(x_1, x_2) \neq (0, 0)$, we have*

$$x^D = \{(x_1, x_2, y_3) \mid y_3 \in \mathbb{F}\}.$$

Proof. Let $\mathbb{F} = \mathrm{GF}(2^n)$. Write $q = 2^n$. For $x = (x_1, x_2, x_3)$, $g = (g_1, g_2, g_3) \in \mathbb{F}^3$, we have

$$g^{-1}xg = (x_1, x_2, x_3 + g_2x_1 - x_2g_1).$$

When $x_2 \neq 0$, we set $g_1 = (y_3 - x_3)/x_2$ and $g_2 = 0$. One gets

$$x \sim_D (x_1, x_2, y_3).$$

When $x_1 \neq 0$, we set $g_1 = 0$ and $g_2 = (y_3 - x_3)/x_1$. One gets

$$x \sim_D (x_1, x_2, y_3). \qquad \square$$

Therefore D has $q^2 + q - 1$ conjugacy classes.

Proposition 2 *Let* $Q = \mathrm{GF}(2^n) \otimes \log_2 Q_2$. *For* $(x_1, x_2) \neq (0, 0)$, *we have*

$$x^Q = \{(x_1, x_2, y_3) \mid y_3 \in \mathbb{F}\}.$$

The proof is the same one given in Proposition 1. Therefore Q has $q^2 + q - 1$ conjugacy classes.

Proposition 3 *If* n *is odd, then we have*

$$\mathrm{GF}(2^n) \otimes \log_2 D_2 \ncong \mathrm{GF}(2^n) \otimes \log_2 Q_2.$$

If n *is even, then we have*

$$\mathrm{GF}(2^n) \otimes \log_2 D_2 \cong \mathrm{GF}(2^n) \otimes \log_2 Q_2.$$

Proof. By counting the number of involutions of D and Q, one can say that the first statement is true.

Case Dihedral. Let $q = 2^n$. When $(x_1, x_2) = (0, 0)$, all x with $x_3 \neq 0$ are central involution. The number of elements of the form $(0, 0, x_3)$ is $q - 1$.

When $x_1 = 0, x_2 \neq 0$, one gets $x^2 \neq 0$ from the third coordinate of x^2. They are of order 4 and the number of elements of the form $(0, x_2, x_3)$ is $(q-1)q$.

When $x_1 \neq 0$ and $x_2 = 0$, such x is of order 2 and the number of elements of the form $(x_1, 0, x_3)$ is $(q-1)q$.

When $x_1 \neq 0$ and $x_2 \neq 0$, if $x^2 = 0$, we have $x_2^2 + x_2 x_1 = 0$. Hence $x_1 = x_2$. The number of elements of the form (x_1, x_1, x_3) is $q(q-1)$.

Therefore, the number of involutions in D is $(q-1)(2q+1)$.

Case Quaternion. We have

$$x^2 = (0, 0, x_1^2 + x_2^2 + x_1 x_2).$$

Let $f(x_1, x_2) = x_1^2 + x_2^2 + x_1 x_2$. When $(x_1, x_2) = (0, 0)$, all x with $x_3 \neq 0$ are central involution. The number of elements of the form $(0, 0, x_3)$ is $q - 1$.

When $x_1 = 0$ and $x_2 \neq 0$, one gets $f(0, x_2) = x_2^2 \neq 0$. All the elements of the form $(0, x_2, x_3)$ are of order 4.

When $x_1 \neq 0$ and $x_2 = 0$, one gets $f(x_1, 0) = x_1^2 \neq 0$. All the elements of the form $(x_1, 0, x_3)$ are of order 4.

When $x_1 \neq 0$ and $x_2 \neq 0$, one gets $f(x_1, x_2) \neq 0$ if n is odd. If $f(x_1, x_2) = 0$ then x_1/x_2 is a primitive element of $\mathrm{GF}(4)$ and is in the coefficient field $\mathrm{GF}(2^n)$ of the group Q. We have $2 \mid n$. They are of order 4.

Let α be a primitive element of $\mathrm{GF}(4)$. If n is even, The set of the nontrivial solutions of the equation

$$f(x_1, x_2) = 0$$

are given by the set $\{(x_1, \alpha x_1), (x_1, (\alpha + 1)x_1) \mid x_1 \in \mathbb{F}\backslash\{0\}\}$. All elements in the set $\{(x_1, \alpha x_1, x_3), (x_1, (\alpha+1)x_1, x_3) \mid x_1 \in \mathbb{F}\backslash\{0\}, x_3 \in \mathbb{F}\}$ are involutions. The number is $2q(q - 1)$.

Therefore, the number of involutions in Q is $q - 1$ if n is odd and $(q - 1)(2q + 1)$ otherwise.

Consequently, if n is odd, then we have

$$D \ncong Q.$$

Let α be a primitive element of $\mathrm{GF}(4)$. If n is even, we define a map f of D to Q by setting

$$f(x) = (x_1, \alpha x_1 + x_2, x_3 + \alpha x_1 x_2).$$

It is easy to see that f is an isomorphism of D onto Q.

Therefore, if n is even, then we have

$$D \cong Q. \qquad \square$$

Proposition 4 *Let $\mathbb{F} = \mathrm{GF}(2^n)$ and let $q = 2^n$. The irreducible characters of D and Q are*

$$\chi_{u,v}(u, v \in \mathbb{F}), \quad \textit{and} \quad \phi_u(u \in \mathbb{F}\backslash\{0\}).$$

where for all $x \in \mathbb{F}^3$,

$$\chi_{u,v}(x) = (-1)^{x_1\cdot u + x_2\cdot v},$$

$$\phi_u(x) = \begin{cases} q(-1)^{x_3\cdot u}, & \textit{if } x_1 = x_2 = 0, \\ 0, & \textit{otherwize,} \end{cases}$$

and the dot product $\cdot$ is the inner product when $\mathbb{F}$ is regarded as the vector space over GF(2) *with the natural bases.*

Proof. Let $G \in \{D, Q\}$. G is a non-abelian group of order q^3. Write $Z = Z(G)$. $Z(G) = \{(0,0,c) \mid c \in \mathbb{F}\} \cong (\mathbb{F}, +)$.

$$G/Z = \{(r,s,0)Z \mid r,s \in \mathbb{F}\} \cong (\mathbb{F}^2, +)$$

and in particular, every element of G is of the form $(x_1, x_2, x_3) \in \mathbb{F}^3$.

By Theorem 9.8 in [1], the irreducible characters of G/Z are $\psi_{u,v}$ $(u, v \in \mathbb{F})$, where

$$\psi_{u,v}(xZ) = (-1)^{x_1\cdot u + x_2\cdot v}$$

The lift to G of $\psi_{u,v}$ is the linear character $\chi_{u,v}$ which appears in the statement of the theorem.

Let $H = \{(x_1, 0, x_3) \mid x_1, x_2 \in \mathbb{F}\}$, so that H is abelian subgroup of order q^2. For $u \in \mathbb{F}\backslash\{0\}$, choose a character ψ_u of H which satisfies

$$\psi_u(0,0,t) = (-1)^{u\cdot t} \quad (t \in \mathbb{F}\backslash\{0\}).$$

We shall calculate $\psi_u \uparrow G$.

Let r be an element with $r \in \mathbb{F}\backslash\{0\}$. By Proposition 1 and 2, we have

$$(r,0,0)^G = \{(r,0,t) \mid t \in \mathbb{F}\}.$$

Then by Proposition 21.23 in [1],

$$\begin{aligned}(\psi_u \uparrow G)(r,0,t) &= \sum_{s\in\mathbb{F}} \psi_u(r,0,s) \\ &= \psi_u(r,0,0) \sum_{s\in\mathbb{F}} \psi_u(0,0,s) \\ &= \psi_u(r,0,0) \sum_{s\in\mathbb{F}} (-1)^{u\cdot s} \\ &= 0.\end{aligned}$$

Also,

$$(\psi_u \uparrow G)(0,0,t) = q\psi_u(0,0,t) = q(-1)^{u\cdot t}, \quad \text{and}$$
$$(\psi_u \uparrow G)(g) = 0 \quad \text{if } g \notin H.$$

We have now established that if $\phi_u = \psi_u \uparrow G$, then ϕ_u takes the values stated in the theorem. We find that

$$\begin{aligned}\langle \phi_u, \phi_u \rangle_G &= \frac{1}{q^3} \sum_{g\in G} \phi_u(g)\overline{\phi_u(g)} \\ &= \frac{1}{q^3} \sum_{g\in Z} \phi_u(g)\overline{\phi_u(g)} \\ &= \frac{1}{q^3} \sum_{g\in Z} q^2 \\ &= 1.\end{aligned}$$

Therefore ϕ_u is irreducible.

Clearly the irreducible characters $\chi_{u,v}(u, v \in \mathbb{F})$ and $\phi_u(u \in \mathbb{F}\backslash\{0\})$ are all distinct, and the sum of the squares of their degrees is

$$q^2 \cdot 1 + (q-1)\cdot q^2 = |G|.$$

Hence we have found all the irreducible characters of G. □

Proposition 5 *Let G be a non-abelian group of order q^3, as above. G is not direct product.*

Proof. This follows from the character table of G. Another proof of this result is the following. Suppose that $G = G_1 \times G_2$ for two proper normal subgroups G_1 and G_2 of G. Let $a \in G$ be an element of order 4. It may be supposed that $a \in G_1$. From $a^G \subseteq G_1$ and $q = |a^G|$ one gets $a^{-1}a^G = Z(G)$. Thus we have $Z(G) \subseteq G_1$. By Lemma 26.1 in [1], $1 \neq G_2 \cap Z(G)$. This yields $G_1 \cap G_2 \neq 1$ a contradiction. Therefore G is not direct product. □

This completes the proof of the Theorem.

References

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Department of Mathematical Information Science
Tokyo University of Science
1-3 Kagurazaka
Shinjuku–ku, Tokyo 162–8601
Japan
E-mail: shousaku_abe@yahoo.co.jp