3-dimensional Riemannian manifolds satisfying $R(X, Y)\cdot R=0$

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§ 1. Introduction

Let (M, g) be a Riemannian manifold with a positive definite metric tensor g. By R we denote the Riemannian curvature tensor. By M_p we denote the tangent space to M at p. Let $X, Y\in M_p$. Then $R(X, Y)$ operates on the tensor algebra as a derivation at each point p. In a locally symmetric space (i.e., $\nabla R=0$), we have $R(X, Y)\cdot R=0$. We consider the converse under some additional conditions.

THEOREM. *Let (M, g) be a complete and irreducible 3-dimensional Riemannian manifold. Assume that the scalar curvature S is positive and bounded away from zero (i.e., $S\geq\varepsilon>0$ for some constant ε). If (M, g) satisfies*

(*) $R(X, Y)\cdot R=0$ *for any $p\in M$ and $X, Y\in M_p$,*

then (M, g) is of positive constant curvature.

This theorem follows from the following

PROPOSITION. *Let (M, g) be a complete 3-dimensional Riemannian manifold satisfying (*). Assume that S is positive and bounded away from zero. Then (M, g) is either*

(1) *a space of positive constant curvature, or*

(2) *locally a product Riemannian manifold of a 2-dimensional space of positive curvature and a real line.*

A consequence of Theorem is as follows:

COROLLARY. *Let (M, g) be a compact and irreducible 3-dimensional Riemannian manifold. If (M, g) satisfies (*) and S is positive, then (M, g) is of positive constant curvature.*

In Theorem the condition on the scalar curvature or something like this is necessary, because of Takagi's example [6].

It may be noticed that (*) is equivalent to $R(X, Y)\cdot R_1=0$, where R_1 denotes the Ricci curvature tensor. In this paper (M, g) is assumed to be connected and of class C^∞.

§ 2. Preliminaries

Let (M, g) be a 3-dimensional Riemannian manifold and assume (*) on

(M, g). Since *dim* $M=3$, $R(X, Y)$ is given by

$$R(X, Y)=R^1X\wedge Y+X\wedge R^1Y-(S/2)X\wedge Y, \tag{2.1}$$

where $g(R^1X, Y)=R_1(X, Y)$ and $(X\wedge Y)Z=g(Y, Z)X-g(X, Z)Y$. Let (K_1, K_2, K_3) be eigenvalues of the Ricci transformation R^1 at a point p. Then (*) is equivalent to (cf. Tanno [7], p. 302)

$$(K_i-K_j)\left(2(K_i+K_j)-S\right)=0\,. \tag{2.2}$$

Therefore we have three cases of eigenvalues of R^1: (K, K, K), $(K, K, 0)$, and $(0, 0, 0)$ at each point p.

[A] If (K, K, K), $K\neq 0$, holds at some point x, then it holds on some open neighborhood U of x. Hence U is an Einstein space, and K is constant on U and on M. Therefore (M, g) is of constant curvature (cf. Takagi and Sekigawa [5]).

[B] From now on we assume that rank $R^1\leq 2$. Let $W=\{x\in M;$ rank $R^1=2$ at $x\}$. By W_0 we denote one component of W. On W_0 we have two C^∞-distributions D_K and D_0 such that

$$D_K=\{X\,;\ R^1X=KX\}\,,$$
$$D_0=\{Z\,;\ R^1Z=0\}\,.$$

For $X, Y\in D_K$ and $Z\in D_0$, by (2.1) we have

$$R(X, Y)=KX\wedge Y\,, \qquad R(Y, Z)=0\,. \tag{2.3}$$

This shows that D_0 is the nullity distribution. Since the index of nullity at each point of M is 1 or 3, the index of nullity of M is 1. Thus, integral curves of D_0 are geodesics, and complete if (M, g) is complete (cf. Clifton and Maltz [2], Abe [1], etc.).

[C] Let $\{E_1, E_2, E_3\}=\{E\}$ be a local field of orthonormal frames such that $E_3\in D_0$ (consequently, $E_1, E_2\in D_K$) and

$$\nabla_{E_3}E_i=0 \qquad i=1, 2, 3\,,$$

where ∇ denotes the Riemannian connection. We call this $\{E\}$ an adapted frame field. If we put

$$\nabla_{E_i}E_j=\sum B_{ijk}E_k\,,$$

then we get $B_{ijk}=-B_{ikj}$ and

$$B_{3ij}=0 \qquad i, j=1, 2, 3\,. \tag{2.4}$$

The second Bianchi identity and (2.3) give

$$(2.5)\qquad E_3K+K(B_{131}+B_{232})=0\,.$$

By (2.4) and $R(E_i, E_3)E_3=\nabla_{E_i}\nabla_{E_3}E_3-\nabla_{E_3}\nabla_{E_i}E_3-\nabla_{[E_i,E_3]}E_3=0$, we get

$$(2.6)\qquad \begin{aligned} &E_3B_{131}+(B_{131})^2+B_{132}B_{231}=0\,,\\ &E_3B_{132}+B_{131}B_{132}+B_{132}B_{232}=0\,,\\ &E_3B_{231}+B_{231}B_{131}+B_{232}B_{231}=0\,,\\ &E_3B_{232}+(B_{232})^2+B_{231}B_{132}=0\,. \end{aligned}$$

(2.5) and $(2.6)_2$, (2.5) and $(2.6)_3$, (2.5) and $(2.6)_{1,4}$ imply

$$(2.7)\qquad B_{132}=C_1(E)K\,,\qquad B_{231}=C_2(E)K\,,$$

$$(2.8)\qquad B_{131}-B_{232}=D(E)K\,,$$

where $C_1(E)$, $C_2(E)$ and $D(E)$ are functions defined on the same domain as $\{E\}$ such that $E_3C_1(E)=E_3C_2(E)=E_3D(E)=0$. By (2.5) and (2.8), we get

$$(2.9)\qquad 2B_{131}=D(E)K-E_3K/K\,.$$

[D] Let $L=x(s)$ be an integral curve of D_0 through $x(0)$ with arc-length parameter s. Then $(2.6)_1$, (2.7) and (2.9) give

$$(2.10)\qquad \frac{1}{2}\frac{d}{ds}\left(\frac{1}{K}\frac{dK}{ds}\right)=HK^2+\frac{1}{4}\left(\frac{1}{K}\frac{dK}{ds}\right)^2,$$

where $H=D(E)^2/4+C_1(E)C_2(E)$. (2.10) implies that H is independent of the choice of the adapted frame fields $\{E\}$. Solving (2.10), we get

$$(2.11)\qquad K_{|L}=K(s)=\gamma \quad\text{or}\quad \pm 1/(\alpha s-\beta)^2 \qquad \text{for } H=0\,,$$

$$(2.12)\qquad K_{|L}=K(s)=\pm 1\Big/\Big[(\alpha s-\beta)^2-H/\alpha^2\Big] \qquad \text{for } H\neq 0$$

where γ, $\alpha\neq 0$, and β are constant on L.

[E] Next we assume that W_0 is oriented. Let $\{E_1, E_2, E_3\}$ be an adapted frame field which is compatible with the orientation. We call it an oriented adapted frame field. Then we see that $f=C_1(E)-C_2(E)$ is independent of the choice of oriented adapted frame fields, and hence f is a C^∞-function on W_0.

[F] $f=0$ holds on an open set $U\subset W_0$, if and only if D_K is integrable on U. This is a geometrical meaning of f.

[G] (cf. Sekigawa [4]) Assume that $E_3K=0$ on W_0. If $f\neq 0$, we put $V=\{x\in W_0\,;\, f(x)\neq 0\}$. Let V_0 be one component of V. $E_3K=0$ and (2.10) imply $H=0$, i.e., $D(E)^2=-4C_1(E)C_2(E)$. We define a function $\theta(E)$ by

$$\cos 2\theta(E)=\Big[C_1(E)+C_2(E)\Big]\Big/f\,,$$
$$\sin 2\theta(E)=D(E)/f\,.$$

Define $\{E^*\}$ by $E_3^*=E_3$ and

$$E_1^* = \cos\theta(E)\,E_1-\sin\theta(E)\,E_2\,,$$
$$E_2^* = \sin\theta(E)\,E_1+\cos\theta(E)\,E_2\,.$$

Then we have $D(E^*)=0$. Furthermore, for two oriented adapted frame fields $\{E\}$ and $\{E'\}$ such that $E_3=E_3'$, we have $E_1^*(E)=\pm E_1^*(E')$ and $E_2^*(E)=\pm E_2^*(E')$. $H=0$ and $D(E^*)=0$ imply $C_1(E^*)\,C_2(E^*)=0$. So we can assume that $C_2(E^*)=0$ [otherwise, change $\{E_1^*, E_2^*, E_3^*\}\to\{E_2^*, -E_1^*, E_3^*\}$]. Then we get

$$B_{132}^* \neq 0\,, \qquad B_{231}^* = B_{131}^* = B_{232}^* = 0\,. \tag{2.13}$$

$R(E_1^*, E_2^*)E_3^*=0$ implies $B_{221}^*=0$ and

$$E_2^*\,B_{132}^* + B_{121}^*\,B_{132}^* = 0\,. \tag{2.14}$$

$R(E_1^*, E_2^*)E_1^* = -KE_2^*$ implies

$$E_2^*\,B_{121}^* + (B_{121}^*)^2 = -K\,. \tag{2.15}$$

§ 3. Proof of Proposition

In the proof we can assume that M is oriented. By [A] of § 2, we assume that rank $R^1\leq 2$. Since $S=2K$ is positive, rank $R^1=2$ on M and $W=W_0=M$. f is defined on M. Since (M, g) is complete and S is bounded away from zero, by (2.11) and (2.12) we have $H=0$ and $E_3K=0$. So we can apply [G] of § 2. Assume that there is a point x_0 such that $f(x_0)\neq 0$. By $B_{2ij}^*=0$, each trajectory of E_2^* is a geodesic in V_0. Let N be a trajectory of E_2^* through x_0 and parametrize it by arc-length parameter t such that $x(0)=x_0$. Put $fK=\pm k$ according to $f(x_0)\gtrless 0$. k is a C^∞-function on M. Put $B_{121}^*=h$ on V_0. Since $B_{132}^*=C_1(E^*)K=fK=\pm k$, on $N\cap V_0=(x(t))\cap V_0$ we have

$$\frac{d\,k(t)}{dt}+h(t)\,k(t)=0\,, \tag{3.1}$$

$$\frac{d\,h(t)}{dt}+h(t)^2=-K(t)\,, \tag{3.2}$$

by (2.14) and (2.15). By the following Lemma we have a contradiction. Hence $f=0$ identically on M. Then [F] of § 2 and Theorem A in [9] show that (M, g) is locally a Riemannian product of a 2-dimensional Riemannian manifold of positive curvature and a real line $\boldsymbol{R}$.

LEMMA. *The following* (i)~(vi) *are not compatible*:

(i) $k(t)$ *and* $K(t)$ *are* C^∞-*functions on* $\boldsymbol{R}$,

(ii) $k(0)>0$,

(iii) $K(t)>0$ *for all* $t\in R$,

(iv) $h(t)$ *is a* C^∞*-function defined on an open interval* $I=\{t: k(t)>0\}$ *containing* 0,

(v) $\dfrac{dk(t)}{dt}+k(t)h(t)=0$ *on* I,

(vi) $\dfrac{dh(t)}{dt}+h(t)^2=-K(t)$ *on* I.

Proof. The first case: $h(0)<0$. By (iii) and (vi) we get

$$\frac{dh(t)}{dt}=-\left(K(t)+h(t)^2\right)<-h(t)^2. \tag{3.3}$$

Let $h^*(t)$ be the solution of

$$\frac{dh^*(t)}{dt}=-h^*(t)^2 \tag{3.4}$$

such that $h^*(0)=h(0)$. Then $h(t)<h^*(t)$ for t: $t>0$ in I. Since $h^*(t)=1/(t-\alpha)$, we get $h(t)<1/(t-\alpha)$, where $\alpha=-1/h(0)$ and $\alpha>t>0$. Since $h(t)$ is decreasing for $t>0$ in I, by (v) $k(t)$ is increasing for $t>0$ in I. Hence, $k(t)>k(0)>0$. Then (v) is

$$\frac{dk(t)}{dt}=k(t)\left(-h(t)\right)>k(0)\left(-\frac{1}{t-\alpha}\right).$$

This shows that if $t\to\alpha-0$, then $dk(t)/dt\to\infty$. This contradicts (i).

The second case: $h(0)=0$. By (vi) we get $dh(0)/dt=-K(0)<0$. Therefore we have some small positive number ε such that $k(\varepsilon)>0$ and $h(\varepsilon)<0$. Hence, this case reduces to the first case.

The third case: $h(0)>0$. For (3.3) and (3.4), we have $h(t)>h^*(t)$ for t: $t<0$. Hence, $h(t)>1/(t-\alpha)$, where $\alpha=-1/h(0)<0$ and $\alpha<t<0$. Then we get

$$\frac{dk(t)}{dt}=k(t)\left(-h(t)\right)<k(0)\left(-\frac{1}{t-\alpha}\right).$$

This implies that if $t\to\alpha+0$, then $dk(t)/dt\to-\infty$. This contradicts (i).

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