A conformal transformation and a special concircular scalar field in a Riemannian manifold

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Introduction

Recently T. Koyanagi [1][2)] has investigated some properties of a Riemannian manifold which admits a scalar field ρ characterized by the property

$$\nabla_l \rho_k = \sigma \rho g_{kl}, \qquad \sigma = \textit{non-zero constant}, \tag{0.1}$$

(such a scalar field ρ is called the special concircular scalar field) where $\rho_k = \nabla_k \rho$ and g_{kl} means the metric tensor of the manifold. He obtained the following

THEOREM A. *Let M be a Riemannian manifold of dimension n which has the curvature tensor satisfying*

$$\nabla_{[m}\nabla_{l]} R_{hijk} = 0$$

and admits the special concircular scalar field ρ defined by (0.1). *Then M is of constant curvature.*

COROLLARY A$_1$. *Let M be an n-dimensional Einstein space* ($n>2$) *which has the scalar curvature $R \neq 0$, the curvature tensor such that*

$$\nabla_{[m}\nabla_{l]} R_{hijk} = 0$$

and admits a proper conformal Killing vector field ξ^i. Then M is of constant curvature.

THEOREM B. *Let M be a Riemannian manifold of dimension n which has the Ricci tensor such that*

$$\nabla_{[l}\nabla_{h]} R_{jk} = 0$$

and admits the special concircular scalar field ρ. Then M is an Einstein space.

A conformal Killing vector field ξ^i satisfies an equation;

$$\underset{\xi}{\mathcal{L}}\, g_{ij} = \nabla_j \xi_i + \nabla_i \xi_j = 2\rho g_{ij},$$

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2) Numbers in brackets refer to the references at the end of the paper.

where $\underset{\xi}{\mathcal{L}}$ means the Lie derivative with respect to ξ^i. In an Einstein space admitting a conformal Killing vector field ξ^i, we obtain the following result; [2]

$$\nabla_j \rho_i = -\frac{R}{n(n-1)} \rho g_{ij},$$

hence ρ is the special concircular scalar field. Therefore Corollary A_1 is evident from the Theorem A.

The purpose of the present paper is to investigate the necessary and sufficient conditions that ρ is the special concircular scalar field if M admits a proper conformal Killing vector field ξ^i. In §1, we give the formulas with respect to a conformal Killing vector field ξ^i. Next in §2, we get the necessary and sufficient conditions and give some corollaries of them. In §3, we shall give some remark when M is a compact orientable Riemannian manifold.

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§1. The formulas with respect to a conformal Killing vector field ξ^i.

Let us suppose an n-dimensional Riemannian manifold M of class $C^r (r \geqq 3)$ which has local coordinates x^i and admits a conformal Killing vector field ξ^i. Then we have the following well-known formulas (see K. Yano [3])

$$\underset{\xi}{\mathcal{L}} g_{ij} = \nabla_j \xi_i + \nabla_i \xi_j = 2\rho g_{ij}, \tag{1.1}$$

$$n\rho = \nabla_i \xi^i, \tag{1.2}$$

$$\begin{aligned} \underset{\xi}{\mathcal{L}} \left\{ {i \atop jk} \right\} &= \frac{1}{2} g^{il} \left[\nabla_j (\underset{\xi}{\mathcal{L}} g_{kl}) + \nabla_k (\underset{\xi}{\mathcal{L}} g_{jl}) - \nabla_l (\underset{\xi}{\mathcal{L}} g_{jk}) \right] \\ &= g^{il} (\rho_j g_{kl} + \rho_k g_{jl} - \rho_l g_{jk}), \end{aligned} \tag{1.3}$$

$$\begin{aligned} \underset{\xi}{\mathcal{L}} \left\{ {i \atop jk} \right\} &= \nabla_j \nabla_k \xi^i + R^i_{kjl} \xi^l \\ &= \rho_j \delta^i_k + \rho_k \delta^i_j - \rho^i g_{jk}, \end{aligned} \tag{1.4}$$

$$\underset{\xi}{\mathcal{L}} R^i_{jkl} = \nabla_l \left(\underset{\xi}{\mathcal{L}} \left\{ {i \atop jk} \right\} \right) - \nabla_k \left(\underset{\xi}{\mathcal{L}} \left\{ {i \atop jl} \right\} \right), \tag{1.5}$$

$$\underset{\xi}{\mathcal{L}} R^i_{jkl} = \delta^i_k \nabla_l \rho_j - \delta^i_l \nabla_k \rho_j + g_{jl} \nabla_k \rho^i - g_{jk} \nabla_l \rho^i, \tag{1.6}$$

(1. 7) $$\underset{\xi}{\mathcal{L}} R_{jk} = -(n-2)\nabla_k \rho_j - g_{jk}\nabla_i \rho^i,$$

(1. 8) $$\underset{\xi}{\mathcal{L}} R = -2(n-1)\nabla_i \rho^i - 2\rho R,$$

(1. 9) $$\nabla_i \rho^i = -\frac{1}{2(n-1)}(2\rho R + \underset{\xi}{\mathcal{L}} R),$$

where ∇_i is the operator of covariant differentiation with respect to g_{ij}, $\left\{ {i \atop jk} \right\}$ Christoffel symbols, R^i_{jkl} the curvature tensor, R_{jk} the Ricci tensor, R the scalar curvature, $\rho^i = g^{ij}\rho_j$ and δ^h_j the Kronecker delta.

§ 2. The necessary and sufficient conditions that ρ is the special concircular scalar field.

First we shall discuss the necessary condition. If ρ is a special concircular scalar field, it satisfies

(2. 1) $$\nabla_k \rho_j = \sigma \rho g_{jk}, \qquad \sigma = \textit{non-zero constant}.$$

Substituting (1. 9) into (1. 7), we obtain

(2. 2) $$\nabla_k \rho_j = \frac{2\rho R + \underset{\xi}{\mathcal{L}} R}{2(n-1)(n-2)} g_{jk} - \frac{1}{(n-2)} \underset{\xi}{\mathcal{L}} R_{jk}.$$

From (2. 1) and (2. 2), we have

(2. 3) $$\underset{\xi}{\mathcal{L}} R_{jk} = \frac{-2(n-1)(n-2)\sigma\rho + 2\rho R + \underset{\xi}{\mathcal{L}} R}{2(n-1)} g_{jk}.$$

Putting

(2. 4) $$\alpha = \frac{-2(n-1)(n-2)\sigma\rho + 2\rho R + \underset{\xi}{\mathcal{L}} R}{2(n-1)},$$

we have

(2. 5) $$\underset{\xi}{\mathcal{L}} R_{jk} = \alpha g_{jk}.$$

From (1. 7) and (2. 5), we obtain

$$\alpha g_{jk} = -(n-2)\nabla_k \rho_j - g_{jk}\nabla_i \rho^i$$

and multiplying both sides of it by g^{jk} and summing with respect to j and k, we get

(2. 6) $$\alpha n = -2(n-1)\nabla_i \rho^i \quad \text{or} \quad \alpha = \frac{2\rho R + \underset{\xi}{\mathcal{L}} R}{n}$$

from (1. 9).

Hence (2. 5) becomes

$$\underset{\xi}{\mathscr{L}} R_{jk}=\frac{2\rho R+\underset{\xi}{\mathscr{L}} R}{n} g_{jk}\,. \tag{2.7}$$

From (2. 4) and (2. 6), we get

$$\sigma=-\frac{2\rho R+\underset{\xi}{\mathscr{L}} R}{2n(n-1)\rho}=non\ zero\ constant\,, \tag{2.8}$$

$$\left(\text{that is, } \rho=-\frac{\underset{\xi}{\mathscr{L}} R}{2(n(n-1)\sigma+R)}\quad (\underset{\xi}{\mathscr{L}} R\neq 0)\,,\quad \sigma=-\frac{R}{n(n-1)}\quad (\underset{\xi}{\mathscr{L}} R=0)\right).$$

Next we shall discuss the sufficient condition. If (2. 7) and (2. 8) are satisfied, we prove that ρ is a special concircular scalar field. From (2. 2), (2. 7) and (2. 8), we have

$$\begin{aligned}\nabla_k\rho_j&=\frac{2\rho R+\underset{\xi}{\mathscr{L}} R}{2(n-1)(n-2)} g_{jk}-\frac{1}{n-2}\underset{\xi}{\mathscr{L}} R_{jk}\\ &=-\frac{n\sigma\rho}{n-2} g_{jk}-\frac{2\rho R+\underset{\xi}{\mathscr{L}} R}{n(n-2)} g_{jk}\\ &=-\frac{n\sigma\rho}{n-2} g_{jk}+\frac{2(n-1)\sigma\rho}{n-2} g_{jk}\\ &=\frac{n-2}{n-2}\sigma\rho g_{jk}=\sigma\rho g_{jk}\,.\end{aligned}$$

Therefore we have

THEOREM 1. *If $M(n\geqq 3)$ admits a proper conformal Killing vector field ξ^i, the necessary and sufficient conditions that ρ is the special concircular scalar field are* (2. 7) *and* (2. 8).

COROLLARY 1. *Let M be an n-dimensional Riemannian manifold* ($n \geqq 3$) *which has the curvature tensor such that $\nabla_{[m}\nabla_{l]}R_{hijk}=0$ and admits a proper conformal Killing vector field ξ^i satisfying* (2. 7) *and* (2. 8). *Then M is of constant curvature.*

PROOF. This is proved evidently from Theorem A and Theorem 1.

COROLLARY 2. *If $M(n\geqq 3)$ admits a proper conformal Killing vector field ξ^i and the scalar curvature $R\neq 0$ is constant, then the necessary and sufficient condition that ρ is the special concircular scalar field is $\underset{\xi}{\mathscr{L}} T_{ij}=0$, where $T_{ij}=R_{ij}-\frac{R}{n}g_{ij}$.*

PROOF. As R is constant, (2. 8) is satisfied. (2. 7) becomes

$$\underset{\xi}{\mathcal{L}} R_{jk} = \frac{2\rho R}{n} g_{jk},$$

and we get

$$\underset{\xi}{\mathcal{L}} R_{jk} - \frac{R}{n} 2\rho g_{jk} = \underset{\xi}{\mathcal{L}} R_{jk} - \frac{R}{n} \underset{\xi}{\mathcal{L}} g_{jk} = 0$$

by (1. 1). Hence (2. 7) becomes

$$\underset{\xi}{\mathcal{L}} \left(R_{jk} - \frac{R}{n} g_{jk} \right) = 0 .$$

Therefore in an Einstein space which has the scalar curvature $R \neq 0$ admitting a proper conformal Killing vector field ξ^i, it is evident that ρ is the special concircular field.

COROLLARY 3. *Let M be a Riemannian manifold of dimension $n \geqq 3$ which has the Ricci tensor $\nabla_{[l} \nabla_{h]} R_{ik} = 0$ and admits a proper conformal Killing vector field ξ^i satisfying* (2. 7) *and* (2. 8). *Then M is an Einstein space.*

PROOF. This is proved evidently from Theorem B and Theorem 1.

COROLLARY 4. *Let M be a Riemannian manifold of dimension $n \geqq 3$ which has the scalar curvature $R = constant \neq 0$ and admits a proper conformal Killing vector field. Then the necessary and sufficient conditions that M is an Einstein space are $\nabla_{[l} \nabla_{h]} R_{ik} = 0$ and $\underset{\xi}{\mathcal{L}} T_{ij} = 0$.*

PROOF. This is proved evidently from Corollaries 2 and 3.

In the paper given by K. Yano and S. Sawaki [4] we find the following formula in a Riemannian manifold admitting conformal Killing vector field ξ^i,

$$\underset{\xi}{\mathcal{L}} T_{ij} = -(n-2) \left(\nabla_j \rho_i - \frac{1}{n} \nabla_l \rho^l g_{ij} \right).$$

From (1. 9), we have

$$\underset{\xi}{\mathcal{L}} T_{ij} = -(n-2) \left\{ \nabla_j \rho_i - \left(- \frac{2\rho R + \underset{\xi}{\mathcal{L}} R}{2n(n-1)\rho} \rho g_{ij} \right) \right\}.$$

Hence the conditions (2. 7) and (2. 8) are equivalent to $\underset{\xi}{\mathcal{L}} T_{ij} = 0$ and (2. 8).

§ 3. Some remark

Let (g_{ij}) be the symmetric matrix of the positive definite metric on M. The following theorem is well known (K. Yano and M. Obata [5], [6]).

THEOREM C. *If a compact Riemannian manifold M of dimension*

$n\geqq 2$ with $R=$const. admits an infinitesimal nonisometric conformal transformation ξ^i: $\underset{\xi}{\mathcal{L}}\, g_{ji}=2\rho g_{ji}$, $\rho\neq$const., and if one of the following conditions is satisfied, then M is isometric to a sphere.

(3. 1) *The vector field ξ^i is a gradient of a scalar.*

(3. 2) $R_i^h\rho^i=k\rho^h$, *k being a constant.*

(3. 3) $\underset{\xi}{\mathcal{L}}\, R_{ji}=\alpha g_{ji}$, *$\alpha$ being a scalar field.*

Its proof is introduced from that there exists ρ such that

$$\nabla_j\rho_i=-\frac{R}{n(n-1)}\rho g_{ij}\,. \tag{3.4}$$

Since from Theorem 1, (3. 3) is equivalent to (3. 4). Now we show that (3. 2) is obtained from (3. 4). Differentiating (3. 4) covariantly with respect to x^k and substituting the resulting equation into the Ricci identity

$$\nabla_k\nabla_j\rho_i-\nabla_j\nabla_k\rho_i=-R_{ijk}^l\rho_l\,,$$

we have

$$\frac{R}{n(n-1)}(\rho_k g_{ij}-\rho_j g_{ik})=R_{ijk}^l\rho_l\,,$$

so that multiplying both sides of it by g^{ij} and summing with respect to i and j, we get

$$R_k^l\rho_l=\frac{R}{n}\rho_k\,.$$

Therefore in case of $n\geqq 3$ we have the following diagram

$$(3.1)\Longrightarrow(3.4)\Longleftrightarrow(3.2)\Longleftrightarrow(3.3)\,.$$

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References

[1] T. KOYANAGI: On a certain property of a Riemannian space admitting a special concircular scalar field, Jour. Fac. Sci. Hokkaido Univ., 22 (1972), 154–157.

[2] H. KÔJYÔ: On conformal Killing tensors of a Riemannian manifold of constant curvature, Hokkaido Math. Jour., 2 (1973), 236–242.

[3] K. YANO: The theory of Lie derivatives and its applications, (North-Holland, 1957).

[4] K. YANO and S. SAWAKI: Riemannian manifolds admitting a conformal transformation group, J. Differential Geometry, 2 (1968) 161–184.

[5] K. YANO and M. OBATA: Sur le groupe de transformations conformes d'une variete de Riemann dont le scalaire de courbure est constant, C. R. Acad. Sc. paris, 260 (1965), 2698-2700.

[6] K. YANO: Integral formulas in Riemannian Geometry, (Dekker, 1970).

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