Integrability conditions for almost quaternion structures

Dedicated to Professor Yoshie Katsurada on her sixtieth birthday

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§ 0. Introduction

Suppose that there are given, in a differentiable manifold, 3 tensor fields F, G and H of type (1, 1) satisfying

$$F^2=-1,\quad G^2=-1,\quad H^2=-1,$$
$$F=GH=-HG,\quad G=HF=-FH,\quad H=FG=-GF.$$

We then call the set (F, G, H) an *almost quaternion structure* and the manifold an *almost quaternion manifold.*

If we can cover the manifold by a system of coordinate neighborhoods with respect to which components of F, G and H are all constant, we say that the almost quaternion structure (F, G, H) is *integrable* and call the structure a *quaernion structure.*

The integrability conditions for almost quaternion structures and the existence of an affine connection with respect to which F, G and H are all parallel have been studied by Bonan [1] and Obata [4], [5].

The main purpose of the present paper is to discuss these problems making use not only of the Nijenhuis tensors $[F, F]$, $[G, G]$, $[H, H]$ but also of the Nijenhuis tensors $[G, H]$, $[H, F]$, $[F, G]$.

§ 1. Preliminaries

Let P and Q be two tensor fields of type (1, 1) in a differentiable manifold. It is well known (Kobayashi and Nomizu [3]) that the expression given by

$$\begin{aligned}(1.1)\qquad [P, Q]\,(X, Y) &= [PX, QY]-P[QX, Y]-Q[X, PY]\\ &\quad +[QX, PY]-Q[PX, Y]-P[X, QY]+(PQ+QP)[X, Y],\end{aligned}$$

X and Y being arbitrary vector fields, defines a tensor field of type (1, 2) and is called the Nijenhuis tensor of P and Q. If $P=Q$, we have

$$(1.2)\qquad [P, P]\,(X, Y)$$

$$=2\left\{[PX, PY]-P[PX, Y]-P[X, PY]+P^2[X, Y]\right\}.$$

Now let L, M and N be tensor fields of type (1, 1). Then we have (Frölicher and Nijenhuis [2])

$$\begin{aligned}(1.3)\qquad &[L, MN](X, Y)+[M, LN](X, Y)\\ &=[L, M](NX, Y)+[L, M](X, NY)\\ &\quad+L[M, N](X, Y)+M[L, N](X, Y),\end{aligned}$$

for arbitrary vector fields X and Y. In fact, we have

$$\begin{aligned}&[L, M](NX, Y)+[L, M](X, NY)\\ &\quad+L[M, N](X, Y)+M[L, N](X, Y)\\ &=[LNX, MY]-L[MNX, Y]-M[NX, LY]\\ &\quad+[MNX, LY]-M[LNX, Y]-L[NX, MY]+(LM+ML)[NX, Y]\\ &\quad+[LX, MNY]-L[MX, NY]-M[X, LNY]\\ &\quad+[MX, LNY]-M[LX, NY]-L[X, MNY]+(LM+ML)[X, NY]\\ &\quad+L\{[MX, NY]-M[NX, Y]-N[X, MY]\\ &\qquad+[NX, MY]-N[MX, Y]-M[X, NY]+(MN+NM)[X, Y]\}\\ &\quad+M\{[LX, NY]-L[NX, Y]-N[X, LY]\\ &\qquad+[NX, LY]-N[LX, Y]-L[X, NY]+(LN+NL)[X, Y]\}\\ &=[LX, MNY]-L[MNX, Y]-MN[X, LY]\\ &\quad+[MNX, LY]-MN[LX, Y]-L[X, MNY]+(LMN+MNL)[X, Y]\\ &\quad+[MX, LNY]-M[LNX, Y]-LN[X, MY]\\ &\quad+[LNX, MY]-LN[MX, Y]-M[X, LNY]+(MLN+LNM)[X, Y]\\ &=[L, MN](X, Y)+[M, LN](X, Y)\end{aligned}$$

and (1.3) is proved.

We introduce here the following notations (Frölicher and Nijenhuis [2]): If S is a tensor field of type (1, 2) and N is a tensor field of type (1, 1), then $S\barwedge N$ is defined to be

$$(1.4)\qquad (S\barwedge N)(X, Y)=S(NX, Y)+S(X, NY)$$

and $N\barwedge S$ to be

$$(1.5)\qquad (N\barwedge S)(X, Y)=NS(X, Y).$$

Then (1.3) can bewritten as

$$\begin{aligned}(1.6)\qquad &[L, MN]+[M, LN]\\ &=[L, M]\barwedge N+L\barwedge[M, N]+M\barwedge[L, N].\end{aligned}$$

Let S be a tensor field of type (1, 2) and M, N tensor fields of type (1, 1). Then we have

$$(S\barwedge M)\barwedge N-(S\barwedge N)\barwedge M=S\barwedge MN-S\barwedge NM. \tag{1.7}$$

In fact

$$\begin{aligned}
&\left\{(S\barwedge M)\barwedge N-(S\barwedge N)\barwedge M\right\}(X, Y)\\
&\quad=(S\barwedge M)(NX, Y)+(S\barwedge M)(X, NY)-(S\barwedge N)(MX, Y)\\
&\qquad-(S\barwedge N)(X, MY)\\
&\quad=S(MNX, Y)+S(NX, MY)+S(MX, NY)+S(X, MNY)\\
&\qquad-S(NMX, Y)-S(MX, NY)-S(NX, MY)-S(X, NMY)\\
&\quad=(S\barwedge MN)(X, Y)-(S\barwedge NM)(X, Y),
\end{aligned}$$

and (1. 7) is proved.

Les S be a tensor field of type (1, 2) and L, N tensor fields of type (1, 1). Then we have

$$(L\barwedge S)\barwedge N=L\barwedge(S\barwedge N). \tag{1.8}$$

In fact

$$\begin{aligned}
&\left\{(L\barwedge S)\barwedge N\right\}(X, Y)\\
&\quad=(L\barwedge S)(NX, Y)+(L\barwedge S)(X, NY)\\
&\quad=L\left\{S(NX, Y)\right\}+L\left\{S(X, NY)\right\}\\
&\quad=L\left\{S(NX, Y)+S(X, NY)\right\}\\
&\quad=L\left\{(S\barwedge N)(X, Y)\right\}\\
&\quad=\left\{L\barwedge(S\barwedge N)\right\}(X, Y)
\end{aligned}$$

for arbitrary vector fields X and Y and consequently (1. 8) is proved.

Equation (1. 7) shows that $(S\barwedge M)\barwedge N$ is not always equal to $S\barwedge(N\barwedge M)$, and equation (1. 8) shows that $(L\barwedge S)\barwedge N$ is always equal to $L\barwedge(S\barwedge N)$, where S is a tensor field of type (1, 2) and L, M, N are tensor fields of type (1, 1).

When a tensor field S of type (1, 2) comes first as in $(S\barwedge M)\barwedge N$, the associative law does not hold, but when S comes second as in $(L\barwedge S)\barwedge N$ the associative law does hold.

§ 2. The almost quaternion structure

We assume that there are given, in a differentiable manifold, three tensor fields F, G, H satisfying

$$F^2=-1, \qquad G^2=-1, \qquad H^2=-1,$$
$$(2.1) \qquad F=GH, \qquad G=HF, \qquad H=FG,$$
$$GH+HG=0, \quad HF+FH=0, \quad FG+GF=0,$$

1 denoting the unit tensor field. In this case we say that the differentiable manifold admits an almost quaternion structure. A manifold admitting an almost quaternion structure is $4n$-dimensional, n being a positive integer.

Now, putting $L=M=F$, $N=G$ in (1, 6), we find

$$[F, FG]+[F, FG]$$
$$=[F, F]\barwedge G+F\barwedge[F, G]+F\barwedge[F, G],$$

that is,

$$(2.2) \qquad [H, F]=F\barwedge[F, G]+\frac{1}{2}[F, F]\barwedge G.$$

On the other hand, putting $L=G$, $M=N=F$ in (1. 6), we find

$$[G, F^2]+[F, GF]$$
$$=[G, F]\barwedge F+G\barwedge[F, F]+F\barwedge[G, F],$$

that is

$$(2.3) \qquad [H, F]=-[F, G]\barwedge F-F\barwedge[F, G]-G\barwedge[F, F].$$

Adding (2, 2) and (2, 3), and dividing the sum by 2, we find

$$(2.4) \qquad [H, F]=-\frac{1}{2}[F, G]\barwedge F-\frac{1}{2}G\barwedge[F, F]+\frac{1}{4}[F, F]\barwedge G.$$

Subtracting (2. 3) from (2. 2), we find

$$(2.5) \qquad [F, G]\barwedge F+2F\barwedge[F, G]+G\barwedge[F, F]+\frac{1}{2}[F, F]\barwedge G=0.$$

Next, putting $L=M=G$, $N=F$ in (1. 6), we find

$$[G, GF]+[G, GF]=[G, G]\barwedge F+G\barwedge[G, F]+G\barwedge[G, F],$$

that is

$$(2.6) \qquad [G, H]=-G\barwedge[F, G]-\frac{1}{2}[G, G]\barwedge F.$$

On the other hand, putting $L=F$, $M=N=G$ in (1. 6), we find

$$[F, G^2]+[G, FG]$$
$$=[F, G]\barwedge G+F\barwedge[G, G]+G\barwedge[F, G],$$

that is,

$$(2.7) \qquad [G, H] = [F, G] \barwedge G + G \barwedge [F, G] + F \barwedge [G, G].$$

Adding (2. 6) and (2. 7) and dividing the sum by 2, we find

$$(2.8) \qquad [G, H] = \frac{1}{2} [F, G] \barwedge G + \frac{1}{2} F \barwedge [G, G] - \frac{1}{4} [G, G] \barwedge F.$$

Subtracting (2. 7) from (2. 6), we find

$$(2.9) \qquad [F, G] \barwedge G + 2G \barwedge [F, G] + F \barwedge [G, G] + \frac{1}{2} [G, G] \barwedge F = 0.$$

Again, putting $L=FG$, $M=F$, $N=G$ in (1. 6), we find

$$\begin{aligned} &[FG, FG] + [F, FGG] \\ &\quad = [FG, F] \barwedge G + FG \barwedge [F, G] + F \barwedge [FG, G], \end{aligned}$$

that is,

$$(2.10) \qquad [H, H] = [F, F] + [H, F] \barwedge G + H \barwedge [F, G] + F \barwedge [G, H].$$

On the other hand, putting $L=FG$, $M=G$, $N=F$ in (1. 6), we find

$$\begin{aligned} &[FG, GF] + [G, FGF] \\ &\quad = [FG, G] \barwedge F + FG \barwedge [G, F] + G \barwedge [FG, F], \end{aligned}$$

that is

$$(2.11) \qquad [H, H] = [G, G] - [G, H] \barwedge F - H \barwedge [F, G] - G \barwedge [H, F].$$

Thus, from (2. 10) and (2. 11), we find

$$\begin{aligned} (2.12) \qquad [H, H] = \frac{1}{2} \{ &[F, F] + [G, G] + [H, F] \barwedge G \\ &- G \barwedge [H, F] - [G, H] \barwedge F + F \barwedge [G, H] \} \end{aligned}$$

and

$$\begin{aligned} (2.13) \qquad &[F, F] - [G, G] + [H, F] \barwedge G + G \barwedge [H, F] \\ &- [G, H] \barwedge F + F \barwedge [G, H] + 2H \barwedge [F, G] = 0. \end{aligned}$$

Finally, putting $L=M=N=F$ in (1. 6), we find

$$\begin{aligned} &[F, F^2] + [F, F^2] \\ &\quad = [F, F] \barwedge F + F \barwedge [F, F] + F \barwedge [F, F], \end{aligned}$$

that is,

$$(2.14) \qquad [F, F] \barwedge F = -2F \barwedge [F, F].$$

Similarly, we get

$$[G, G]\barwedge G = -2G\barwedge[G, G]\,, \tag{2.15}$$

$$[H, H[\barwedge H = -2H\barwedge[H, H]\,. \tag{2.16}$$

§ 3. Theorems

We now prove the

THEOREM 3.1. *If*

$$[F, F]=0\,, \quad [G, G]=0\,,$$

then

$$[F, G]=0\,.$$

PROOF. Since $[F, F]=0$, we have, from (2.2),

$$[H, F]=F\barwedge[F, G]\,, \tag{3.1}$$

and from (2.5),

$$[F, G]\barwedge F = -2F\barwedge[F, G]\,. \tag{3.2}$$

Since $[G, G]=0$, we have from (2.6),

$$[G, H] = -G\barwedge[F, G]\,, \tag{3.3}$$

and from (2.9),

$$[F, G]\barwedge G = -2G\barwedge[F, G]\,. \tag{3.4}$$

We substitute $[F, F]=0$, $[G,G]=0$, (3.1) and (3.3) into (2.13) and find

$$(F\barwedge[F, G])\barwedge G+G\barwedge(F\barwedge[F, G])$$
$$-(G\barwedge[F, G])\barwedge F-F\barwedge(G\barwedge[F, G])+2H\barwedge[F, G]=0\,,$$

from which

$$(F\barwedge[F, G])\barwedge G-(G\barwedge[F, G])\barwedge F=0\,, \tag{3.5}$$

since

$$G\barwedge(F\barwedge[F, G]) = -F\barwedge(G\barwedge[F, G]) = -H\barwedge[F, G]\,,$$

by virtue of $GF=-FG=-H$.

Now, using (1.8), (3.2) and (3.4), we find, from (3.5),

$$F\barwedge([F, G]\barwedge G)-G\barwedge([F, G]\barwedge F)=0\,,$$
$$-2F\barwedge(G\barwedge[F, G])+2G\barwedge(F\barwedge[F, G])=0\,,$$
$$-4FG\barwedge[F, G]=0\,,$$

that is,

$$H\barwedge[F, G]=0, \tag{3.6}$$

Since $H^2=-1$, we have, from (3.6),

$$[F, G]=0.$$

Thus the theorem is proved.

THEOREM 3.2. *If*

$$[F, F]=0, \quad [G, G]=0,$$

then

$$[G, H]=0, \quad [H, F]=0, \quad [F, G]=0$$

and

$$[H, H]=0.$$

PROOF. By Theorem 3.1, we have $[F, G]=0$, and consequently, we have, from (3.1) and (3.3),

$$[H, F]=0,$$

and

$$[G, H]=0$$

respectively, and consequently, from (2.12),

$$[H, H]=0.$$

We next prove

THEOREM 3.3. *If*

$$[F, F]=0, \quad [F, G]=0,$$

then

$$[G, G]=0.$$

PROOF. Putting $L=M=G$, $N=H$ in (1.6), we find

$$\begin{aligned}&[G, GH]+[G, GH]\\ &\quad=[G, G]\barwedge H+G\barwedge[G, H]+G\barwedge[G, H],\end{aligned}$$

from which, using $[G, GH]=[G, F]=0$,

$$[G, G]\barwedge H=-2G\barwedge[G, H]. \tag{3.7}$$

From $[F, G]=0$ and (2.7), we find

$$[G, H]=F\barwedge[G, G]$$

and consequently we find, from (3. 7),

$$[G, G]\barwedge H=-2G\barwedge(F\barwedge[G, G])=-2GF\barwedge[G, G],$$

that is,

$$[G, G]\barwedge H=2H\barwedge[G, G]. \tag{3.8}$$

On the other hand, putting $S=[G, G]$, $M=F$, $N=G$ in (1. 7), we find

$$\begin{aligned}&([G, G]\barwedge F)\barwedge G-([G, G]\barwedge G)\barwedge F\\&=[G, G]\barwedge FG-[G, G]\barwedge GF,\end{aligned}$$

from which,

$$2[G, G]\barwedge H=([G, G]\barwedge F)\barwedge G-([G, G]\barwedge G)\barwedge F. \tag{3.9}$$

But, we have, from $[F, G]=0$ and (2. 9),

$$[G, G]\barwedge F=-2F\barwedge[G, G]. \tag{3.10}$$

Thus, substituting (2. 10) and (3. 10) into (3. 9), we find

$$2[G, G]\barwedge H=-2(F\barwedge[G, G])\barwedge G+2(G\barwedge[G, G])\barwedge F,$$

or, using (1. 8),

$$[G, G]\barwedge H=-F\barwedge([G, G]\barwedge G)+G\barwedge([G, G]\barwedge F)$$

from which, using (2. 15) and (3. 10),

$$\begin{aligned}[G, G]\barwedge H&=2F\barwedge(G\barwedge[G, G])-2G\barwedge(F\barwedge[G, G])\\&=4FG\barwedge[G, G],\end{aligned}$$

that is,

$$[G, G]\barwedge H=4H\barwedge[G, G]. \tag{3.11}$$

Comparing (3. 8) and (3. 11), we see that

$$H\barwedge[G, G]=0,$$

from which, H^2 being equal to -1,

$$[G, G]=0,$$

and consequently the theorem is proved.

Combining Theorems 3. 2 and 3. 3, we obtain

THEOREM 3. 4. *If*

$$[F, F]=0, \quad [F, G]=0,$$

then

$$[G,G]=0\,,\quad [H,H]=0\,,\quad [G,H]=0\,,\quad [H,F]=0\,.$$

We now prove

THEOREM 3.5. *If*

$$[F,G]=0\,,\quad [F,H]=0\,,$$

then

$$[F,F]=0\,.$$

PROOF. From (2.3) and the assumptions, we have

$$G\barwedge[F,F]=0\,,$$

from which

$$[F,F]=0\,.$$

Thus, combining Theorems 3.4 and 3.5, we have

THEOREM 3.6. *If*

$$[F,G]=0\,,\quad [F,H]=0\,,$$

then

$$[F,F]=0\,,\quad [G,G]=0\,,\quad [H,H]=0\,,\quad [G,H]=0\,.$$

We next prove

THEOREM 3.7. *If*

$$[F,F]=0\,,\quad [G,H]=0\,,$$

then

$$[G,G]=0\,.$$

PROOF. First of all, from (2.2) and the assumptions, we have

$$[F,H]=F\barwedge[F,G]\,. \tag{3.12}$$

On the other hand, putting $L=M=G$ and $N=H$ in (1.6) and using the assumptions, we find

$$2[F,G]=[G,G]\barwedge H\,. \tag{3.13}$$

Also, putting $L=N=G$ and $M=H$ in (1.6), and using the assumptions, we find

$$[F,G]=-H\barwedge[G,G]\,,$$

from which

(3. 14) $$H\barwedge[F,G]=[G,G]$$

We have also, from (2. 13) and the assumptions,

$$-[G,G]+[H,F]\barwedge G+G\barwedge[H,F]+2H\barwedge[F,G]=0.$$

Substituting (3. 12) and (3. 14) into this equation, we find

$$F\barwedge[F,G]\barwedge G=0,$$

from which

(3. 15) $$[F,G]\barwedge G=0.$$

Thus, from (3. 14) and (3. 15), we find

(3. 16) $$[G,G]\barwedge G=0.$$

On the other hand, we have, from (2. 15), and (3. 16)

$$G\barwedge[G,G]=0,$$

from which

$$[G,G]=0,$$

which proves the theorem.

Combining Theorems 3. 2 and 3. 7, we obtain

THEOREM 3. 8. *If*

$$[F,F]=0, \quad [G,H]=0,$$

then

$$[G,G]=0, \quad [H,H]=0, \quad [F,G]=0, \quad [F,H]=0.$$

From Theorems 3. 2, 3. 4, 3. 6 and 3. 8, we have

PHEOREM 3. 9. *If two of six Nijenhuis tensors*:

$$[F,F], \quad [G,G], \quad [H,H], \quad [G,H], \quad [H,F], \quad [F,G]$$

vanish, then the others vanish too.

§ 4. Affine connections in an almost quaternion manifold.

In this section, we prove

THEOREM 4. 1. (Obata [4]) *In order that there exists, in an almost quaternion manifold, a symmetric affine connection* ∇ *such that*

$$\nabla F=0, \quad \nabla G=0, \quad \nabla H=0,$$

it is necessary and sufficient that

$$[F, F]=0, \quad [G, G]=0.$$

PROOF. We first introduce, in the almost quaternion manifold, a symmetric affine connection $\overset{0}{\nabla}$ and denote the components of the connection by $\overset{0}{\Gamma}{}_j{}^h{}_i$, for example, we immerse the manifold in a sufficiently high dimensional Euclidean space, consider the induced Riemannian metric and form the Levi-Civita connection $\overset{0}{\nabla}$ with respcct to this metric.

We put

$$\overset{1}{\Gamma}{}_j{}^h{}_i=\overset{0}{\Gamma}{}_j{}^h{}_i+\overset{1}{T}{}_{ji}{}^h, \tag{4.1}$$

where

$$\overset{1}{T}{}_{ji}{}^h=-\frac{1}{4}\left\{F_i{}^t\overset{0}{\nabla}_t F_j{}^h+(\overset{0}{\nabla}_j F_i{}^t)F_t{}^h\right\} -\frac{1}{4}(\overset{0}{\nabla}_j F_i{}^t+\overset{0}{\nabla}_i F_j{}^t)F_t{}^h. \tag{4.2}$$

(See Walker [6] or Yano [7]).

Then denoting by $\overset{1}{\nabla}_j$ the operator of covariant differentiation with respect to $\overset{1}{\Gamma}{}_j{}^h{}_i$, we see that

$$\begin{aligned}\overset{1}{\nabla}_j F_i{}^h &=\overset{0}{\nabla}_j F_i{}^h+T_{js}{}^h F_i{}^s-T_{ji}{}^s F_s{}^h\\ &=\overset{0}{\nabla}_j F_i{}^h\\ &\quad+\frac{1}{4}\overset{0}{\nabla}_i F_j{}^h-\frac{1}{4}(\overset{0}{\nabla}_j F_s{}^t)F_i{}^s F_t{}^h-\frac{1}{4}(\overset{0}{\nabla}_j F_s{}^t+\overset{0}{\nabla}_s F_j{}^t)F_i{}^s F_t{}^h\\ &\quad+\frac{1}{4}F_i{}^t(\overset{0}{\nabla}_t F_j{}^s)F_s{}^h-\frac{1}{4}\overset{0}{\nabla}_j F_i{}^h-\frac{1}{4}(\overset{0}{\nabla}_j F_i{}^h+\overset{0}{\nabla}_i F_j{}^h)\\ &=\overset{0}{\nabla}_j F_i{}^h-\frac{1}{4}\overset{0}{\nabla}_j F_i{}^h-\frac{1}{4}\overset{0}{\nabla}_j F_i{}^h-\frac{1}{4}\overset{0}{\nabla}_j F_i{}^h-\frac{1}{4}\overset{0}{\nabla}_j F_i{}^h,\end{aligned}$$

that is,

$$\overset{1}{\nabla}_j F_i{}^h=0 \tag{4.3}$$

and that

$$\begin{aligned}\overset{1}{\Gamma}{}_j{}^h{}_i-\overset{1}{\Gamma}{}_i{}^h{}_j &=\overset{1}{T}{}_{ji}{}^h-\overset{1}{T}{}_{ij}{}^h\\ &=\frac{1}{4}\Big\{F_j{}^t\overset{0}{\nabla}_t F_i{}^h-F_i{}^t\overset{0}{\nabla}_t F_j{}^h\\ &\qquad-(\overset{0}{\nabla}_j F_i{}^t-\overset{0}{\nabla}_i F_j{}^t)F_t{}^h\Big\},\end{aligned}$$

that is,

$$\overset{1}{\Gamma}{}_{j}{}^{h}{}_{i}-\overset{1}{\Gamma}{}_{i}{}^{h}{}_{j}=\frac{1}{8}[F,F]_{ji}{}^{h}, \tag{4.4}$$

where $[F,F]_{ji}{}^{h}$ are components of the Nijenhuis tensor $[F,F]$ formed with F.

Thus, if $[F,F]=0$, then $\overset{1}{\Gamma}{}_{j}{}^{h}{}_{i}$ define a symmetric connection $\overset{1}{\nabla}$ such that $\overset{1}{\nabla}F=0$.

We next put

$$\Gamma_{j}{}^{h}{}_{i}=\overset{1}{\Gamma}{}_{j}{}^{h}{}_{i}+T_{ji}{}^{h}, \tag{4.5}$$

where

$$\begin{aligned} T_{ji}{}^{h} &= -\frac{1}{4}\left\{G_{i}{}^{t}\overset{1}{\nabla}_{t}G_{j}{}^{h}+(\overset{1}{\nabla}_{j}G_{i}{}^{t})G_{t}{}^{h}+H_{i}{}^{t}\overset{1}{\nabla}_{t}H_{j}{}^{h}+(\overset{1}{\nabla}_{j}H_{i}{}^{t})H_{t}{}^{h}\right\} \\ &\quad +\frac{1}{4}(F_{i}{}^{s}\overset{1}{\nabla}_{s}G_{j}{}^{t}+F_{s}{}^{t}\overset{1}{\nabla}_{j}G_{i}{}^{s})H_{t}{}^{h}-\frac{1}{4}(\overset{1}{\nabla}_{j}G_{i}{}^{t}+\overset{1}{\nabla}_{i}G_{j}{}^{t})G_{t}{}^{h}, \end{aligned} \tag{4.6}$$

which can also be written as

$$\begin{aligned} T_{ji}{}^{h} &= -\frac{1}{2}(\overset{1}{\nabla}_{j}G_{i}{}^{t})G_{t}{}^{h} \\ &\quad -\frac{1}{4}\left\{G_{i}{}^{t}\overset{1}{\nabla}_{t}G_{j}{}^{h}+\overset{1}{\nabla}_{i}G_{j}{}^{t})\,G_{t}{}^{h}+H_{i}{}^{t}\overset{1}{\nabla}_{t}H_{j}{}^{h}-F_{i}{}^{s}(\overset{1}{\nabla}_{s}G_{j}{}^{t})\,H_{t}{}^{h}\right\}, \end{aligned} \tag{4.7}$$

since

$$\overset{1}{\nabla}_{j}H_{i}{}^{t}-F_{s}{}^{t}\overset{1}{\nabla}_{j}G_{i}{}^{s}=\overset{1}{\nabla}_{j}(F_{s}{}^{t}G_{i}{}^{s})-F_{s}{}^{t}\overset{1}{\nabla}_{j}G_{i}{}^{s}=0$$

by virtue of $\overset{1}{\nabla}_{j}F_{s}{}^{t}=0$.

Now denoting by ∇_{j} the operator of covariant differentiation with respect to $\Gamma_{j}{}^{h}{}_{i}$, we see that

$$\nabla_{j}F_{i}{}^{h}=\overset{1}{\nabla}_{j}F_{i}{}^{h}+T_{ja}{}^{h}F_{i}{}^{a}-T_{ji}{}^{a}F_{a}{}^{h},$$

that is,

$$\nabla_{j}F_{i}{}^{h}=T_{ja}{}^{h}F_{i}{}^{a}-T_{ji}{}^{a}F_{a}{}^{h}. \tag{4.8}$$

On the other hand, we have

$$\begin{aligned} T_{ja}{}^{h}F_{i}{}^{a} = -\frac{1}{2}(\overset{1}{\nabla}_{j}G_{a}{}^{t})F_{i}{}^{a}G_{t}{}^{h}-\frac{1}{4}\Big\{&G_{a}{}^{t}\overset{1}{\nabla}_{t}G_{j}{}^{h}+(\overset{1}{\nabla}_{a}G_{j}{}^{t})G_{t}{}^{h} \\ &+H_{a}{}^{t}\overset{1}{\nabla}_{t}H_{j}{}^{h}-F_{a}{}^{s}(\overset{1}{\nabla}_{s}G_{j}{}^{t})H_{t}{}^{h}\Big\}F_{i}{}^{a}, \end{aligned}$$

or, using $\overset{1}{\nabla}_j F_i^h = 0$,

$$T_{ja}{}^h F_i^a = -\frac{1}{2}(\overset{1}{\nabla}_j H_i^t)G_t^h + \frac{1}{4}H_i^t\overset{1}{\nabla}_t G_j^h - \frac{1}{4}F_i^a(\overset{1}{\nabla}_a G_j^t)G_t^h - \frac{1}{4}G_i^t\overset{1}{\nabla}_t H_j^h - \frac{1}{4}(\overset{1}{\nabla}_i G_j^t)H_t^h, \tag{4.9}$$

and

$$T_{ji}{}^a F_a^h = -\frac{1}{2}(\overset{1}{\nabla}_j G_i^t)F_a^h G_t^a - \frac{1}{4}\Big\{G_i^t\overset{1}{\nabla}_t G_j^a + (\overset{1}{\nabla}_i G_j^t)G_t^a + H_i^t\overset{1}{\nabla}_t H_j^a - F_i^s(\overset{1}{\nabla}_s G_j^t)H_t^a\Big\}F_a^h,$$

or using $\overset{1}{\nabla}_j F_i^h = 0$,

$$T_{ji}{}^a F_a^h = -\frac{1}{2}(\overset{1}{\nabla}_j G_i^t)H_t^h - \frac{1}{4}G_i^t(\overset{1}{\nabla}_t H_j^h) - \frac{1}{4}(\overset{1}{\nabla}_i G_j^t)H_t^h + \frac{1}{4}H_i^t(\overset{1}{\nabla}_t G_j^h) - \frac{1}{4}F_i^s(\overset{1}{\nabla}_s G_j^t)G_t^h. \tag{4.10}$$

Thus, from (4.8), (4.9) and (4.10), we find

$$\nabla_j F_i^h = \frac{1}{2}\Big\{(\overset{1}{\nabla}_j H_i^t)G_t^h + (\overset{1}{\nabla}_j G_i^t)H_t^h\Big\}.$$

But

$$\begin{aligned}
&(\overset{1}{\nabla}_j H_i^t)G_t^h + (\overset{1}{\nabla}_j G_i^t)H_t^h \\
&= \Big\{\overset{1}{\nabla}_j(F_s^t G_i^s)\Big\}G_t^h + (\overset{1}{\nabla}_j G_i^t)H_t^h \\
&= F_s^t(\overset{1}{\nabla}_j G_i^s)G_t^h + (\overset{1}{\nabla}_j G_i^t)H_t^h \\
&= -(\overset{1}{\nabla}_j G_i^s)H_s^h + (\overset{1}{\nabla}_j G_i^t)H_t^h \\
&= 0,
\end{aligned}$$

and consequently

$$\nabla_j F_i^h = 0. \tag{4.11}$$

For the covariant derivative of G_i^h with respect to ∇, we have

$$\nabla_j G_i^h = \overset{1}{\nabla}_j G_i^h + T_{ja}{}^h G_i^a - T_{ji}{}^a G_a^h. \tag{4.12}$$

On the other hand, we have

$$T_{ja}{}^h G_i^a = -\frac{1}{2}(\overset{1}{\nabla}_j G_a^t)G_i^a G_t^h - \frac{1}{4}G_a^t\overset{1}{\nabla}_t G_j^h + (\overset{1}{\nabla}_a G_j^t)G_t^h + H_a^t\overset{1}{\nabla}_t H_j^h - F_a^s(\overset{1}{\nabla}_s G_j^t)H_t^h G_i^a$$

or

$$T_{ja}{}^{h}G_{i}{}^{a}=-\frac{1}{2}\overset{1}{\nabla}_{j}G_{i}{}^{h}+\frac{1}{4}\overset{1}{\nabla}_{i}G_{j}{}^{h}-\frac{1}{4}(\overset{1}{\nabla}_{a}G_{j}{}^{t})G_{i}{}^{a}G_{t}{}^{h}-\frac{1}{4}F_{i}{}^{t}\overset{1}{\nabla}_{t}H_{j}{}^{h}-\frac{1}{4}H_{i}{}^{s}(\overset{1}{\nabla}_{s}G_{j}{}^{t})H_{t}{}^{h}, \tag{4.13}$$

and

$$T_{ji}{}^{a}G_{a}{}^{h}=\frac{1}{2}(\overset{1}{\nabla}_{j}G_{i}{}^{h})-\frac{1}{4}\left\{G_{i}{}^{t}\overset{1}{\nabla}_{t}G_{j}{}^{a}+(\overset{1}{\nabla}_{i}G_{j}{}^{t})G_{t}{}^{a}+H_{i}{}^{t}\overset{1}{\nabla}_{t}H_{j}{}^{a}-F_{i}{}^{s}(\overset{1}{\nabla}_{s}G_{j}{}^{t})H_{t}{}^{a}\right\}G_{a}{}^{h},$$

or using $\overset{1}{\nabla}_{j}F_{i}{}^{h}=0$,

$$T_{ji}{}^{a}G_{a}{}^{h}=\frac{1}{2}(\overset{1}{\nabla}_{j}G_{i}{}^{h})-\frac{1}{4}(\overset{1}{\nabla}_{t}G_{j}{}^{a})\,G_{i}{}^{t}G_{a}{}^{h}+\frac{1}{4}\overset{1}{\nabla}_{i}G_{j}{}^{h}-\frac{1}{4}H_{i}{}^{t}(\overset{1}{\nabla}_{t}H_{j}{}^{a})G_{a}{}^{h}-\frac{1}{4}F_{i}{}^{s}(\overset{1}{\nabla}_{s}H_{j}{}^{h}). \tag{4.14}$$

Thus, from (4.12), (4.13) and (4.14), we find

$$\nabla_{j}G_{i}{}^{h}=-\frac{1}{4}H_{i}{}^{s}\left\{(\overset{1}{\nabla}_{s}G_{j}{}^{t})H_{t}{}^{h}-(\overset{1}{\nabla}_{s}H_{j}{}^{t})\,G_{t}{}^{h}\right\}.$$

But

$$\begin{aligned}&(\overset{1}{\nabla}_{s}G_{j}{}^{t})H_{t}{}^{h}-(\overset{1}{\nabla}_{s}H_{j}{}^{t})G_{t}{}^{h}\\ &=(\overset{1}{\nabla}_{s}G_{j}{}^{t})H_{t}{}^{h}-\left\{\overset{1}{\nabla}_{s}(F_{a}{}^{t}G_{j}{}^{a})\right\}G_{t}{}^{h}\\ &=(\overset{1}{\nabla}_{s}G_{j}{}^{t})H_{t}{}^{h}-(\overset{1}{\nabla}_{s}G_{j}{}^{a})G_{t}{}^{h}F_{a}{}^{t}\\ &=0,\end{aligned}$$

and consequently

$$\nabla_{j}G_{i}{}^{h}=0. \tag{4.15}$$

We have proved that if $[F, F]=0$, then $\overset{1}{\Gamma}_{j}{}^{h}{}_{i}$ difine a symmetric connection $\overset{1}{\nabla}$ such that $\overset{1}{\nabla}F=0$. If $\overset{1}{\nabla}$ is a symmetric connection, then we have, from (4.6),

$$T_{ji}{}^{h}-T_{ij}{}^{h}=\frac{1}{8}[G, G]_{ji}{}^{h}+\frac{1}{8}[H, H]_{ji}{}^{h}-\frac{1}{4}\left\{F_{j}{}^{s}\overset{1}{\nabla}_{s}G_{i}{}^{t}-F_{i}{}^{s}\overset{1}{\nabla}_{s}G_{j}{}^{t}-(\overset{1}{\nabla}_{j}G_{i}{}^{s}-\overset{1}{\nabla}_{i}G_{j}{}^{s})F_{s}{}^{t}\right\}H_{t}{}^{h},$$

$[G, G]_{ji}{}^{h}$ and $[H, H]_{ji}{}^{h}$ being components of $[G, G]$ and $[H, H]$ respectively. But components $[F, G]_{ji}{}^{h}$ of $[F, G]$ being given by

$$[F, G]_{ji}{}^{h} = F_j{}^t \overset{1}{\nabla}_t G_i{}^h - F_i{}^t \overset{1}{\nabla}_t G_j{}^h - (\overset{1}{\nabla}_j F_i{}^t - \overset{1}{\nabla}_i F_j{}^t) G_t{}^h$$
$$+ G_j{}^t \overset{1}{\nabla}_t F_i{}^h - G_i{}^t \overset{1}{\nabla}_t F_j{}^h - (\overset{1}{\nabla}_j G_i{}^t - \overset{1}{\nabla}_i G_j{}^t) F_t{}^h ,$$

or

$$[F, G]_{ji}{}^{h} = F_j{}^t \overset{1}{\nabla}_t G_i{}^h - F_i{}^t \overset{1}{\nabla}_t G_j{}^h - (\overset{1}{\nabla}_j G_i{}^t - \overset{1}{\nabla}_i G_j{}^t) F_t{}^h ,$$

since $\overset{1}{\nabla}$ is a symmetric connection and $\overset{1}{\nabla}_j F_i{}^h = 0$, we have

$$T_{ji}{}^{h} - T_{ij}{}^{h} = \frac{1}{8}[G, G]_{ji}{}^{h} + \frac{1}{8}[H, H]_{ji}{}^{h} - \frac{1}{4}[F, G]_{ji}{}^{t} H_t{}^h . \tag{4.16}$$

Thus, if we assume that $[F, F]=0$ and $[G, G]=0$, or $[F, F]=0$ and $[F, G]=0$, then, by Theorem 3. 2 or Theorem 3. 3,

$$T_{ji}{}^{h} - T_{ij}{}^{h} = 0 ,$$

and consequently ∇ is a symmetric connection such that

$$\nabla F = 0 , \quad \nabla G = 0$$

and hence

$$\nabla H = \nabla (FG) = 0 .$$

Thus, the converse being evident, the proof of the theorem is completed.

Combining Theorems 3. 9 and 4. 1, we have

THEOREM 4. 2. *In order that there exists, in an almost quaternion manifold, a symmetric affine connection such that*

$$\nabla F = 0 , \quad \nabla G = 0 , \quad \nabla H = 0 ,$$

it is necessary and sufficient that two of Nijenhuis tensors:

$$[F, F] , \quad [G, G] , \quad [H, H] , \quad [G, H] , \quad [H, F] , \quad [F, G]$$

vanish.

§ 5. Affine connections in an almost quaternion manifold. (continued)

In the proof of Theorem 4. 1, we first introduced, in an almost quaternion manifold, a symmetric connection $\overset{0}{\nabla}$ with components $\overset{0}{\Gamma}{}_j{}^h{}_i$ and put

$$\overset{1}{\Gamma}{}_j{}^h{}_i = \overset{0}{\Gamma}{}_j{}^h{}_i + \overset{1}{T}{}_{ji}{}^h, \tag{5.1}$$

where $\overset{1}{T}{}_{ji}{}^h$ is given by (4. 2) and showed that

$$\overset{1}{\nabla}{}_j F_i{}^h = 0. \tag{5.2}$$

Since $\overset{0}{\nabla}$ is a symmetric connection, denoting by

$$\overset{1}{S}{}_{ji}{}^h = \overset{1}{\Gamma}{}_j{}^h{}_i - \overset{1}{\Gamma}{}_i{}^h{}_j \tag{5.3}$$

the torsion tensor of $\overset{1}{\nabla}$, we have, from (4. 2),

$$\overset{1}{S}{}_{ji}{}^h = \frac{1}{8}[F, F]_{ji}{}^h. \tag{5.4}$$

We next put

$$\Gamma_j{}^h{}_i = \overset{1}{\Gamma}{}_j{}^h{}_i + \overset{1}{T}{}_{ji}{}^h, \tag{5.5}$$

where $\overset{1}{T}{}_{ji}{}^h$ is given by (4. 6) and showed that

$$\nabla_j F_i{}^h = 0, \quad \nabla_j G_i{}^h = 0. \tag{5.6}$$

Denoting by

$$S_{ji}{}^h = \Gamma_j{}^h{}_i - \Gamma_i{}^h{}_j \tag{5.7}$$

the torsion tensor of ∇, we have, from (5. 4) and (5. 5),

$$S_{ji}{}^h = \frac{1}{8}[F, F]_{ji}{}^h + T_{ji}{}^h - T_{ij}{}^h. \tag{5.8}$$

We shall now compute $T_{ji}{}^h - T_{ij}{}^h$. From (4. 6), we find

$$\begin{aligned} &T_{ji}{}^h - T_{ij}{}^h \\ &= \frac{1}{4}\{G_j{}^t \overset{1}{\nabla}{}_t G_i{}^h - G_i{}^t \overset{1}{\nabla}{}_t G_j{}^h - (\overset{1}{\nabla}{}_j G_i{}^t - \overset{1}{\nabla}{}_i G_j{}^t) G_t{}^h\} \\ &\quad + \frac{1}{4}\{H_j{}^t \overset{1}{\nabla}{}_t H_i{}^h - H_i{}^t \overset{1}{\nabla}{}_t H_j{}^h - (\overset{1}{\nabla}{}_j H_i{}^t - \overset{1}{\nabla}{}_i H_j{}^t) H_t{}^h\} \\ &\quad - \frac{1}{4}\{F_j{}^s \overset{1}{\nabla}{}_s G_i{}^t - F_i{}^s \overset{1}{\nabla}{}_s G_j{}^t - (\overset{1}{\nabla}{}_j G_i{}^s - \overset{1}{\nabla}{}_i G_j{}^s) F_s{}^t\} H_t{}^h. \end{aligned} \tag{5.9}$$

On the other hand, we have

$$\begin{aligned} &G_j{}^t \overset{1}{\nabla}{}_t G_i{}^h - G_i{}^t \overset{1}{\nabla}{}_t G_j{}^h - (\overset{1}{\nabla}{}_j G_i{}^t - \overset{1}{\nabla}{}_i G_j{}^t) G_t{}^h \\ &= \frac{1}{2}[G, G]_{ji}{}^h - \overset{1}{S}{}_{ji}{}^h + G_j{}^c G_i{}^b \overset{1}{S}{}_{cb}{}^h \\ &\qquad - (G_j{}^a \overset{1}{S}{}_{ai}{}^t - G_i{}^a \overset{1}{S}{}_{aj}{}^t) G_t{}^h \end{aligned}$$

and

$$H_j{}^t \overset{1}{\nabla}_t H_i{}^h - H_i{}^t \overset{1}{\nabla}_t H_j{}^h - (\overset{1}{\nabla}_j H_i{}^t - \overset{1}{\nabla}_i H_j{}^t) G_t{}^h$$
$$= \frac{1}{2}[H, H]_{ji}{}^h - \overset{1}{S}_{ji}{}^h + H_j{}^c H_i{}^b \overset{1}{S}_{cb}{}^h$$
$$-(H_j{}^a \overset{1}{S}_{ai}{}^t - H_i{}^a \overset{1}{S}_{aj}{}^t) H_t{}^h,$$

and consequently

$$\left\{G_j{}^t \overset{1}{\nabla}_t G_i{}^h - G_i{}^t \overset{1}{\nabla}_t G_j{}^h - (\overset{1}{\nabla}_j G_i{}^t - \overset{1}{\nabla}_i G_j{}^t) G_t{}^h\right\} \tag{5.10}$$
$$+\left\{H_j{}^t \overset{1}{\nabla}_t H_i{}^h - H_i{}^t \overset{1}{\nabla}_t H_j{}^h - (\overset{1}{\nabla}_j H_i{}^t - \overset{1}{\nabla}_i H_j{}^t) H_t{}^h\right\}$$
$$= \frac{1}{2}[G, G]_{ji}{}^h + \frac{1}{2}[H, H]_{ji}{}^h - \frac{1}{4}[F, F]_{ji}{}^h,$$

since

$$G_j{}^c G_i{}^b \overset{1}{S}_{cb}{}^h = \frac{1}{8} F_t{}^c H_j{}^t F_s{}^b H_i{}^s [F, F]_{cb}{}^h$$
$$= -\frac{1}{8} H_j{}^t H_i{}^s [F, F]_{ts}{}^h$$
$$= -H_j{}^t H_i{}^s \overset{1}{S}_{ts}{}^h$$

and

$$G_j{}^a \overset{1}{S}_{ai}{}^t G_t{}^h = -\frac{1}{8} F_b{}^a H_j{}^b [F, F]_{ai}{}^t H_s{}^h F_t{}^s$$
$$= -\frac{1}{8} H_j{}^b [F, F]_{bi}{}^s H_s{}^h$$
$$= -H_j{}^a \overset{1}{S}_{ai}{}^t H_t{}^h$$

because of

$$F_t{}^c F_s{}^b [F, F]_{cb}{}^h = -[F, F]_{ts}{}^h$$

and

$$F_b{}^a [F, F]_{ai}{}^t F_t{}^s = [F, F]_{bi}{}^s.$$

We also have

$$\left\{F_j{}^s \overset{1}{\nabla}_s G_i{}^t - F_i{}^s \overset{1}{\nabla}_s G_j{}^t - (\overset{1}{\nabla}_j G_i{}^s - \overset{1}{\nabla}_i G_j{}^s) F_s{}^t\right\} H_t{}^h$$
$$= \left\{[F, G]_{ji}{}^t + F_j{}^c G_i{}^b \overset{1}{S}_{cb}{}^t + G_j{}^c F_i{}^b \overset{1}{S}_{cb}{}^t\right.$$
$$\left. -(G_j{}^a \overset{1}{S}_{ai}{}^s - G_i{}^a \overset{1}{S}_{aj}{}^s) F_s{}^t - (F_j{}^a \overset{1}{S}_{ai}{}^s - F_i{}^a \overset{1}{S}_{aj}{}^s) G_s{}^t\right\} H_t{}^h$$

$$= [F, G]_{ji}{}^t H_t{}^h + F_j{}^c G_i{}^b \overset{1}{S}_{cb}{}^t H_t{}^h + G_j{}^c F_i{}^b \overset{1}{S}_{cb}{}^t H_t{}^h$$
$$-(G_j{}^a \overset{1}{S}_{ai}{}^s - G_i{}^a \overset{1}{S}_{aj}{}^s) G_s{}^h + (F_j{}^a \overset{1}{S}_{ai}{}^s - F_i{}^a \overset{1}{S}_{aj}{}^s) F_s{}^h$$
$$= [F, G]_{ji}{}^t H_t{}^h + \frac{1}{4}[F, F]_{ji}{}^h + 2(G_j{}^a \overset{1}{S}_{ai}{}^t - G_i{}^a \overset{1}{S}_{aj}{}^t) G_t{}^h ,$$

and consequently

$$\left\{F_j{}^s \overset{1}{\nabla}_s G_i{}^t - F_i{}^s \overset{1}{\nabla}_s G_j{}^t - (\overset{1}{\nabla}_j G_i{}^s - \overset{1}{\nabla}_i G_j{}^s) F_s{}^t\right\} H_t{}^h$$
$$= \left\{H \barwedge [F, G] + \frac{1}{4}[F, F] + \frac{1}{4} G \barwedge [F, F] \barwedge G\right\}_{ji}{}^h , \tag{5.11}$$

since

$$F_j{}^c G_i{}^b \overset{1}{S}_{cb}{}^t H_t{}^h = -\frac{1}{8} F_j{}^c G_i{}^b [F, F]_{cb}{}^t G_a{}^h F_t{}^a$$
$$= -\frac{1}{8} G_i{}^b [F, F]_{jb}{}^a G_a{}^h$$
$$= G_i{}^b \overset{1}{S}_{bj}{}^a G_a{}^h ,$$
$$G_j{}^c F_i{}^b \overset{1}{S}_{cb}{}^t H_t{}^h = -G_j{}^a \overset{1}{S}_{ai}{}^s G_s{}^h ,$$

and

$$F_j{}^a \overset{1}{S}_{ai}{}^s F_s{}^h = \frac{1}{8} F_j{}^a [F, F]_{ai}{}^s F_s{}^h$$
$$= \frac{1}{8}[F, F]_{ji}{}^h .$$

Thus, from (5.8), (5.9), (5.10) and (5.11), we find

$$S = \frac{1}{8}\left\{[G, G] + [H, H] - 2H \barwedge [F, G] + \frac{1}{2} G \barwedge [F, F] \barwedge G\right\} , \tag{5.12}$$

S being the torsion tensor of ∇.

On the other hand, we have, from (2.2),

$$[H, F] = F \barwedge [F, G] + \frac{1}{2}[F, F] \barwedge G$$

and consequently

$$G \barwedge [H, F] = -H \barwedge [F, G] + \frac{1}{2} G \barwedge [F, F] \barwedge G ,$$

hence

$$\frac{1}{2} G \barwedge [F, F] \barwedge G = G \barwedge [H, F] + H \barwedge [F, G] .$$

Substituting this into (5. 12), we find

$$S=\frac{1}{8}\Big\{[G, G]+[H, H]+G\barwedge[H, F]-H\barwedge[F, G]\Big\}. \tag{5. 13}$$

If we let G, H, F play the roles of F, G, H respectively in the discussion above, we obtain a connection $'\nabla$ such that

$$'\nabla G=0, \quad '\nabla H=0, \quad '\nabla F=0$$

and the torsion tensor $'S$ of $'\nabla$ is given by

$$'S=\frac{1}{8}\Big\{[H, H]+[F, F]+H\barwedge[F, G]-F\barwedge[G, H]\Big\}$$

and if we let H, F, G play the roles of F, G, H respectively in the discussion above, we obtain a connection $''\nabla$ such that

$$''\nabla H=0, \quad ''\nabla F=0, \quad ''\nabla G=0$$

and the torsion tensor $''S$ of $''\nabla$ is given by

$$''S=\frac{1}{8}\Big\{[F, F]+[G, G]+F\barwedge[G, H]-G\barwedge[H, F]\Big\}.$$

Thus if we define a connection by

$$\frac{1}{3}(\Gamma_j{}^h{}_i+{}'\Gamma_j{}^h{}_i+{}''\Gamma_j{}^h{}_i),$$

$'\Gamma_j{}^h{}_i$ and $''\Gamma_j{}^h{}_i$ being respectively components of the connections $'\nabla$ and $''\nabla$, the covariant derivatives of F, G and H with respect to this connection are zero and the torsion tensor of this connection is given by

$$\frac{1}{12}\Big\{[F, F]+[G, G]+[H, H]\Big\}$$

(Obata [5]).

§6. Discussions in terms of complex coordinates

Assume that a $4n$-dimensional differentiable manifold V admits an almost quaternion structure F, G, H and that

$$[F, F]=0. \tag{6. 1}$$

Then the manifold is complex analytic and is covered by a system of complex coordinate neighborhoods U; z^κ, $z^{\bar\kappa}$ ($z^{\bar\kappa}=\bar z^\kappa$), ($\kappa, \lambda, \mu, \cdots=1, 2, \cdots, 2n$; $\bar\kappa, \bar\lambda, \bar\mu, \cdots=2n+1, 2n+2, \cdots, 4n$) with respect to which the tensor field F

of type (1, 1) has components of the form

$$F=\begin{pmatrix} iE & 0 \\ 0 & -iE \end{pmatrix}, \tag{6.2}$$

where E is the $2n\times 2n$ unit matrix.

We represent the components of the tensor field G of type (1, 1) with respect to this complex coordinate system by

$$G=\begin{pmatrix} G_1 & G_2 \\ G_3 & G_4 \end{pmatrix},$$

where G_1, G_2, G_3 and G_4 are $2n\times 2n$ matrices.

Then, from $FG+GF=0$, we have

$$\begin{pmatrix} iE & 0 \\ 0 & -iE \end{pmatrix}\begin{pmatrix} G_1 & G_2 \\ G_3 & G_4 \end{pmatrix}+\begin{pmatrix} G_1 & G_2 \\ G_3 & G_4 \end{pmatrix}\begin{pmatrix} iE & 0 \\ 0 & -iE \end{pmatrix}=0,$$

that is,

$$\begin{pmatrix} iG_1+iG_1 & iG_2-iG_2 \\ -iG_3+iG_3 & -iG_4-iG_4 \end{pmatrix}=0,$$

from which

$$G_1=0, \quad G_4=0.$$

Thus G has the form

$$G=\begin{pmatrix} 0 & G' \\ G'' & 0 \end{pmatrix}, \tag{6.3}$$

that is, G is hybrid. (Yano [8])

On the other hand, we have, from $G^2=-1$,

$$G'G''=G''G'=-E, \tag{6.4}$$

and from $H=FG$, we see that H has components

$$H=\begin{pmatrix} 0 & iG' \\ -iG'' & 0 \end{pmatrix} \tag{6.5}$$

with respect to this complex coordinate system $z^h=(z^\kappa, z^{\bar\kappa})$, $(h, i, j \cdots =1, 2, \cdots, 4n)$.

We now consider the condition

$$[G, G]=0. \tag{6.6}$$

In terms of components, (6.6) can be written as

(6. 7) $$G_j^{\ t}\partial_t G_i^{\ h} - G_i^{\ t}\partial_t G_j^{\ h} - (\partial_j G_i^{\ t} - \partial_i G_j^{\ t})G_t^{\ h} = 0\,,$$

where $G_i^{\ h}$ are components of G and $\partial_j = \partial/\partial z^j$.

It will be easily verified that (6. 7) is equivalent to

(6. 8) $$\partial_\mu G_\lambda^{\ \bar\kappa} - \partial_\lambda G_\mu^{\ \bar\kappa} = 0\,,$$
$$G_\mu^{\ \bar\alpha}\partial_{\bar\alpha} G_\lambda^{\ \bar\kappa} - G_\lambda^{\ \bar\alpha}\partial_{\bar\alpha} G_\mu^{\ \bar\kappa} = 0\,.$$

But the second equation of (6. 8) is equivalent to the first. Because the second equation is equivalent to

$$(G_\mu^{\ \bar\alpha}\partial_{\bar\alpha} G_\lambda^{\ \bar\kappa} - G_\lambda^{\ \bar\alpha}\partial_{\bar\alpha} G_\mu^{\ \bar\kappa})G_{\bar\omega}^{\ \mu} = 0\,,$$

or to

$$-\partial_{\bar\omega} G_\lambda^{\ \bar\kappa} + G_\lambda^{\ \bar\alpha} G_\mu^{\ \bar\kappa}\partial_{\bar\alpha} G_{\bar\omega}^{\ \mu} = 0\,,$$

which is equivalent to

$$G_{\bar\nu}^{\ \lambda}(-\partial_{\bar\omega} G_\lambda^{\ \bar\kappa} + G_\lambda^{\ \bar\alpha} G_\mu^{\ \bar\kappa}\partial_{\bar\alpha} G_{\bar\omega}^{\ \mu}) = 0\,,$$

or to

$$G_\lambda^{\ \bar\kappa}(\partial_{\bar\omega} G_{\bar\nu}^{\ \lambda} - \partial_{\bar\nu} G_{\bar\omega}^{\ \lambda}) = 0\,,$$

that is, to

$$\partial_{\bar\omega} G_{\bar\nu}^{\ \lambda} - \partial_{\bar\nu} G_{\bar\omega}^{\ \lambda} = 0\,,$$

which is equivalent to the first equation.

We next consider the condition

(6. 9) $$[F, G] = 0\,.$$

In terms of components, (6. 9) can be written as

(6. 10) $$F_j^{\ t}\partial_t G_i^{\ h} - F_i^{\ t}\partial_t G_j^{\ h} - (\partial_j F_i^{\ t} - \partial_i F_j^{\ t})G_t^{\ h}$$
$$+ G_j^{\ t}\partial_t F_i^{\ h} - G_i^{\ t}\partial_t F_j^{\ h} - (\partial_j G_i^{\ t} - \partial_i G_j^{\ t})F_t^{\ h} = 0\,,$$

or as

(6. 11) $$F_j^{\ t}\partial_t G_i^{\ h} - F_i^{\ t}\partial_t G_j^{\ h} - (\partial_j G_i^{\ t} - \partial_i G_j^{\ t})F_t^{\ h} = 0\,,$$

$F_i^{\ h}$ being constant.

It will be easily verified (Obata [4]) that (6. 11) is equivalent to

$$\partial_\mu G_\lambda^{\ \bar\kappa} - \partial_\lambda G_\mu^{\ \bar\kappa} = 0\,.$$

Thus we have

THEOREM 6. 1. *Under the condition*

$$[F, F] = 0\,,$$

the conditions

$$[G, G]=0 \quad \text{and} \quad [F, G]=0$$

are equivalent.

We now assume that

$$[F, F]=0, \quad [G, G]=0, \tag{6.12}$$

and choose a complex coordinate system $z^h=(z^\kappa, z^{\bar{\kappa}})$ with respect to which the tensor field F has components of the form (6. 2). In this complex coordinate system, the tensor field G has components of the form

$$G=\begin{pmatrix} 0 & G_{\bar{\lambda}}{}^{\kappa} \\ G_{\lambda}{}^{\bar{\kappa}} & 0 \end{pmatrix}, \tag{6.13}$$

where

$$G_{\bar{\lambda}}{}^{\kappa} G_{\mu}{}^{\bar{\lambda}} = -\delta_{\mu}^{\kappa}, \quad G_{\bar{\mu}}{}^{\kappa} G_{\kappa}{}^{\bar{\lambda}} = -\delta_{\bar{\mu}}^{\bar{\lambda}}.$$

We proved in Theorem 4. 1 that under the assumption (6. 12) we can find out a symmetric affine connection ∇ such that

$$\nabla F=0, \quad \nabla G=0, \quad \nabla H=0.$$

We denote the coefficients of ∇ by $\Gamma_j{}^h{}_i$.

Writing down the equations

$$\nabla_j F_i{}^h = \partial_j F_i{}^h + \Gamma_j{}^h{}_t F_i{}^t - \Gamma_j{}^t{}_i F_t{}^h = 0$$

in the complex coordinate system $z^h=(z^\kappa, \bar{z}^{\bar{\kappa}})$, we find that $\Gamma_j{}^h{}_i$ are all zero except

$$\Gamma_{\mu}{}^{\kappa}{}_{\lambda}, \quad \Gamma_{\bar{\mu}}{}^{\bar{\kappa}}{}_{\bar{\lambda}} = \overline{\Gamma_{\mu}{}^{\kappa}{}_{\lambda}}.$$

Writing down next the equations

$$\nabla_j G_i{}^h = \partial_j G_i{}^h + \Gamma_j{}^h{}_t G_i{}^t - \Gamma_j{}^t{}_i G_t{}^h = 0$$

in the complex coordinate system, we find

$$\Gamma_{\mu}{}^{\kappa}{}_{\lambda} = -(\partial_{\mu} G_{\lambda}{}^{\bar{\alpha}}) G_{\bar{\alpha}}{}^{\kappa}, \quad \Gamma_{\bar{\mu}}{}^{\bar{\kappa}}{}_{\bar{\lambda}} = -(\partial_{\bar{\mu}} G_{\bar{\lambda}}{}^{\alpha}) G_{\alpha}{}^{\bar{\kappa}}. \tag{6.14}$$

We now compute the components

$$R_{kji}{}^h = \partial_k \Gamma_{ji}{}^h - \partial_j \Gamma_k{}^h{}_i + \Gamma_k{}^h{}_t \Gamma_j{}^t{}_i - \Gamma_j{}^h{}_t \Gamma_k{}^t{}_i$$

of the curvature tensor of the connection ∇ and find that the components are all zero except

$$R_{\bar{\nu}\mu\lambda}{}^{\kappa} = -R_{\mu\bar{\nu}\lambda}{}^{\kappa}, \quad R_{\bar{\nu}\mu\bar{\lambda}}{}^{\bar{\kappa}} = -R_{\mu\bar{\nu}\bar{\lambda}}{}^{\bar{\kappa}},$$

where

$$R_{\bar{\nu}\mu\lambda}{}^{\kappa}=\partial_{\bar{\nu}}\Gamma_{\mu}{}^{\kappa}{}_{\lambda}\,,\quad R_{\nu\bar{\mu}\bar{\lambda}}{}^{\bar{\kappa}}=\partial_{\nu}\Gamma_{\bar{\mu}}{}^{\bar{\kappa}}{}_{\bar{\lambda}}\,.$$

On the other hand, M Obata [4] proved

Theorem. In order that there exists a coordinate system in which the components of the tensors F, G, H defining an almost quaternion structure are all constant, it is necessary and sufficient that

$$[F,F]=0\,,\quad [G,G]=0$$

and

$$\partial_{\bar{\nu}}\left\{(\partial_{\mu}G_{\lambda}{}^{\bar{\alpha}})G_{\bar{\alpha}}{}^{\kappa}\right\}=0$$

in a complex coordinate system in which F has components

$$F=\begin{pmatrix} iE & 0 \\ 0 & -iE \end{pmatrix}.$$

Thus we have

THEOREM 6.2. (Obata [4]) *A necessary and sufficient condition that an almost quaternion structure* (F, G, H) *is integrable is that*

$$[F,F]=0\,,\quad [G,G]=0$$

and

$$R_{kji}{}^{h}=0\,,$$

where $R_{kji}{}^{h}$ *are components of the curvature tensor of a symmetric affine connection* ∇ *such that* $\nabla F=0$, $\nabla G=0$.

Combining Theorems 3.9 and 6.2 we have

THEOREM 6.3. *A necessary and sufficient condition that an almost quaternion structure* (F, G, H) *is integrable is that two of Nijenhuis tensors*

$$[F,F],\quad [G,G],\quad [H,H],\quad [G,H],\quad [H,F],\quad [F,G]$$

vanish and

$$R_{kji}{}^{h}=0\,,$$

where $R_{kji}{}^{h}$ *are those in the theorem above.*

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