On radicals of group rings of Frobenius groups

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1. Introduction

Throughout the present paper, $\mathfrak{G}$ will represent a Frobenius group with a Frobenius subgroup $\mathfrak{H}$ and a Frobenius kernel $\mathfrak{N}$. Then, by [3; (25. 2)], $\mathfrak{G}$ is a semi-direct product of $\mathfrak{H}$ and $\mathfrak{N}$. Let p be a prime divisor of $|\mathfrak{G}|$, and K an algebraically closed field of characteristic p. The purpose of this paper is to determine the K-dimension $[J(K\mathfrak{G})\colon K]$ of the radical $J(K\mathfrak{G})$ of the group ring $K\mathfrak{G}$. If p is a divisor of $|\mathfrak{N}|$ and $\mathfrak{P}$ is a p-Sylow subgroup of $\mathfrak{N}$ (of $\mathfrak{G}$), then by Thompson's theorem ([3; (25. 10]), $\mathfrak{P}$ is a normal subgroup of $\mathfrak{G}$, and hence $[J(K\mathfrak{G})\colon K]=(\mathfrak{G}\colon \mathfrak{P})\cdot(|\mathfrak{P}|-1)$ ([2; Ex. 64. 1]). Therefore, in this paper, we shall restrict our attention to the case that p is a divisor of $|\mathfrak{H}|$. In §2, we shall prove that $[J(K\mathfrak{G})\colon K]=[J(K\mathfrak{H})\colon K]$, and in the subsequent sections from §3, we shall study the dimension of $J(K\mathfrak{H})$. We shall use freely Tsushima's theorem ([6; Prop. 1]) and Zassenhaus' theorems ([8; Satz 8 and Satz 16]). In this paper, the groups of type A'), $\cdots$, E'), F), G) in the sense of [8] will be called the groups of type $A, \cdots$, type E, type F, type G, respectively. Moreover, every module is a left module and of finite dimension over K.

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2. $[J(K\mathfrak{G})\colon K]=[J(K\mathfrak{H})\colon K]$

Two irreducible $K\mathfrak{N}$-modules $\mathfrak{T}_1$, $\mathfrak{T}_2$ are said to be conjugate if $\mathfrak{T}_1$ is isomorphic to a $K\mathfrak{N}$-module $X\otimes\mathfrak{T}_2$ $(\subseteq\mathfrak{T}_2^{\mathfrak{G}}=K\mathfrak{G}\otimes_{K\mathfrak{N}}\mathfrak{T}_2)$ for some $X\in\mathfrak{G}$. Let c be the number of p-regular classes of $\mathfrak{H}$, and $d+1$ the number of conjugate classes of $\mathfrak{N}$. At first, we shall state the following:

LEMMA 1 (cf. [3; (25. 4)]). *If $\mathfrak{T}$ is a non-trivial irreducible $K\mathfrak{N}$-module, then $T^{\mathfrak{G}}$ is an irreducible $K\mathfrak{G}$-module.*

PROOF. By [6; Lemma 2], it suffices to prove that for every $X\in\mathfrak{H}^*=\mathfrak{H}-1$, $X\otimes\mathfrak{T}(\subseteq\mathfrak{T}^{\mathfrak{G}})$ is not isomorphic to $\mathfrak{T}$ as a $K\mathfrak{N}$-module. Let $\mathfrak{K}(\neq 1)$ be a conjugate class of $\mathfrak{N}$ and Y an element of $\mathfrak{K}$. Then, $\cup_{X\in\mathfrak{H}}\mathfrak{K}^X$ is a conjugate class of $\mathfrak{G}$ containing Y. Since $C_{\mathfrak{G}}(Z)\subseteq\mathfrak{N}$ for every $Z\in\mathfrak{N}^*$, $(\mathfrak{G}\colon C_{\mathfrak{G}}(Y))=|\mathfrak{H}|\cdot(\mathfrak{N}\colon C_{\mathfrak{N}}(Y))$ and hence $|\mathfrak{K}|\cdot|\mathfrak{H}|=|\cup_{X\in\mathfrak{H}}\mathfrak{K}^X|$. Thus, $\mathfrak{K}^X\neq\mathfrak{K}$ for every

$X \in \mathfrak{H}$. Accordingly, $|\mathfrak{H}|$ is a divisor of d and $1+d/|\mathfrak{H}|$ is the number of orbits of a permutation group $\mathfrak{H}$ acting on the set of all conjugate classes of $\mathfrak{N}$. Then, by Brauer's lemma ([3; (12.1)]), $1+d/|\mathfrak{H}|$ is the number of orbits of a permutation group $\mathfrak{H}$ acting on the set of all complex irreducible characters of $\mathfrak{N}$. Hence, noting that p is not a divisor of $|\mathfrak{N}|$, $X \otimes \mathfrak{T}$ can not be isomorphic to $\mathfrak{T}$ as a $K\mathfrak{N}$-module for every element $X \in \mathfrak{H}^*$ (cf. [2; p. 600, Remark (1)]).

REMARK. By the proof of Lemma 1, we readily see the following:

(1) $|\mathfrak{H}|$ is a divisor of d.

(2) $\mathfrak{N}$ contains $1+d/|\mathfrak{H}|$ conjugate classes of $\mathfrak{G}$.

(3) $d/|\mathfrak{H}|$ is the number of non-trivial and non-conjugate irreducible $K\mathfrak{N}$-modules.

LEMMA 2. *$c+d/|\mathfrak{H}|$ is the number of p-regular conjugate classes of $\mathfrak{G}$.*

PROOF. Let $\mathfrak{C}_1=1, \mathfrak{C}_2, \cdots, \mathfrak{C}_h$ be all the conjugate classes of $\mathfrak{H}$, and X_j an element of $\mathfrak{C}_j$. Then, $\hat{\mathfrak{C}}_j=\cup_{X\in\mathfrak{N}}\mathfrak{C}_j^X$ is a conjugate class of $\mathfrak{G}$ containing X_j. Since $\mathfrak{N}=\mathfrak{G}-\cup_{Y\in\mathfrak{G}}(\mathfrak{H}^Y-1)$, we have a disjoint union $\mathfrak{G}=\mathfrak{N}\cup\hat{\mathfrak{C}}_2\cup\cdots\cup\hat{\mathfrak{C}}_h$. By Remark (2), $\mathfrak{N}$ contains $1+d/|\mathfrak{H}|$ conjugate classes of $\mathfrak{G}$. Hence, $c+d/|\mathfrak{H}|$ is the number of p-regular conjugate classes.

Since $\mathfrak{H}$ is homomorphic to $\mathfrak{G}$, every irreducible $K\mathfrak{H}$-module may be regarded as an irreducible $K\mathfrak{G}$-module. Concerning irreducible representations of $K\mathfrak{G}$, we shall prove the following:

THEOREM 3. *Let $\mathfrak{S}_1, \cdots, \mathfrak{S}_c$ be all the non-isomorphic irreducible $K\mathfrak{H}$-modules, and $\mathfrak{T}_1, \cdots, \mathfrak{T}_{d/|\mathfrak{H}|}$ all the non-trivial and non-conjugate irreducible $K\mathfrak{N}$-modules. Then, $\mathfrak{S}_1, \cdots, \mathfrak{S}_c$ (as $K\mathfrak{G}$-modules), $\mathfrak{T}_1^{\mathfrak{G}}, \cdots, \mathfrak{T}_{d/|\mathfrak{H}|}^{\mathfrak{G}}$ exhaust the non-isomorphic irreducible $K\mathfrak{G}$-modules.*

PROOF. By Lemma 2, $c+d/|\mathfrak{H}|$ is the number of all non-isomorphic irreducible $K\mathfrak{G}$-modules. Accordingly, by Lemma 1, it suffices to prove that the above modules exhaust all the non-isomorphic ones. Since $\mathfrak{T}_i$ is a non-trivial $K\mathfrak{N}$-module and $\mathfrak{S}_j$ is a trivial $K\mathfrak{N}$-module, it follows $\mathrm{Hom}_{K\mathfrak{G}}(\mathfrak{T}_i^{\mathfrak{G}}, \mathfrak{S}_j) \cong \mathrm{Hom}_{K\mathfrak{N}}(\mathfrak{T}_i, \mathfrak{S}_j)=0$, and hence $\mathfrak{T}_i^{\mathfrak{G}}$ is not isomorphic to $\mathfrak{S}_j$. While, $\mathfrak{T}_i$ is not conjugate to $\mathfrak{T}_j$ for every $i \neq j$, and so $\mathrm{Hom}_{K\mathfrak{G}}(\mathfrak{T}_i^{\mathfrak{G}}, \mathfrak{T}_j^{\mathfrak{G}}) \cong \mathrm{Hom}_{K\mathfrak{N}}(\mathfrak{T}_i, \mathfrak{T}_j^{\mathfrak{G}}) \cong \sum_{X\in\mathfrak{H}} \oplus \mathrm{Hom}_{K\mathfrak{N}}(\mathfrak{T}_i, X \otimes \mathfrak{T}_j)=0$, which means that $\mathfrak{T}_i^{\mathfrak{G}}$ is not isomorphic to $\mathfrak{T}_j^{\mathfrak{G}}$ for every $i \neq j$.

The next is fundamental in our whole study.

THEOREM 4. *$[J(K\mathfrak{G}): K]=[J(K\mathfrak{H}): K]$, and so $J(K\mathfrak{G})=J(K\mathfrak{H})E$ where $E=|\mathfrak{N}|^{-1}\sum_{X\in\mathfrak{N}}X$.*

PROOF. By Th. 3, $[J(K\mathfrak{G}): K]=|\mathfrak{G}|-\sum_{i=1}^{c}[\mathfrak{S}_i: K]^2-\sum_{i=1}^{d/|\mathfrak{H}|}[\mathfrak{T}_i^{\mathfrak{G}}: K]^2$

$=|\mathfrak{G}|-(|\mathfrak{H}|-[J(K\mathfrak{H})\colon K])-|\mathfrak{H}|(\sum_{i=1}^{d/|\mathfrak{H}|}[\mathfrak{T}_i\colon K]^2|\mathfrak{H}|)=|\mathfrak{G}|-|\mathfrak{H}|+[J(K\mathfrak{H})\colon K]$
$-|\mathfrak{H}|(|\mathfrak{N}|-1)=[J(K\mathfrak{H})\colon K]$.

COROLLARY 5 (D. A. R. Wallace [6]). *If $\mathfrak{H}$ is a p-Sylow subgroup of $\mathfrak{G}$, then* $[J(K\mathfrak{G})\colon K]=|\mathfrak{H}|-1$

3. Type A

In this section, we shall determine the dimension of $J(K\mathfrak{H})$ when $\mathfrak{H}$ is of type A, namely, $\mathfrak{H}$ is the group generated by two elements A and B with the defining relations:

(1) $A^m=1,\ B^n=A^t,\ BAB^{-1}=A^r$;

(2) $(r-1, m)=r_0,\ r_0t=m$;

(3) $r^\nu\not\equiv 1\ (m)$ for $1\leq\nu<n$, and $r^n\equiv 1\ (m)$;

(4) $(n, t)=1$ and every prime divisor of n divides r_0.

If p is a prime divisor of t, then $[J(K\mathfrak{H})\colon K]=n[J(K\langle A\rangle)\colon K]=n(m-m')$, where $m=m'p^e$ and $(m', p)=1$: Henceforth, we shall assume that p is a divisor of r_0n and $r_0n=n'p^e$ with $(n', p)=1$. Noting that $(r_0, t)=1$ and hence $\mathfrak{H}'=\langle A^{r_0}, B^{n'}\rangle$ is a normal subgroup of $\mathfrak{H}$ of index n', we can see $[J(K\mathfrak{H})\colon K]=n'[J(K\mathfrak{H}')\colon K]$. Thus, to our end, it suffices to determine the dimension $[J(K\mathfrak{H}')\colon K]$. Let ζ be a primitive t-th root of 1 in K, Θ_k a linear representation of $\langle A^{r_0}\rangle$ in K defined by $A^{r_0}\to\zeta^k$, $\mathfrak{H}'_k=\{X\in\mathfrak{H}'\mid\Theta_k^{(X)}=\Theta_k$ i.e. $\Theta_k^{(X)}(Y)=\Theta_k(XYX^{-1})$ for all $Y\in\langle A^{r_0}\rangle\}$, and $c_k=\min\{f>0\mid kr^{n'f}\equiv k\,(t)\}$. Then, $\mathfrak{H}'_k=\langle A^{r_0}, B^{n'c_k}\rangle$ and Θ_k can be extended to a linear representation $\hat{\Theta}_k$ of $\mathfrak{H}'_k$ by $\hat{\Theta}_k(A^{r_0i}B^{n'c_kj})=\Theta_k(A^{r_0i})$.

THEOREM 6. Let $\{\Theta_{k_1}=1, \cdots, \Theta_{k_s}\}$ *be the set of all non-$\mathfrak{H}'$-conjugate representations of $\langle A^{r_0}\rangle$ in K. Then, $\{\hat{\Theta}_{k_1}^{\mathfrak{H}'}=1, \cdots, \hat{\Theta}_{k_s}^{\mathfrak{H}'}\}$ is the set of all distinct irreducible representations of $\mathfrak{H}'$.*

PROOF. Let $\mathfrak{M}_i$ (resp. $\hat{\mathfrak{M}}_i$) be a representation module of Θ_{k_i} (resp. $\hat{\Theta}_{k_i}$). Then, by [6; Lemma 2], $\hat{\mathfrak{M}}_i^{\mathfrak{H}'}$ is an irreducible $K\mathfrak{H}'$-module and $\mathrm{Hom}_{K\mathfrak{H}'}(\hat{\mathfrak{M}}_i^{\mathfrak{H}'}, \hat{\mathfrak{M}}_j^{\mathfrak{H}'})\cong\mathrm{Hom}_{K\mathfrak{H}'_i}(\hat{\mathfrak{M}}_i, \hat{\mathfrak{M}}_j^{\mathfrak{H}'})\subseteq\mathrm{Hom}_{K\langle A^{r_0}\rangle}(\mathfrak{M}_i, \hat{\mathfrak{M}}_j^{\mathfrak{H}'})\cong\sum_h\oplus\mathrm{Hom}_{K\langle A^{r_0}\rangle}(\mathfrak{M}_i, G_h\otimes\mathfrak{M}_j)$ $=0$ for $i\neq j$, where $\mathfrak{H}'=\cup_h G_h\mathfrak{H}'_j$ is a left coset decomposition of $\mathfrak{H}'$ modulo $\mathfrak{H}'_j$. Hence, $\hat{\mathfrak{M}}_i^{\mathfrak{H}'}$ is not isomorphic to $\hat{\mathfrak{M}}_j^{\mathfrak{H}'}$ for $i\neq j$. Since p is not a divisor of t, $\langle A^{r_0}\rangle$ is the set of p-regular elements of $\mathfrak{H}'$. If Θ_i is conjugate to Θ_j then A^{r_0i} is conjugate to A^{r_0j}, and conversely. Hence, s is the number of p-regular classes and $\{\hat{\Theta}_{k_i}^{\mathfrak{H}'}|1\leq i\leq s\}$ is the set of all irreducible representations.

COROLLARY 7. (1) *If p is a divisor of t, then $[J(K\mathfrak{H})\colon K]=n(m-m')$, where $m=m'p^e$ and $(m', p)=1$.*

(2) *If p is a divisor of r_0n, then $[J(K\mathfrak{H})\colon K]=mn-n'(\sum_{j=1}^{s}c_j^2)$, where $r_0n=n'p^e$ and $(n', p)=1$.*

4. Modular irreducible representations of a finite group

Let $\mathfrak{A}$ be a finite group, $\mathfrak{B}$ a normal subgroup of $\mathfrak{A}$ with $(\mathfrak{A}:\mathfrak{B})=p$, $\{\mathfrak{M}_1, \cdots, \mathfrak{M}_a\}$ the set of all irreducible $K\mathfrak{B}$-modules which can be extended to $K\mathfrak{A}$-modules $\{\hat{\mathfrak{M}}_1, \cdots, \hat{\mathfrak{M}}_a\}$, and $\{\mathfrak{M}_{a+1}, \cdots, \mathfrak{M}_{a+b}\}$ the set of all non-conjugate irreducible $K\mathfrak{B}$-modules which can not be extended to $K\mathfrak{A}$-modules*).

THEOREM 8. *$\{\hat{\mathfrak{M}}_1, \cdots, \hat{\mathfrak{M}}_a, \mathfrak{M}_{a+1}^{\mathfrak{A}}, \cdots, \mathfrak{M}_{a+b}^{\mathfrak{A}}\}$ is the set of all irreducible $K\mathfrak{A}$-modules and $a+pb$ is the number of all irreducible $K\mathfrak{B}$-modules.*

PROOF. Let $\mathfrak{L}$ be an irreducible $K\mathfrak{A}$-module, and $\mathfrak{T}$ a composition factor of $\mathfrak{L}$ as a $K\mathfrak{B}$-module. Then, the $K\mathfrak{A}$-module $\mathfrak{L}$ is a composition factor of $\mathfrak{T}^A$. If $\mathfrak{B}=\mathrm{I}(\mathfrak{T})\ (=\{X\in\mathfrak{A}\,|\,X\otimes\mathfrak{T}\cong\mathfrak{T}\})$, then $\mathfrak{T}^A$ is irreducible and $\mathfrak{T}^A$ is isomorphic to $\mathfrak{L}$ by [6; Lemma 3]. If $\mathrm{I}(\mathfrak{T})=\mathfrak{A}$, then $\mathfrak{T}$ can be extended to $\mathfrak{L}$ by [6; Lemma 3]. Therefore, it remains only to prove that the above modules are all distinct. If $1\leq i\leq a$ and $a+1\leq j\leq a+b$, then $\mathrm{Hom}_{K\mathfrak{A}}(\mathfrak{M}_j^{\mathfrak{A}}, \hat{\mathfrak{M}}_i)\cong \mathrm{Hom}_{K\mathfrak{B}}(\mathfrak{M}_j, \mathfrak{M}_i)=0$. While, if $a+1\leq i\neq j\leq a+b$, then $\mathrm{Hom}_{K\mathfrak{A}}(\mathfrak{M}_i^{\mathfrak{A}}, \mathfrak{M}_j^{\mathfrak{A}})\cong \mathrm{Hom}_{K\mathfrak{B}}(\mathfrak{M}_i, \mathfrak{M}_j^{\mathfrak{A}})\cong \sum_{t=0}^{p-1}\oplus\mathrm{Hom}_{K\mathfrak{B}}(\mathfrak{M}_i, X^t\otimes\mathfrak{M}_j)=0$, where $\mathfrak{A}=\cup_{t=0}^{p-1}X^t\mathfrak{B}$ is a left coset decomposition of $\mathfrak{A}$ modulo $\mathfrak{B}$. Hence, the first assertion has been proved. Regarding $\mathfrak{A}/\mathfrak{B}$ as a permutation group acting on the set of all irreducible $K\mathfrak{B}$-modules, we can see that the lengths of the orbits are 1 or p. Hence, by [6; Lemma 3], $a+pb$ is the number of all distinct irreducible $K\mathfrak{B}$-modules.

5. Type B

In this section, we shall determine the dimension of $J(K\mathfrak{H})$ when $\mathfrak{H}$ is of type B, namely, $\mathfrak{H}$ is the group generated by elements A, B and R with the defining relations:

(1) $\langle A, B\rangle$ is of type A;
(2) $RAR^{-1}=A^l$, $RBR^{-1}=B^l$;
(3) $l^2\equiv 1\ (m)$, $l\equiv 1\ (n)$, and $l\equiv -1\ (4)$;
(4) $n\equiv 0\ (2)$ and $R^2=B^{nr_0/2}$.

If p is odd, then $[J(K\mathfrak{H})\colon K]=2[J(K\langle A, B\rangle)\colon K]$ and we can reduce the problem to that in §3. Thus, we may, and shall, restrict our attension to the case $p=2$. Let $n=2^e n'$ with $(2, n')=1$. Then, $\mathfrak{H}'=\langle A^{r_0}, B^{n'}, R\rangle$ is a normal subgroup of $\mathfrak{H}=\langle A^{r_0}, B, R\rangle$**) with $(\mathfrak{H}\colon \mathfrak{H}')=n'$, and hence $[J(K\mathfrak{H})\colon K]=n'[J(K\mathfrak{H}')\colon K]$. Obiously, $\mathfrak{H}''=\langle A^{r_0}, B^{n'}\rangle$ is a normal subgroup of $\mathfrak{H}'$ with $(\mathfrak{H}'\colon \mathfrak{H}'')=2$. Now, let ζ be a primitive t-th root of 1 in K, η

*) "A $K\mathfrak{B}$-module $\mathfrak{M}$ can be extend to a $K\mathfrak{A}$-module $\hat{\mathfrak{M}}$" means that there exists a $K\mathfrak{A}$-module $\hat{\mathfrak{M}}$ such that a $K\mathfrak{B}$-module $\hat{\mathfrak{M}}$ is isomorphic to a $K\mathfrak{B}$-module $\mathfrak{M}$.

**) $\mathfrak{H}=\langle A^{r_0}, B, R\rangle$ by $(r_0, t)=1$.

a primitive r'-th root of 1 where $r_0=2^f r'$ with $(2, r')=1$, s the number of non-$\mathfrak{H}''$-conjugate linear representations of $\langle A^{r_0}\rangle$, and $\{\Lambda_{k_i,j}|1\leq i\leq s,\ 0\leq k_i\leq t-1,\ 0\leq j\leq r'-1\}$ the set of all irreducible representations of $\mathfrak{H}''$ defined by

$$A^{r_0}\longrightarrow Y=\begin{pmatrix} b_1 & & & 0\\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ 0 & & & b_{c_i}\end{pmatrix},\quad B^{n'}\longrightarrow Z=\eta^j\begin{pmatrix} 0 & & & & 1\\ 1 & 0 & & & \\ & \cdot & \cdot & & 0\\ 0 & & \cdot & \cdot & \\ & & & 1 & 0\end{pmatrix}$$

where $b_\alpha=\zeta^{k_i r_\alpha}$, $r_\alpha=r^{nr_0-n'(\alpha-1)}$, $1\leq\alpha\leq c_i=\min\{g>0|k_i\equiv k_i r^{n'g}\ (t)\}$*). Then, the irreducible representations of $\mathfrak{H}'$ are given in the following

THEOREM 9. *$\Lambda_{k_i,j}$ can be extended to an irreducible representation $\hat{\Lambda}_{k_i,j}$ of $\mathfrak{H}'$ if and only if*

$$(1)\ \begin{cases} k_i l\equiv k_i\ (t)\\ jl\equiv j\ (r')\end{cases}\quad \text{or}\quad (2)\ \begin{cases} c_i\equiv 0\ (2),\quad k_i r^{n'c_i/2}\equiv k_i l\ (t)\\ jl\equiv j\ (r')\end{cases}$$

PROOF. At first, we shall notice that c_i is a divisor of $\tau=2^{e-1}r_0$ and hence $Z^\tau=I_{c_i}$ (the identity matrix of degree c_i). By $A^{r^{\tau n'}}=R^2AR^{-2}=A^{l^2}=A$, we see that $(r^n)^\tau\equiv 1\ (t)$, which implies that c_i is a divisor of τ. If $\Lambda_{k_i,j}$ can be extended, then there exists a regular matrix X of degree c_i such that $X^2=Z^\tau=I_{c_i}$, $XY=Y^lX$ and $XZ=Z^lX$. Since $r^n\equiv 1\ (t)$ and $l\equiv 1\ (n)$, it follows $l\equiv 1\ (c_i)$ and $Z^l=\eta^{j(l-1)}Z$. By $XZ=\eta^{j(l-1)}ZX$, we have $jl\equiv j\ (r')$ and

$$X=\begin{pmatrix} a_0 & a_{c_i-1} & & a_1\\ a_1 & \cdot & \cdot & \\ & \cdot & \cdot & a_{c_i-1}\\ a_{c_i-1} & & a_1 & a_0\end{pmatrix}\begin{pmatrix} 1 & & & 0\\ & \eta^{j(l-1)} & & \\ & & \eta^{2j(l-1)} & \\ & & & \ddots & \\ 0 & & & & \eta^{(c_i-1)j(l-1)}\end{pmatrix}.$$

Since $XY=Y^lX$ and X is regular, $k_i r^{n'\beta}\equiv k_i l\ (t)$ for a non zero a_β, where $0\leq\beta<c_i$. Let γ be an integer such that $k_i r^{n'\gamma}\equiv k_i l\ (t)$ and $0\leq\gamma<c_i$. Then, $k_i r^{n'(\beta-\gamma)}\equiv k_i\ (t)$ and hence $\beta\equiv\gamma\ (c_i)$. Therefore, $\beta=\gamma$ and the only a_β is non zero. By $X^2=I_{c_i}$, we have

$$(1)\ \begin{cases} k_i l\equiv k_i\ (t)\\ jl\equiv j\quad (r')\end{cases}\quad \text{or}\quad (2)\ \begin{cases} c_i\equiv 0\ (2),\quad k_i r^{n'c_i/2}\equiv k_i l\ (t)\\ jl\equiv j\ (r')\end{cases}$$

*) This assertion follows by the next proof:
$\Lambda_{k_i,j}$ is irreducible by Th. 6. If $\Lambda_{k_i,a}$ is equivalent to $\Lambda_{k_j,b}$, then $i=j$, $\Lambda_{k_i,a}(B^{n'c_i})=\Lambda_{k_i,b}(B^{n'c_i})$ and hence $ac_i\equiv bc_i\ (r')$. Thus, $a=b$ by noting that c_i is a divisor of 2^e. Let $\mathfrak{K}_1, \mathfrak{K}_2,\cdots,\mathfrak{K}_s$ be conjugate classes of $\mathfrak{H}''$ which is contained in $\langle A^{r_0}\rangle$. Then $\{A^{2^f ti}\cdot\mathfrak{K}_j|0\leq i\leq r'-1, 1\leq j\leq s\}$ is the set of all 2-regular classes of $\mathfrak{H}''$.

Conversely, if $k_i l \equiv k_i\ (t)$ and $jl \equiv j\ (r')$, then $\Lambda_{k_i,j}$ can be extended to an irreducible representation $\hat{\Lambda}_{k_i,j}$ of $\mathfrak{H}'$ by $R \to I_{c_i}$. While, if $c_i \equiv 0\ (2)$, $k_i r^{n'c_i/2} \equiv k_i l\ (t)$, and $jl \equiv j\ (r')$, then $\Lambda_{k_i,j}$ can be extended to an irreducible representation $\hat{\Lambda}_{k_i,j}$, of $\mathfrak{H}'$ by $R \to \begin{pmatrix} 0 & I_{c_i/2} \\ I_{c_i/2} & 0 \end{pmatrix}$.

Now, combining Th. 8 with Th. 9, we readily obtain

COROLLARY 10. (1) *If* $p>2$, *then* $[J(K\mathfrak{H}) : K] = 2[J(K\langle A, B\rangle) : K]$, *where* $\langle A, B\rangle$ *is of type A*.

(2) *If* $p=2$, *then* $[J(K\mathfrak{H}) : K] = 2nm - 2n'r'(\sum_{i=1}^{s} c_i^2) + n'u(\sum_{i\in\Omega} c_i^2)$, *where* $u = |\{0 \leq j \leq r'-1 \mid jl \equiv j\ (r')\}|$ *and* $\Omega = \left\{1 \leq i \leq s \,\middle|\, \begin{cases} c_i \equiv 0\ (2) \\ k_i r^{n'c_i/2} \equiv k_i l\ (t) \end{cases} \text{or } k_i l \equiv k_i\ (t)\right\}$.

6. Type C

In this section, we shall give the dimension of $J(K\mathfrak{H})$ when $\mathfrak{H}$ is of type C, namely, $\mathfrak{H}$ is a group generated by elements A, B, P and Q with defining relations:

(1) $\langle A, B\rangle$ is of type A;

(2) $\langle P, Q\rangle$ is the quaternion group of order 8;

(3) $n \equiv m \equiv 1\ (2)$, $n \not\equiv 0\ (3)$ and $m \equiv 0\ (3)$;

(4) $APA^{-1} = Q$, $AQA^{-1} = PQ$, $PB = BP$, $BQ = QB$.

At first, we shall prove that $\langle A^3, B\rangle$ is a normal subgroup of $\mathfrak{H}$. Since $A^{-1}B^2A = A^{(r+1)(r-1)}B^2$ and $(r, 3) = 1$, $A^{-1}B^2A$ is an element of $\langle A^3, B\rangle$. Noting that the order $r_0 n$ of B is odd, we can see that $A^{-1}BA$ is an element of $\langle A^3, B\rangle$, namely, $\langle A^3, B\rangle$ is a normal subgroup of $\mathfrak{H}$. Accordingly, if $p>3$ then $[J(K\mathfrak{H}) : K] = 24[J(K\langle A^3, B\rangle) : K] = 8[J(K\langle A, B\rangle) : K]$. If $p=2$, then $[J(K\mathfrak{H}) : K] = nm[J(K\langle P, Q\rangle) : K] = 7nm$. Henceforth, we shall restrict our attention to the case $p=3$. Since $\mathfrak{H}' = \langle A, P, Q\rangle$ is a normal subgroup of $\mathfrak{H}$ with $(\mathfrak{H} : \mathfrak{H}') = n$, we have $[J(K\mathfrak{H}) : K] = n[J(K\mathfrak{H}') : K]$. To our end, it suffices therefore to give the dimension of $J(K\mathfrak{H}')$. We consider the normal subgroup $\mathfrak{H}'' = \langle A^3\rangle \times \langle P, Q\rangle$ of $\mathfrak{H}'$ whose index is 3. Let ζ be a primitive m'-th root of 1 in K, where $m = 3^e m'$ with $(3, m') = 1$, and Θ_j a linear representation defined by $A^3 \to \zeta^j$. Let λ be a primitive 4-th root of 1 in K, and $\Gamma_0, \Gamma_1, \cdots, \Gamma_4$ irreducible representations defined by

$$\begin{cases} \Gamma_0(P) = 1 \\ \Gamma_0(Q) = 1 \end{cases}, \begin{cases} \Gamma_1(P) = 1 \\ \Gamma_1(Q) = -1 \end{cases}, \begin{cases} \Gamma_2(P) = -1 \\ \Gamma_2(Q) = 1 \end{cases}, \begin{cases} \Gamma_3(P) = -1 \\ \Gamma_3(Q) = -1 \end{cases}, \begin{cases} \Gamma_4(P) = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} \\ \Gamma_4(Q) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \end{cases}.$$

Then, $\{\Delta_{ij} = \Gamma_i \otimes \Theta_j \mid 0 \leq i \leq 4,\ 0 \leq j \leq m'-1\}$ is the set of all irreducible representations of $\mathfrak{H}''$.

THEREM 11. *Δ_{ij} can be extended if and only if $i=0$ or 4.*

PROOF. (1) $i=0$: Let σ be an element of K such that $\sigma^3=\zeta$. Then Δ_{0j} can be extended to $\hat{\Delta}_{0j}$ by $A\to\sigma^j$.

(2) $i=1,2,3$: If Δ_{ij} $(i=1,2,3)$ can be extended, then $\Delta_{ij}(P)=\Delta_{ij}(Q)=1$, which is a contradiction.

(3) $i=4$: Let α be an element of K such that $\alpha^3=(2(1+\lambda))^{-1}\zeta^j$. Then Δ_{4j} can be extended to $\hat{\Delta}_{4j}$ by $A\to\begin{pmatrix} \alpha & -\alpha \\ -\lambda\alpha & -\lambda\alpha \end{pmatrix}$.

The next is a combination of Ths. 8 and 11.

COROLLARY 12. (1) If $p>3$, then $[J(K\mathfrak{H})\colon K]=8[J(K\langle A,B\rangle)\colon K]$.

(2) If $p=3$, then $[J(K\mathfrak{H})\colon K]=8nm-14nm'$, where $m=3^e m'$ and $(3,m')=1$.

(3) If $p=2$, then $[J(K\mathfrak{H})\colon K]=7nm$.

7. Type D

In this section, we shall determine the dimension of $J(K\mathfrak{H})$ when $\mathfrak{H}$ is of type D, namely, $\mathfrak{H}$ is the group generated by elements A, B, P and Q with defining relations:

(1) $\langle A,B\rangle$ is of type A;

(2) $\langle P,Q\rangle$ is the quaternion group of order 8;

(3) $n\equiv m\equiv 1$ (2) and $n\equiv 0$ (3);

(4) $AP=PA$, $AQ=QA$, $BPB^{-1}=Q$ and $BQB^{-1}=PQ$.

If $p>3$ then $[J(K\mathfrak{H})\colon K]=24[J(K\langle A,B^3\rangle)\colon K]=8[J(K\langle A,B\rangle)\colon K]$ and our problem can be reduced to that in §3. If $p=2$ then $[J(K\mathfrak{H})\colon K]=nm[J(K\langle P,Q\rangle)\colon K]=7nm$. In what follows, we shall restrict our attention to the case $p=3$. If $r_0n=3^e n'$ with $(3,n')=1$, then $\mathfrak{H}'=\langle A^{r_0},B^{n'},P,Q\rangle$ is a normal subgroup of $\mathfrak{H}$ with $(\mathfrak{H}\colon\mathfrak{H}')=n'$ and $\mathfrak{H}''=\langle A^{r_0},B^{3n'}\rangle\times\langle P,Q\rangle$ is a normal subgroup of $\mathfrak{H}'$ with $(\mathfrak{H}'\colon\mathfrak{H}'')=3$. Now, let ζ be a primitive t-th root of 1 in K, and Λ_{k_j} $(1\leq j\leq s)$ a representation of $\langle A^{r_0},B^{3n'}\rangle$ defined by

$$A^{r_0}\longrightarrow\begin{pmatrix} b_1 & & & 0 \\ & b_2 & & \\ & & \ddots & \\ 0 & & & b_{c_j} \end{pmatrix},\quad B^{3n'}\longrightarrow\begin{pmatrix} 0 & & & 1 \\ 1 & 0 & & \\ & 1 & \ddots & \\ 0 & & 1 & 0 \end{pmatrix},$$

where $b_i=\zeta^{k_j r'_i}$, $r'_i=r^{r_0n-3n'i+3n'}$, $1\leq i\leq c_j=\min\{f>0|k_jr^{3n'f}\equiv k_j\ (t)\}$. Then, $\{\Delta_{ij}=\Gamma_i\otimes\Lambda_{k_j}|0\leq i\leq 4,\ 1\leq j\leq s\}$ is the set of all irreducible representations of $\mathfrak{H}''$, where $\{\Gamma_i|0\leq i\leq 4\}$ is the set of all irreducible representations of $\langle P,Q\rangle$, which has given in §6. Concerning irreducible representations of

$\mathfrak{H}'$, we have the following:

THEOREM 13. *Δ_{ij} can be extended if and only if $k_j \equiv k_j r^{n'}$ (t) and $i=0$ or 4.*

PROOF. (1) $i=0$: If Δ_{0j} can be extended then there exists a regular matrix $X=(a_{\alpha\beta})$ of degree c_j such that $X\Delta_{0j}(A^{r_0})=\Delta_{0j}(A^{r_0})^{\nu}X$ and $\nu=r^{n'}$. There holds then $a_{\alpha\beta}b_\beta=a_{\alpha\beta}b_\alpha^\nu$ with some α, β. Since X is regular, we obtain $b_\beta=b_\alpha^\nu$, and hence $k_j\nu^{(-3\alpha+3\beta+1)}\equiv k_j$ (t). Noting that $\nu^{3^e}\equiv 1$ (t) and $(3^e, -3\alpha+3\beta+1)=1$, it follows $k_j\equiv k_j r^{n'}$ (t). Conversely, if $k_j\equiv k_j r^{n'}$ (t) then $c_j=1$ and so Δ_{0j} can be extended to a linear representation $\hat{\Delta}_{0j}$ of $\mathfrak{H}'$ by $B^{n'}\to 1$.

(2) $i=1, 2, 3$: If Δ_{ij} $(i=1,2,3)$ can be extended then $\Delta_{ij}(P)=\Delta_{ij}(Q)=I_{c_j}$, which is a contradiction.

(3) $i=4$: If Δ_{4j} can be extended then, by making use of the same argument as in (1), we have $k_j\equiv k_j r^{n'}$ (t). Conversely, if $k_j\equiv k_j r^{n'}$ (t) and $n'\equiv 1$ (3), then Δ_{4j} can be extended to a representation $\hat{\Delta}_{4j}$ by $B^{n'}\to\begin{pmatrix} a & -a \\ -a\lambda & -a\lambda\end{pmatrix}$, where $a^3=(2(1+\lambda))^{-1}$. While, if $k_j\equiv k_j r^{n'}$ (t) and $n'\equiv 2$ (3), then Δ_{4j} can be extended to a representation $\hat{\Delta}_{4j}$ by $B^{n'}\to\begin{pmatrix} a & a\lambda \\ -a & a\lambda\end{pmatrix}$, where $a^3=(2(1-\lambda))^{-1}$.

By Ths. 8 and 13, we obtain the following:

COROLLARY 14. (1) *If $p>3$ then $[J(K\mathfrak{H}): K]=8[J(K\langle A, B\rangle): K]$.*

(2) *If $p=3$ then $[J(K\mathfrak{H}): K]=8nm+10vn'-24n'(\sum_{j=1}^{s}c_j^2)$, where $v=|\{1\leq j\leq s \mid k_j r^{n'}\equiv k_j\ (t)\}|$.*

(3) *If $p=2$ then $[J(K\mathfrak{H}): K]=7nm$.*

8. Type E

In this section, we shall give the dimension of $J(K\mathfrak{H})$ when $\mathfrak{H}$ is of type E, namely, $\mathfrak{H}$ is the group generated by elements A, B, P, Q and R with defining relations:

(1) $\langle A, B, P, Q\rangle$ is of type C;

(2) $R^2=P^2$, $(RQ)^2=1$;

(3) $RAR^{-1}=A^l$, $RBR^{-1}=B^l$;

(4) $l^2\equiv 1$ (m), $l\equiv 1$ (n) and $l\equiv -1$ (3).

If p is odd then $[J(K\mathfrak{H}): K]=2[J(K\langle A, B, P, Q\rangle): K]$. We may restrict therefore our attention to the case $p=2$. Since $\mathfrak{H}'=\langle A, P, Q, R\rangle$ is a normal subgroup of $\mathfrak{H}$ whose index is odd n, $[J(K\mathfrak{H}): K]=n[J(K\mathfrak{H}'): K]$. Obviously, $\langle P, Q\rangle$ is a normal subgroup of $\mathfrak{H}''=\langle A, P, Q\rangle$ and $K\mathfrak{H}''/J(K\mathfrak{H}'')\cong K(\mathfrak{H}''/\langle P, Q\rangle)\cong K\langle A\rangle$. Thus, the set of all irreducible representations of $\mathfrak{H}''$ coincides with that of all irreducible representations of $\langle A\rangle$. Now, let

ω be a primitive m-th root of 1 in K, and Θ_j an irreducible representation of $\mathfrak{H}''$ defined by $A \to \omega^j$, $P \to 1$, $Q \to 1$. Then, $\{\Theta_j | 0 \leq j \leq m-1\}$ is the set of all irreducible representations of $\mathfrak{H}''$ in K.

THEOREM 15. *Θ_j can be extended if and only if $j \equiv jl$ (m).*

PROOF. If $j \equiv jl$ (m) then Θ_j can be extended to a representation $\hat{\Theta}_j$ of $\hat{\mathfrak{G}}'$ by $R \to 1$. The converse is almost evident.

Combining Th. 15 with Th. 8, we obtain the following:

COROLLARY 16. (1) *If p is odd then $[J(K\mathfrak{H}): K] = 2[J(K\langle A, B, P, Q\rangle): K]$, where $\langle A, B, P, Q\rangle$ is of type C.*

(2) *If $p=2$ then $[J(K\mathfrak{H}): K] = n(14m+u)$, where $u = |\{0 \leq j \leq m-1 | j \equiv jl \ (m)\}|$.*

9. Ordinary representations of SL(2, 5)

In this section, we recall the character table of SL(2, 5) and two irreducible (ordinary) representations of degree 2 of SL(2, 5), which will be need in §§ 10 and 11. These results were given by I. Schur [5; p. 128].

Character table of SL(2, 5)

** \ *	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}0&-1\\1&-1\end{pmatrix}$	$\begin{pmatrix}1&-1\\1&0\end{pmatrix}$	$\begin{pmatrix}2&0\\0&-2\end{pmatrix}$	$\begin{pmatrix}1&0\\1&1\end{pmatrix}$	$\begin{pmatrix}-1&0\\1&-1\end{pmatrix}$	$\begin{pmatrix}1&0\\2&1\end{pmatrix}$	$\begin{pmatrix}-1&0\\2&-1\end{pmatrix}$
φ_1	1	1	1	1	1	1	1	1	1
φ_2	2	-2	-1	1	0	$-\bar{\varepsilon}$	$\bar{\varepsilon}$	$-\varepsilon$	ε
φ_3	2	-2	-1	1	0	$-\varepsilon$	ε	$-\bar{\varepsilon}$	$\bar{\varepsilon}$
φ_4	3	3	0	0	-1	$\bar{\varepsilon}$	$\bar{\varepsilon}$	ε	ε
φ_5	3	3	0	0	-1	ε	ε	$\bar{\varepsilon}$	$\bar{\varepsilon}$
φ_6	4	4	1	1	0	-1	-1	-1	-1
φ_7	4	-4	1	-1	0	-1	1	-1	1
φ_8	5	5	-1	-1	1	0	0	0	0
φ_9	6	-6	0	0	0	1	-1	1	-1

* representatives of conjugate classes, ** characters

$\varepsilon = (1+\sqrt{5})/2$, $\bar{\varepsilon} = (1-\sqrt{5})/2$

SL(2, 5) is generated by two elements P and Q with the defining relations: $P^2 = Q^3 = (PQ)^5$, $P^4 = 1$, and has two irreducible representations Φ_2 and Φ_3 defined by

$$\Phi_2(P) = \begin{pmatrix} -\eta & \eta \\ -\eta-\eta^{-1} & \eta \end{pmatrix}, \quad \Phi_2(Q) = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \quad \text{and}$$

$$\Phi_3(P)=\begin{pmatrix}-\eta^2 & \eta^2\\ -\eta^2-\eta^{-2} & \eta^2\end{pmatrix},\quad \Phi_3(Q)=\begin{pmatrix}0 & 1\\ -1 & 1\end{pmatrix},$$

where η is a primitive 5-th root of 1.

10. Type F

In this section, we shall give the dimension of $J(K\mathfrak{H})$ when $\mathfrak{H}$ is of type F, namely, $\mathfrak{H}=\langle A, B\rangle\times\langle P, Q\rangle$, where $\langle A, B\rangle$ is of type A, $(|\langle A, B\rangle|, 30)=1$ and $\langle P, Q\rangle$ $(\cong \mathrm{SL}(2, 5))$ has the following relations: $P^2=Q^3=(PQ)^5$, $P^4=1$. If $p>5$ then $[J(K\mathfrak{H}): K]=120[J(K\langle A, B\rangle): K]$ and our problem is reduced to that in §3. If $p\leq 5$ then $[J(K\mathfrak{H}): K]=nm[J(K\langle P, Q\rangle): K]$. The dimension of $J(K\langle P, Q\rangle)$ for $p\leq 5$ is given in the next theorem.

THEOREM 17. (1) *If $p=5$ then* $[J(K\cdot\mathrm{SL}(2, 5)): K]=65$.

(2) *If $p=3$ then* $[J(K\cdot\mathrm{SL}(2, 5)): K]=41$.

(3) *If $p=2$ then* $[J(K\cdot\mathrm{SL}(2, 5)): K]=95$.

PROOF. Let Φ_ν be a representation whose character is φ_ν, and $\bar{\Phi}_\nu$ (resp. $\bar{\varphi}_\nu$) a modular representation (resp. character) associated with Φ_ν (resp. φ_ν).

(1) By Brauer's result [1; p. 588], $\{1, 2, 3, 4, 5\}$ is the set of all degrees of irreducible representations of $\mathrm{SL}(2, 5)$.

(2) Since $\mathrm{SL}(2, 5)$ coincides with its commutator subgroup, $\bar{\varphi}_1$ is only the linear character. Thus, by the character table of $\mathrm{SL}(2, 5)$, $\bar{\varphi}_2$ and $\bar{\varphi}_3$ are different irreducible characters. Since φ_4, φ_5 and φ_9 belong to blocks of defect 0, $\bar{\varphi}_4$, $\bar{\varphi}_5$, $\bar{\varphi}_9$ are irreducible. Next, we shall prove that $\bar{\varphi}_6$ is irreducible. Regarding φ_6 as an irreducible character of the alternative group A_5, we consider an irreducible character $\hat{\varphi}_6$ of the symmetric group S_5 which is an extendion of φ_6. Then, $\bar{\hat{\varphi}}_6$ is an extendion of $\bar{\varphi}_6$ and irreducible by [4; p. 31]. Now, let Ψ be a composition factore of $\bar{\Phi}_6$. Then, $I(\Psi)=A_5$ or S_5. If $I(\Psi)=A_5$ then $\Psi^{S_5}=\bar{\hat{\Phi}}_6$, and then $\bar{\Phi}_6$ has two conjugate composition factors Ψ, Ψ'. However, by the character table of $\mathrm{SL}(2, 5)$ Ψ, Ψ' are different from $\bar{\Phi}_2$, $\bar{\Phi}_3$. Accordingly, $\mathrm{SL}(2, 5)$ has at least eight irreducible modular representations, which is contrary to the fact that there are only seven 3-regular classes in $\mathrm{SL}(2, 5)$. We have seen therefore $I(\Psi)=S_5$. Then, by making use of the same argument as in [3; (9.12)], Ψ can be extended to an irreducible representation $\hat{\Psi}$ of S_5. Since $\hat{\Psi}$ is a composition factor of Ψ^{S_5}, Ψ^{S_5} is not irreducible. Noting that $\bar{\hat{\Phi}}_6$ is a composition factor of Ψ^{S_5} and $\Psi^{S_5}|_{A_5}$ $=\Psi\oplus\Psi$, we can easily see that $\bar{\Phi}_6$ is equivalent to the irreducible representation Ψ.

(3) By making use of the same argument as in (2), we see that $\bar{\varphi}_1$, $\bar{\varphi}_2$, $\bar{\varphi}_3$ are irreducible. Since $\mathrm{PSL}(2, 5)$ is isomorphic to A_5, φ_6 can be regarded

as an irreducible character of A_5 and belongs to a block of defect 0 (as a character of A_5). Hence $\{\bar{\Phi}_1, \bar{\Phi}_2, \bar{\Phi}_3, \bar{\Phi}_6\}$ is the set of all irreducible representations of SL(2, 5) for $p=2$.

COROLLARY 18. (1) *If $p>5$ then $[J(K\mathfrak{H}): K]=120[J(K\langle A, B\rangle): K]$, where $\langle A, B\rangle$ is of type A.*

(2) *If $p=5$ then $[J(K\mathfrak{H}): K]=65nm$.*

(3) *If $p=3$ then $[J(K\mathfrak{H}): K]=41nm$.*

(4) *If $p=2$ then $[J(K\mathfrak{H}): K]=95nm$.*

11. Type G

In this section, we shall give the dimension of $J(K\mathfrak{H})$ when $\mathfrak{H}$ is of type G, namely, $\mathfrak{H}$ is the group generated by elements A, B, P, Q and R with defining relations:

(1) $\langle A, B, P, Q\rangle$ is of type F;

(2) $R^2=(RP)^4=P^2$;

(3) $RAR^{-1}=A^l$, $RBR^{-1}=B^l$;

(4) $l^2\equiv 1$ (m), $l\equiv 1$ (n).

If p is odd then $[J(K\mathfrak{H}): K]=2[J(K\langle A, B, P, Q\rangle): K]$ and we can reduce our problem to that in §10. In what follows, we shall restrict our attention to the case $p=2$. Since $\mathfrak{H}'=\langle A, P, Q, R\rangle$ is a normal subgroup of $\mathfrak{H}$ with $(\mathfrak{H}: \mathfrak{H}')=n$ (odd) and $[J(K\mathfrak{H}): K]=n[J(K\mathfrak{H}'): K]$, it suffices to determine the dimension of $J(K\mathfrak{H}')$. Let ω be a primitive m-th root of 1 in K, and Θ_j a linear character of $\langle A\rangle$ in K defined by $A\to\omega^j$. We consider $\mathfrak{H}''=\langle P, Q\rangle\times\langle A\rangle$, which is a normal subgroup of $\mathfrak{H}'$ with $(\mathfrak{H}': \mathfrak{H}'')=2$. Then, $\{\Delta_{ij}=\bar{\Phi}_i\otimes\Theta_j \mid i=1, 2, 3, 6,\ 0\leq j\leq m-1\}$ is the set of all irreducible representations of $\mathfrak{H}''$ (cf. the proof of Th. 17). Noting that $\varphi_6=\varphi_2\cdot\varphi_3$ (cf. the character table of SL(2, 5)), we obtain the following:

THEOREM 19. *Δ_{ij} can be extended if and only if $j\equiv jl$ (m).*

PROOF. If there exists a regular matrix X such that $X^2=(X\Delta_{ij}(P))^4=\Delta_{ij}(P)^2$ and $X\Delta_{ij}(A)X^{-1}=\Delta_{ij}(A)$, then $j\equiv jl$ (m). Conversely, if $j\equiv jl$ (m) then Δ_{ij} can be extended to a representation $\hat{\Delta}_{ij}$ of $\mathfrak{H}'$ such that $\hat{\Delta}_{ij}(R)$ is the identity matrix.

By Th. 8 and Th. 19, we readily obtain the following:

COROLLARY 20. (1) *If p is odd then $[J(K\mathfrak{H}): K]=2[J(K\langle A, B, P, Q\rangle): K]$, where $\langle A, B, P, Q\rangle$ is of type F.*

(2) *If $p=2$ then $[J(K\mathfrak{H}): K]=n(190m+25u)$, where $u=|\{0\leq j\leq m-1 \mid j\equiv jl\ (m)\}|$.*

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References

[1] R. BRAUER and C. NESBITT: On the modular characters of groups, Ann. of Math. 42 (1941), 556–590.

[2] C. W. CURTIS and I. REINER: Representation theory of finite groups and associative algebras, Wiley, New York, 1962.

[3] W. FEIT: Characters of finite groups, Benjamin, New York, 1967.

[4] G. de B. ROBINSON: Representation theory of the symmetric groups, Univ. of Toronto, 1961.

[5] I. SCHUR: Untersuchungen über die Darstellungen der endlichen Gruppen durch gebrochene lineare Substitutionen, J. Reine Angew. Math. 132 (1907), 85–137.

[6] Y. TSUSHIMA: Radicals of group algebras, Osaka J. Math. 4 (1967), 179–182.

[7] D. A. R. WALLACE: Note on the radical of a group algebra, Proc. Cambridge Philos. Soc. 54 (1958), 128–130.

[8] H. ZASSENHAUS: Über endlichen Fastkörper, Abh. Math. Sem. Hansischen Univ. 11 (1936), 187–220.

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