Kählerian manifolds with vanishing Bochner curvature tensor satisfying $R(X, Y)\cdot R_1=0$

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1. Introduction. Let (M, J, g) be a Kählerian manifold of complex dimension n with the almost complex structure J and the Kählerian metric g.

The Bochner curvature tensor B of M is defined as follows:

$$B(X, Y)=R(X, Y)-\frac{1}{2n+4}[R^1X\wedge Y+X\wedge R^1Y+R^1JX\wedge JY$$
$$+JX\wedge R^1JY-2g(JX, R^1Y)J-2g(JX, Y)R^1\circ J]$$
$$+\frac{\text{trace } R^1}{(2n+4)(2n+2)}[X\wedge Y+JX\wedge JY-2g(JX, Y)J]$$

for any tangent vectors X and Y, where R and R^1 are the Riemannian curvature tensor of M and a field of symmetric endomorphism which corresponds to the Ricci tensor R_1 of M, that is, $g(R^1X, Y)=R_1(X, Y)$, respectively. $X\wedge Y$ denotes the endomorphism which maps Z upon $g(Y, Z)X-g(X, Z)Y$.

The tensor B has the properties similar to those of Weyl's conformal curvature tensor of a Riemannian manifold. For example, we can classify the restricted homogeneous holonomy groups of Kählerian manifolds with vanishing B, which seems to be an analogy of Kurita's theorem for the holonomy groups of conformally flat Riemannian manifolds [3], [5].

On the other hand, K. Sekigawa and one of the authors of present paper [4] classified conformally flat manifolds satisfying the condition

$$(*)\qquad R(X, Y)\cdot R_1=0 \quad \text{for any tangent vectors } X \text{ and } Y,$$

where the endomorphism $R(X, Y)$ operates on R_1 as a derivation of the tensor algebra at each point of M.

In this paper, we shall prove

THEOREM. Let (M, J, g) be a connected Kählerian manifold of complex dimension n $(n\geqq 2)$ with vanishing Bochner curvature tensor satisfying the condition $(*)$, Then M is one of the following manifolds;

(I) A space of constant holomorphic sectional curvature.

(II) A locally product manifold of a space of constant holomorphic sectional curvature $K(\neq 0)$ and a space of constant holomorphic sectional curvature $-K$.

2. Preliminaries. Let (M, J, g) be a Kählerian manifold with vanishing B. Then its curvature tensor R is written as follows:

$$\begin{aligned}(2.1)\qquad R(X, Y)=&\frac{1}{2n+4}[R^1X\wedge Y+X\wedge R^1Y+R^1JX\wedge JY\\&+JX\wedge R^1JY-2g(JX, R^1Y)J-2g(JX, Y)R^1\circ J]\\&-\frac{\text{trace } R^1}{(2n+4)(2n+2)}[X\wedge Y+JX\wedge JY-2g(JX, Y)J]\end{aligned}$$

There are following relations among g, J and R^1:

$$J^2=-I,\qquad g(JX, Y)+g(X, JY)=0,$$
$$R^1\circ J=J\circ R^1,\quad g(R^1X, Y)=g(X, R^1Y).$$

Then, at a point $x\in M$, we can take an orthonormal basis $\{e_1, \cdots, e_n, Je_1, \cdots, Je_n\}$ of tangent space $T_x(M)$ such that

$$(2.2)\qquad R^1e_i=\lambda_ie_i,\quad R^1Je_i=\lambda_iJe_i\qquad \text{for } i=1, \cdots, n.$$

And we have

$$(2.3)\qquad \begin{cases} R(e_i, Je_i)=\sigma_ie_i\wedge Je_i+\tau_iJ-\dfrac{1}{n+2}R^1\circ J & (i=1, \cdots, n)\\ R(e_i, e_j)=\sigma_{ij}(e_i\wedge e_j+Je_i\wedge Je_j), & \\ R(e_i, Je_j)=\sigma_{ij}(e_i\wedge Je_j-Je_i\wedge e_j) & (i, j=1, \cdots, n,\ i\neq j),\end{cases}$$

where we have put

$$\begin{cases}\sigma_{ij}=\dfrac{1}{2(n+1)(n+2)}[(n+1)(\lambda_i+\lambda_j)-\Lambda],\\ \sigma_i\ =\dfrac{1}{(n+1)(n+2)}[2(n+1)\lambda_i-\Lambda],\\ \tau_i\ =\dfrac{1}{(n+1)(n+2)}[\Lambda-(n+1)\lambda_i],\\ \Lambda\ =\lambda_1+\lambda_2+\cdots+\lambda_n.\end{cases}$$

3. Proof of theorem. At a point $x\in M$, we take an orthnormal basis $\{e_1, \cdots, e_n, Je_1, \cdots, Je_n\}$ of $T_x(M)$ satisfying (2.2). Now by the equation (*), (2.3) and

$$\big(R(X, Y)\cdot R_1\big)(Z, W)=-R_1\big(R(X, Y)Z, W\big)-R_1\big(Z, R(X, Y)W\big),$$

we have

$$(3.1)\qquad (\lambda_i-\lambda_j)\sigma_{ij}=0 \qquad \text{for } i\neq j.$$

LEMMA 3.1. At each point of M, R^1 has at most two distinct characteristic roots, which cannot have the same sign.

PROOF. If there exists an integer r $(1\leqq r<n)$ such that $\lambda_1=\cdots=\lambda_r=\lambda$, $\lambda_{r+1}\neq\lambda, \cdots, \lambda_n\neq\lambda$, then (3.1) implies

$$(n+1)(\lambda+\lambda_{r+1})-\Lambda=0$$
$$\cdots\cdots$$
$$(n+1)(\lambda+\lambda_n)-\Lambda=0.$$

Hence $\lambda_{r+1}=\cdots=\lambda_n=\mu$. Again (3.1) implies $(n+1-r)\lambda+(r+1)\mu=0$, from which we have $\lambda\mu<0$.

If M is Einstein, then the condition (*) is automatically satisfied and, by (2.1), it is easily seen that M is a space of constant holomorphic sectional curvature.

Henceforth, we assume that M is not Einstein. Then, by lemma 3.1, there exists a point $x_0\in M$ and an integer r $(1\leqq r<n)$ such that, changing the indices of $\lambda_1, \cdots, \lambda_n$ suitablly, they satisfy

$$(3.2)\qquad \begin{cases} \lambda_1=\cdots=\lambda_r=\lambda>0, \quad \lambda_{r+1}=\cdots=\lambda_n=\mu<0, \\ (n-r+1)\lambda=-(r+1)\mu \end{cases}$$

at x_0. Next, we take a point x in a neighborhood of x_0. By lemma 3.1 and the continuity of characteristic roots of R^1, when x is sufficiently near x_0, we may conclude that, with the same r, (3.2) is satisfied at x. Let W be the set of points $x\in M$ such that R^1 have two distinct characteristic roots at x, which is an open set. By W_0 we denote the connected component of W containing x_0. Then r is constant on W_0, and $\lambda(x)$ and $\mu(x)$ are differentiable functions. Then, we have the following two distributions:

$$T_1(x)=\left\{X\in T_x(M):\ R^1X=\lambda(x)X\right\},$$
$$T_2(x)=\left\{X'\in T_x(M):\ R^1X'=\mu(x)X'\right\},$$

which are differentiable, J-invariant, mutually orthogonal and complementally. Let X, $Y\in T_1$ and X', $Y'\in T_2$, we have

$$\begin{cases} R(X, Y)=K[X\wedge Y+JX\wedge JY-2g(JX, Y)J_1], \\ R(X', Y')=-K[X'\wedge Y'+JX'\wedge JY'-2g(JX', Y')J_2], \\ R(X, X')=0 \end{cases}$$

by (2. 1) and (3. 2), where

$$K=\frac{1}{2(n+1)(n+2)}[(2n+2-r)\lambda-(n-r)\mu]\neq 0$$

and J_1 and J_2 are defined by $J_1X=JX$, $J_1X'=0$ and $J_2X=0$, $J_2X'=JX'$, respectively. Then, by [5], T_1 and T_2 are parallel, K is constant and $W_0=M$. That is, M is a locally product manifold of a r-dimensional space of constant holomorphic sectional curvature $4K$ and an $(n-r)$-dimensional space of constant holomorphic sectional curvature $-4K$.

REMARK. For a Kählerian manifold with vanishing B, the condition (*) is equivalent to $R(X, Y)\cdot R=0$.

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