Distribution of zeros and deficiency of a canonical product of genus one

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1. Introduction. It is known that a simple geometrical restriction on the distribution of zeros is enough to make zero a Nevanlinna deficient value. This interesting phenomenon was firstly found by Edrei and Fuchs [1]. Soon after Edrei, Fuchs and Hellerstein [2] extended the results. One of their results can be stated in the following manner.

THEOREM A. *Let $g(z)$ be a canonical product of genus one and having zeros $\{a_k\}$ in the sector*

$$|\pi-\arg a_k|\leqq\frac{\pi}{60}.$$

Then

$$\delta(0, g)\geqq\frac{A}{1+A}$$

with a positive constant A.

Their formulation has a more general form. Of course $\pi/60$ is far from the best. Indeed in [3] we proved the following.

THEOREM B. *Let $g(z)$ be a canonical product of genus one and with only zeros $\{a_k\}$ in the sector*

$$|\pi-\arg a_k|\leqq\beta<\frac{\pi}{4}.$$

Then

$$\delta(0, g)\geqq\frac{A(\beta)}{1+A(\beta)},$$

where

$$A(\beta)=\frac{1}{\pi}\int_1^\infty\frac{1}{s^2}\left\{\frac{s+\frac{\sqrt{2}}{2}}{s^2+\sqrt{2}\,s+1}-\frac{s\sin 2\beta+\sin\beta}{s^2+2s\cos\beta+1}\right\}ds.$$

In this paper we shall prove the following

THEOREM 1. *Let $g(z)$ be a canonical product of genus one and with* *only zeros $\{a_k\}$ in the sector*

$$|\pi-\arg a_k|\leqq\frac{\pi}{4}.$$

Then

$$\delta(0, g)\geqq\frac{A}{1+A},$$

where

$$A=\frac{1}{4\pi}\int_1^\infty\frac{2\sqrt{2}s^2+2\sqrt{3}s+\sqrt{2}}{s^2(s^4+\sqrt{6}s^3+3s^2+\sqrt{6}s+1)}ds.$$

THEOREM 2. *Under the same assumptions in Theorem* 1

$$1\leqq\mu\leqq\rho\leqq2,$$

wher ρ and μ indicate the order and the lower order of $g(z)$, respectively.

It is an open problem to decide how wide the best possible opening of allowable sectors is. It is our conjecture that the result remains true when the opening of the sector is less than π. There are several evidences supporting the conjecture. The result does not hold if the opening of the sector is equal to π, which was shown in [2].

We shall give an evidence, which proves the conjecture under a symmetrical condition on the distribution of zeros. In order to state the result we need a preparation.

Let us consider eight points

$$|a|e^{i\alpha_j},\ j=1, 2, \cdots, 8$$

satisfying conditions:

$$\begin{aligned}
&\pi/2+\varepsilon\leqq\alpha_1\leqq\alpha_2\leqq\alpha_3\leqq\alpha_4\leqq\pi,\\
&\alpha_8=-\alpha_1,\quad \alpha_7=-\alpha_2,\quad \alpha_6=-\alpha_3,\quad \alpha_5=-\alpha_4 \pmod{2\pi},\\
&\alpha_1+\alpha_2=5\pi/4+3\varepsilon/2,\\
&\alpha_2+\alpha_3=3\pi/2+\varepsilon,\\
&\alpha_3+\alpha_4=7\pi/4+\varepsilon/2.
\end{aligned}$$

Here ε is an arbitrary fixed positive number satisfying $0<\varepsilon<\pi/2$. Then a set of these eight points is called an eight point group. If a point P satisfies

$$|\pi-\arg P|\leqq\frac{\pi}{2}-\varepsilon,$$

we can make an eight point group. We require the following condition S: every zero a_k of $g(z)$ satisfies

$$|\pi-\arg a_k|\leqq\frac{\pi}{2}-\varepsilon$$

and other seven points of the eight point group formed by a_k are also zeros of $g(z)$.

THEOREM 3. *Let $g(z)$ be a canonical product of genus one with condition S. Then*

$$\delta(0, g)\geqq\frac{A(\varepsilon)}{1+A(\varepsilon)}$$

with a positive constant $A(\varepsilon)$.

2. Lemmas. In order to prove Theorems 1 and 2 we need two lemmas.

LEMMA 1. *Let $\phi(y, \theta, \alpha)$ be*

$$\frac{1}{2}\log L(\alpha)+2y\cos\alpha\cos\theta\,,$$

$$L(\alpha)=1-4y\cos\alpha\cos\theta+2y^2(\cos 2\theta+4\cos^2\alpha)$$
$$-4y^3\cos\alpha\cos\theta+y^4.$$

Then

$$\phi(y, \theta, \alpha)\leqq\phi(y, \theta, 3\pi/4)$$

for $3\pi/4\leqq\alpha\leqq\pi$, $0\leqq\theta\leqq\pi/6$ *and* $y\geqq 0$.

PROOF. Let us consider $\partial\phi/\partial\alpha$. Then

$$\frac{\partial\phi(y, \theta, \alpha)}{\partial\alpha}=\frac{2y^2\sin\alpha}{L(\alpha)}\{2\cos\alpha\cos 2\theta+y\cos\theta(3-4\cos^2\theta$$
$$-4\cos^2\alpha)+4y^2\cos\alpha\cos^2\theta-y^3\cos\theta\}\,.$$

Evidently $\partial\phi(y, \theta, \alpha)/\partial\alpha\leqq 0$ under the given conditions. Hence $\phi(y, \theta, \alpha)$ is monotone decreasing there. This gives the desired result.

LEMMA 2. $\phi(y, \theta, 3\pi/4)<0$ *for* $y>0$, $0\leqq\theta\leqq\pi/6$.

PROOF.

$$\frac{\partial\phi(y, \theta, 3\pi/4)}{\partial y}=\frac{y^2}{L(3\pi/4)}\{\sqrt{2}\cos\theta(1-2\cos 2\theta)$$
$$-2y\cos 2\theta-\sqrt{2}\,y^2\cos\theta\}\,.$$

Hence $\partial\psi/\partial y<0$ for $y>0$, $0\leqq\theta\leqq\pi/6$. Thus

$$\psi(0,\ \theta,\ 3\pi/4)=0$$

implies the desired result.

3. Proof of Theorem 1. Let us consider

$$G(z)=g(z)\overline{g(\bar{z})}\,.$$

Then $N(r, 0, G)=2N(r, 0, g)$ and $m(r, G)\leqq m(r, g(z))+m(r, \overline{g(\bar{z})})=2\mathrm{m}(r, g)$. Hence

$$1-\delta(0, G)=\varlimsup_{r\to\infty}\frac{N(r, 0, G)}{m(r, G)}\geqq\varlimsup_{r\to\infty}\frac{N(r, 0, g)}{m(r, g)}=1-\delta(0, g)\,.$$

Hence it is sufficient to prove the result for $G(z)$. Evidently

$$\begin{aligned}\log|G(re^{i\theta})|&=\sum_{k=1}^{\infty}\left\{\log\left|1-\frac{2r}{|a_k|}e^{i\theta}\cos\alpha_k+\frac{r^2}{|a_k|^2}e^{2i\theta}\right|\right.\\&\qquad\left.+2\Re\left(\frac{r}{|a_k|}e^{i\theta}\cos\alpha_k\right)\right\}\\&=\sum_{k=1}^{\infty}\psi\left(\frac{r}{|a_k|},\ \theta,\ \alpha_k\right).\end{aligned}$$

By Lemma 1

$$\log|G(re^{i\theta})|\leqq\sum_{k=1}^{\infty}\psi\left(\frac{r}{|a_k|},\ \theta,\ \frac{3\pi}{4}\right)$$

for $0\leqq\theta\leqq\pi/6$, $r\geqq0$ and by Lemma 2

$$\sum_{k=1}^{\infty}\psi\left(\frac{r}{|a_k|},\ \theta,\ \frac{3\pi}{4}\right)<0$$

for $0\leqq\theta\leqq\pi/6$, $r>0$. Let

$$h(z)=\prod_{k=1}^{\infty}\left(1+\frac{z}{|a_k|}\right)\mathrm{e}^{-z/|a_k|}\,.$$

Put

$$H(z)=h(ze^{\pi i/4})h(ze^{-\pi i/4})\,.$$

Then

$$\log|H(re^{i\theta})|=\sum_{k=1}^{\infty}\psi\left(\frac{r}{|a_k|},\ \theta,\ \frac{3}{4}\pi\right).$$

Thus

$$m(r, 0, G) \geqq -\frac{1}{\pi}\int_0^{\pi/6}\log|G(re^{i\theta})|d\theta$$

$$\geqq -\frac{1}{\pi}\int_0^{\pi/6}\log|H(re^{i\theta})|d\theta .$$

The right hand side integral is equal to

$$-\frac{1}{\pi}\int_{\pi/4}^{\pi/4+\pi/6}\log|h(re^{i\theta})|d\theta - \frac{1}{\pi}\int_{-\pi/4}^{-\pi/4+\pi/6}\log|h(re^{i\theta})|d\theta .$$

Evidently $h(\bar{z})=\overline{h(z)}$. Hence we have

$$m(r, 0, G) \geqq -\frac{1}{\pi}\int_{\pi/4-\pi/6}^{\pi/4+\pi/6}\log|h(re^{i\theta})|d\theta .$$

Now by Shea's representation [4] we have

$$m(r, 0, G) > \frac{1}{4\pi}\int_0^\infty \frac{r^2}{t^2}N(t, 0, G)K(t, r)dt ,$$

$$\frac{1}{2}K(t, r) = \frac{r\sin\frac{5\pi}{12}+t\sin\frac{5\pi}{6}}{t^2+2tr\cos\frac{5\pi}{12}+r^2} - \frac{r\sin\frac{\pi}{12}+t\sin\frac{\pi}{6}}{t^2+2tr\cos\frac{\pi}{12}+r^2} .$$

Then

$$K(t, r) = \frac{r(2\sqrt{2}\,t^2+2\sqrt{3}\,tr+\sqrt{2}\,r^2)}{t^4+\sqrt{6}\,t^3r+3t^2r^2+\sqrt{6}\,tr^3+r^4} .$$

Hence

$$m(r, 0, G) \geqq \frac{1}{4\pi}\int_0^\infty \frac{N(sr, 0, G)}{s^2}K(s, 1)ds$$

$$\geqq N(r, 0, G)\frac{1}{4\pi}\int_1^\infty \frac{K(s, 1)}{s^2}ds$$

$$\equiv AN(r, 0, G) .$$

Evidently $A>0$. Therefore

$$1-\delta(0, G) = \overline{\lim_{r\to\infty}}\frac{N(r, 0, G)}{N(r, 0, G)+m(r, 0, G)}$$

$$\leqq \frac{1}{1+A} ,$$

which gives the desired fact.

4. Proof of Theorem 2. Using the same notations as in §3, we have

$$2m(r, g) \geqq m(r, G) \geqq m(r, 0, G)$$
$$\geqq \frac{1}{2\pi}\int_0^\infty \frac{N(sr, 0, g)}{s^2} K(s, 1) ds .$$

Further

$$\frac{1}{2} K(s, 1) \geqq \frac{\sin\frac{5\pi}{12} - \sin\frac{\pi}{12}}{s^2 + 2s\cos\frac{\pi}{12} + 1}$$
$$\geqq \frac{\sin\frac{5\pi}{12} - \sin\frac{\pi}{12}}{2\left(1 + \cos\frac{\pi}{12}\right)} \equiv \pi M$$

for $0 \leqq s \leqq 1$. Hence

$$m(r, g) \geqq Mr\int_0^r \frac{N(t, 0, g)}{t^2} dt ,$$

$$M = \frac{\sqrt{2}}{4\pi\left(1 + \cos\frac{\pi}{12}\right)} .$$

Since $g(z)$ is of genus one,

$$\lim_{r\to\infty}\int_0^r \frac{N(t, 0, g)}{t^1} dt = \infty .$$

Hence

$$\lim_{r\to\infty} \frac{m(r, g)}{r} = \infty ,$$

which implies $1 \leqq \mu$. $\mu \leqq \rho \leqq 2$ is known.

5. Lemmas. In order to prove Theorem 3 we need two Lemmas. Let

$$\phi(y, \theta, \alpha) = \frac{1}{2}\log L(\alpha) L(\beta) + 2y\cos\theta(\cos\alpha + \cos\beta)$$

with $\beta = 7\pi/4 + \varepsilon/2 - \alpha$, $0 < \varepsilon < \pi/4$ *and* $\alpha \in I = [3\pi/4 + \varepsilon/2,\ 7\pi/8 + \varepsilon/4]$.

LEMMA 3. *$\phi(y, \theta, \pi)$ is monotone decreasing for $\alpha \in I$, if $y \geqq 0$ and $\theta \in J_1 = [\pi/4 - \varepsilon/2,\ \pi/4]$ or $\theta \in J_2 = [-\pi/4 + \varepsilon/2,\ -\pi/4 + \varepsilon]$.*

PROOF. Let us consider $\partial\phi(y,\theta,\alpha)/\partial\alpha$. Then

$$\frac{\partial\phi(y,\theta,\alpha)}{\partial\alpha}=\frac{2y^2}{L(\alpha)L(\beta)}(P+Qy+Ry^2+Sy^3+Ty^4+Uy^5+Vy^6+Wy^7),$$

$$\begin{aligned}
P&=\cos 2\theta(\sin 2\alpha-\sin 2\beta),\\
Q&=-\cos\theta(\sin\alpha-\sin\beta+2\sin\alpha\cos 2\alpha-2\sin\beta\cos 2\beta)\\
&\quad-2\cos\theta\cos 2\theta(\sin\alpha-\sin\beta+2\cos\beta\sin 2\alpha-2\cos\alpha\sin 2\beta),\\
R&=\sin 2\alpha-\sin 2\beta+2\sin(\alpha-\beta)+4\cos\beta\sin\alpha\cos 2\alpha\\
&\quad-4\cos\alpha\sin\beta\cos 2\beta+\cos 2\theta(3\sin 2\alpha-3\sin 2\beta+2\sin 2(\alpha-\beta)\\
&\quad+6\sin(\alpha-\beta)+4\cos\beta\sin\alpha\cos 2\alpha-4\cos\alpha\sin\beta\cos 2\beta)\\
&\quad+\cos^2 2\theta\left\{4\sin(\alpha-\beta)+2\sin 2\alpha-2\sin 2\beta\right\},\\
S&=\cos\theta(-3\sin\alpha+3\sin\beta-2\sin\alpha\cos 2\beta+2\sin\beta\cos 2\alpha\\
&\qquad-4\cos\beta\sin 2\alpha+4\cos\alpha\sin 2\beta-4\sin\alpha\cos 2\alpha\\
&\qquad+4\sin\beta\cos 2\beta-4\sin\alpha\cos 2\beta+4\sin\beta\cos 2\beta\cos 2\alpha)\\
&\quad+\cos\theta\cos 2\theta(-6\sin\alpha+6\sin\beta-8\cos\beta\sin 2\alpha+8\cos\alpha\sin 2\beta\\
&\qquad-4\sin\alpha\cos 2\alpha+4\sin\beta\cos 2\beta-4\sin\alpha\cos 2\beta\\
&\qquad+4\sin\beta\cos 2\alpha)\\
&\quad+\cos\theta\cos^2 2\theta(-4\sin\alpha+4\sin\beta),\\
T&=2\sin(2\alpha-2\beta)+2\sin 2\alpha-2\sin 2\beta+4\cos\alpha\cos\beta(\sin 2\alpha-\sin 2\beta)\\
&\quad+\cos 2\theta\left\{3\sin 2\alpha-3\sin 2\beta+4\cos\alpha\cos\beta(\sin 2\alpha-\sin 2\beta)\right.\\
&\qquad\left.+4\sin(\alpha-\beta)+8\cos\alpha\cos\beta\sin(\alpha-\beta)\right\}\\
&\quad+\cos^2 2\theta\left\{4\sin(\alpha-\beta)+2\sin 2\alpha-2\sin 2\beta\right\},\\
U&=\cos\theta\left\{-3\sin\alpha+3\sin\beta-4\cos\beta\sin 2\alpha+4\cos\alpha\sin 2\beta\right.\\
&\qquad\left.-2\cos 2\beta(\sin\alpha-\sin\beta)-2\cos 2\alpha(\sin\alpha-\sin\beta)\right\}\\
&\quad+4\cos\theta\cos 2\theta(-\sin\alpha+\sin\beta-\cos\beta\sin 2\alpha+\cos\alpha\sin 2\beta),\\
V&=\sin 2\alpha-\sin 2\beta+2\sin(\alpha-\beta)\\
&\quad+\cos 2\theta\left\{\sin 2\alpha-\sin 2\beta+2\sin(\alpha-\beta)\right\},\\
W&=-\cos\theta(\sin\alpha-\sin\beta).
\end{aligned}$$

Evidently $P=-2\cos 2\theta\sin(\beta-\alpha)\cos(\alpha+\beta)\leqq 0$. Let us consider Q, which is equal to $-\cos\theta(\sin\alpha-\sin\beta)\{3-4\sin^2\alpha-4\sin\alpha\sin\beta-4\sin^2\beta\}$ $-2\cos\theta\cos 2\theta(\sin\alpha-\sin\beta)(1+4\cos\alpha\cos\beta)$.

Let $f(\alpha)$ be $3-4\sin^2\alpha-4\sin\alpha\sin\beta-4\sin^2\beta$. By $d\beta/d\alpha=-1$ we have

$$\begin{aligned}
\frac{d}{d\alpha}f(\alpha)&=+4\sin(\beta-\alpha)\left\{2\cos(\alpha+\beta)-1\right\}\\
&=4\sin(\beta-\alpha)\left\{2\sin\left(\frac{\pi}{4}+\frac{\varepsilon}{2}\right)-1\right\}\geqq 0.
\end{aligned}$$

Hence $f(\alpha)$ is monotone increasing for $\alpha\in I$. Further

$$f\left(\frac{3\pi}{4}+\frac{\varepsilon}{2}\right)=1+2\sin\varepsilon>0\,.$$

Hence $f(\alpha)>0$ for $\alpha\in I$. Thus $Q\leqq 0$. Let us consider R, which is equal to

$$\begin{aligned}-\sin(\beta-\alpha)\Big[&1+\cos(\alpha+\beta)+4\cos\alpha\cos\beta\cos(\alpha+\beta)\\&+\cos 2\theta\,\{3+3\cos(\alpha+\beta)+2\cos(\beta-\alpha)+4\cos\alpha\cos\beta\cos(\alpha+\beta)\}\\&+\cos^2 2\theta\,\{4+4\cos(\alpha+\beta\}\Big].\end{aligned}$$

This is evidently non-positive. Let us consider S.

$$\begin{aligned}S=&-\cos\theta\,(\sin\alpha-\sin\beta)\\&\Big[3+4\sin\alpha\sin\beta+8\cos\alpha\cos\beta+4\cos 2\alpha\cos 2\beta+2f(\alpha)\\&\quad+\cos 2\theta\,\{8+16\cos\alpha\cos\beta+2f(\alpha)\}+4\cos^2 2\theta\Big]\\&\leqq 0\,.\end{aligned}$$

Consider T. Then

$$\begin{aligned}T=&-2\sin(\beta-\alpha)\\&\Big[\cos(\beta-\alpha)+2\cos(\alpha+\beta)+8\cos\alpha\cos\beta\cos(\alpha+\beta)\\&\quad+\cos 2\theta\,\{2+3\cos(\alpha+\beta)+4\cos\alpha\cos\beta\,(1+\cos(\alpha+\beta))\}\\&\quad+\cos^2 2\theta\,\{2+2\cos(\alpha+\beta)\}\Big]\\&\leqq 0\,.\end{aligned}$$

U is equal to

$$\begin{aligned}-(\sin\alpha-\sin\beta)\cos\theta\Big\{&3+2\cos 2\alpha+2\cos 2\beta+8\cos\alpha\cos\beta\\&+4\cos 2\theta\,(1+2\cos\alpha\cos\beta)\Big\},\end{aligned}$$

which is non-positive. V is equal to

$$-2\sin(\beta-\alpha)\Big[1+\cos(\alpha+\beta)+\cos 2\theta\,\{1+\cos(\alpha+\beta)\}\Big]$$

which is non-positive. Trivially $W\leqq 0$. Hence

$$\frac{\partial\phi(y,\theta,\alpha)}{\partial\alpha}\leqq 0$$

for $\alpha\in I$, $y\geqq 0$ and $\theta\in J_1\cup J_2$. This implies the desired fact.

Let

$$\Phi(y, \theta, \alpha) = \frac{1}{2}\log L(\alpha)L(\beta)L(\gamma)L(\delta)$$
$$+2y\cos\theta(\cos\alpha+\cos\beta+\cos\gamma+\cos\delta),$$

where

$$\alpha+\beta=\frac{5\pi}{4}+\frac{3}{2}\varepsilon,\quad \beta+\gamma=\frac{3}{2}\pi+\varepsilon,\quad \gamma+\delta=\frac{7}{4}\pi+\frac{\varepsilon}{2}$$

and

$$\frac{\pi}{2}+\varepsilon\leqq\alpha\leqq\frac{5}{8}\pi+\frac{3}{4}\varepsilon.$$

Φ is constructed from eight prime factors corresponding to an eight point group, if $y=r/|a_k|$, $|z|=r$. Let $\Phi=\Phi_1+\Phi_2$, where

$$\Phi_1=\log\left|1-\frac{z}{|a_k|e^{i\alpha}}\right|\left|1-\frac{z}{|a_k|e^{i\beta}}\right|\left|1-\frac{z}{|a_k|e^{i\gamma}}\right|\left|1-\frac{z}{|a_k|e^{i\delta}}\right|$$
$$+\Re\left(\frac{z}{|a_k|e^{i\alpha}}+\frac{z}{|a_k|e^{i\beta}}+\frac{z}{|a_k|e^{i\gamma}}+\frac{z}{|a_k|e^{i\delta}}\right).$$

Then by the rotation $z=\zeta\exp\{(-\pi/4+\varepsilon/2)i\}$ we have

$$\phi\left(\frac{|\zeta|}{|a_k|}, \Theta, \alpha+\frac{\pi}{4}-\frac{\varepsilon}{2}\right)=\Phi_1\left(\frac{|z|}{|a_k|}, \theta, \alpha\right),$$
$$\Theta=\arg\zeta,\quad \theta=\arg z,\quad \Theta=\theta+\pi/4-\varepsilon/2.$$

By Lemma 3 we have

$$\Phi_1\left(\frac{|z|}{|a_k|}, \theta, \alpha\right)\leqq\Phi_1\left(\frac{|z|}{|a_k|}, \theta, \frac{\pi}{2}+\varepsilon\right)$$

for $0\leqq\theta\leqq\varepsilon/2$. Similarly

$$\Phi_2\left(\frac{|z|}{|a_k|}, \theta, \alpha\right)\leqq\Phi_2\left(\frac{|z|}{|a_k|}, \theta, \frac{\pi}{2}+\varepsilon\right)$$

for $0\leqq\theta\leqq\varepsilon/2$. Hence

$$\Phi\left(\frac{|z|}{|a_k|}, \theta, \alpha\right)\leqq\Phi\left(\frac{|z|}{|a_k|}, \theta, \frac{\pi}{2}+\varepsilon\right)$$

for $0\leqq\theta\leqq\varepsilon/2$.

LEMMA 4.

$$\sum_{k=1}^{\infty}\Phi\left(\frac{|z|}{|a_k|}, \theta, \frac{\pi}{2}+\varepsilon\right)<0$$

for $|z|>0$, $0\leqq\theta\leqq\varepsilon/2$. *Here the summation is taken over all the eight point groups formed by the zeros of* $g(z)$. *Further* $0<\varepsilon<\pi/4$ *is assumed.*

PROOF. In this case $\alpha=\pi/2+\varepsilon$, $\beta=\gamma=3\pi/4+\varepsilon/2$ and $\delta=\pi$. Hence we can use the Valiron representation.

$$\sum_{k=1}^{\infty}\Phi\left(\left|\frac{z}{a_k}\right|, \theta, \frac{\pi}{2}+\varepsilon\right)=-\int_0^{\infty}\frac{n(sr)}{s^2}Kds,$$

$$K=K_1+K_2, \qquad K_2(-\theta)=K_1(\theta),$$

$$K_1(\theta)=\frac{s\cos 2\theta+\cos\theta}{s^2+2s\cos\theta+1}+2\frac{s\sin(\varepsilon+2\theta)+\cos\left(\frac{\pi}{4}-\frac{\varepsilon}{2}-\theta\right)}{s^2+2s\cos\left(\frac{\pi}{4}-\frac{\varepsilon}{2}-\theta\right)+1}$$

$$+\frac{-s\cos(2\varepsilon+2\theta)+\sin(\varepsilon+\theta)}{s^2+2s\sin(\varepsilon+\theta)+1}.$$

Hence it is sufficient to prove $K_1>0$ and $K_2>0$ for $0\leqq\theta\leqq\varepsilon/2$. We firstly consider K_1, whose numerator can be represented by

$$A_1s^5+A_2s^4+A_3s^3+A_4s^2+A_5s+A_6,$$

$$A_1=\cos 2\theta-\cos(2\varepsilon+2\theta)+2\sin(\varepsilon+2\theta),$$

$$A_2=\cos\theta-2\cos\theta\cos(2\varepsilon+2\theta)+2\cos\left(\frac{\pi}{4}-\frac{\varepsilon}{2}-\theta\right)$$
$$+\sin(\theta+\varepsilon)\left\{2\cos 2\theta+4\cos\theta+1+4\sin\varepsilon+\theta)\right\}$$
$$+2\cos\left(\frac{\pi}{4}-\frac{\varepsilon}{2}-\theta\right)\left\{\cos 2\theta-\cos(2\varepsilon+2\theta)\right\},$$

$$A_3=2\cos\left(\frac{\pi}{4}-\frac{\varepsilon}{2}-\theta\right)\left\{3\cos\theta-2\cos\theta\cos(2\varepsilon+2\theta)\right\}$$
$$+2\cos 2\theta-2\cos(2\varepsilon+2\theta)+\left\{6\cos\theta+8\sin(\varepsilon+\theta)\right\}\sin(\varepsilon+\theta)$$
$$+2\cos\left(\frac{\pi}{4}-\frac{\varepsilon}{2}-\theta\right)(2\cos 2\theta+3)\sin(\varepsilon+\theta),$$

$$A_4=2\cos 2\theta-2\cos\theta\cos(2\varepsilon+2\theta)+4\cos\left(\frac{\pi}{4}-\frac{\varepsilon}{2}-\theta\right)$$
$$+2\sin(\varepsilon+\theta)\left\{3+\cos 2\theta+4\cos\theta\sin(\varepsilon+\theta)\right\}$$
$$+2\cos\left(\frac{\pi}{4}-\frac{\varepsilon}{2}-\theta\right)\left\{\cos 2\theta-\cos(2\varepsilon+2\theta)+8\sin\theta\sin(\varepsilon+\theta)\right\},$$

$$A_5=\cos 2\theta-\cos(2\varepsilon+2\theta)+2\sin(\varepsilon+2\theta)+4\cos\theta\sin(\varepsilon+\theta)$$
$$+2\cos\left(\frac{\pi}{4}-\frac{\varepsilon}{2}-\theta\right)\left\{3\cos\theta+3\sin(\varepsilon+\theta)\right\},$$

$$A_6=\cos\theta+\sin(\varepsilon+\theta)+2\cos\left(\frac{\pi}{4}-\frac{\varepsilon}{2}-\theta\right).$$

It is very easy to prove that each $A_j>0$ for $0\leqq\theta\leqq\varepsilon/2$ and $0<\varepsilon<\pi/4$. Hence $K_1>0$ there. Similarly $K_2>0$ there. Thus we have the desired result.

LEMMA 5.

$$\frac{2\sin 2\varepsilon}{s^2+2s\cos\varepsilon+1}+\frac{\sin 2\varepsilon}{s^2+2s\sin\varepsilon+1}-\frac{\sin 3\varepsilon}{s^2+2s\sin 3\varepsilon/2+1}>0$$

for $s\geqq 0$, $0<\varepsilon<\pi/4$.

PROOF. Consider the numerator, which has the form

$$As^4+Bs^3+Cs^2+Ds+E,$$

$$A=E=3\sin 2\varepsilon-\sin\varepsilon,$$

$$B=2\sin 2\varepsilon\left(\cos\varepsilon+2\sin\varepsilon+3\sin\frac{3}{2}\varepsilon\right)$$
$$-2\sin 3\varepsilon(\cos 2\varepsilon+\sin\varepsilon),$$

$$C=6\sin 2\varepsilon-2\sin 3\varepsilon+8\sin\varepsilon\sin 2\varepsilon\sin\frac{3}{2}\varepsilon$$
$$+4\sin 2\varepsilon\sin\frac{3}{2}\varepsilon\cos\varepsilon-4\sin\varepsilon\sin 3\varepsilon\cos 2\varepsilon,$$

$$D=2\sin 2\varepsilon\left(\cos\varepsilon+3\sin\frac{3}{2}\varepsilon+2\sin\varepsilon\right)$$
$$-2\sin 3\varepsilon(\cos 2\varepsilon+\sin\varepsilon).$$

By a simple calculation we have

$$A=E=3\sin\varepsilon+A',$$
$$B=-6\sin\varepsilon+B',$$
$$C=6\sin\varepsilon+C',$$
$$D=-2\sin\varepsilon+D',$$

where A', B', C', D' are positive for $0<\varepsilon<\pi/4$. Hence it is sufficient to prove

$$m(s)\equiv 3s^4-6s^3+6s^2-2s+3>0$$

for $s \geqq 0$. To this end differentiate $m(s)$ twice

$$m'(s) = 12s^3 - 18s^2 + 12s - 2\,,$$
$$m''(s) = 36s^2 - 36s + 12\,.$$

$m''(s) > 0$ implies that $m'(s)$ has only one zero a and $m'(s) < 0$ for $0 \leqq s < a$ and $m'(s) > 0$ for $s > a$. Hence $m(s) > m(a)$. Consider

$$8m(s) - 2sm'(s) + m'(s) = 6s^2 + 22\,.$$

This is evidently positive. Hence $m(a) > 0$. Thus $m(s) > 0$, which is the desired result.

6. Proof of Theorem 3. By Theorem 1 it is enough to show the result for $0 < \varepsilon < \pi/4$. Let b_k be any representative of the kth eight point group formed by a zero of $g(z)$. Let $h(z)$ be

$$\prod_{k=1}^{\infty}\left(1 + \frac{z}{|b_k|}\right)e^{-z/|b_k|}$$

Let

$$G(z) = \left[h(z)\,h\{ze^{\left(-\frac{\pi}{4}+\frac{\varepsilon}{2}\right)i}\}\,h\{ze^{\left(\frac{\pi}{4}-\frac{\varepsilon}{2}\right)i}\}\right]^2 h\{ze^{\left(-\frac{\pi}{2}+\varepsilon\right)i}\}\,h\{ze^{\left(\frac{\pi}{2}-\varepsilon\right)i}\}\,.$$

Then

$$\log|G(re^{i\theta})| = \sum_{k=1}^{\infty}\Phi\left(\frac{r}{|b_k|},\, \theta,\, \frac{\pi}{2}+\varepsilon\right).$$

Hence by Lemma 3 and Lemma 4 for $r > 0$, $0 \leqq \theta \leqq \varepsilon/2$

$$\log|g(re^{i\theta})| \leqq \log|G(re^{i\theta})| < 0\,.$$

Evidently $N(r, 0, g) = N(r, 0, G) = 8N(r, 0, h)$. Further

$$\begin{aligned}
m(\mathrm{r}, 0, g) &\geqq -\frac{1}{\pi}\int_0^{\varepsilon/2}\log|g(re^{i\theta})|\,d\theta \\
&\geqq -\frac{1}{\pi}\int_0^{\varepsilon/2}\log|G(re^{i\theta})|\,d\theta \\
&= -\frac{2}{\pi}\int_0^{\varepsilon/2}\log|h(re^{i\theta})|\,\mathrm{d}\theta - \frac{1}{\pi}\int_{-\pi/2+\varepsilon}^{-\pi/2+3\varepsilon/2}\log|h(re^{i\theta})|\,d\theta \\
&\quad - \frac{1}{\pi}\int_{\pi/2-\varepsilon}^{\pi/2-\varepsilon/2}\log|h(re^{i\theta})|\,d\theta - \frac{2}{\pi}\int_{-\pi/4+\varepsilon/2}^{-\pi/4+\varepsilon}\log|h(re^{i\theta})|\,d\theta \\
&\quad - \frac{2}{\pi}\int_{\pi/4-\varepsilon/2}^{\pi/4}\log|h(re^{i\theta})|\,d\theta\,.
\end{aligned}$$

By $h(\bar{z})=\overline{h(z)}$ we have

$$m(r, 0, g)\geqq -\frac{2}{\pi}\int_0^{\varepsilon/2}\log|h(re^{i\theta})|d\theta-\frac{1}{\pi}\int_{\pi/2-3\varepsilon/2}^{\pi/2-\varepsilon}\log|h(re^{i\theta})|d\theta$$
$$-\frac{2}{\pi}\int_{\pi/4-\varepsilon}^{\pi/4}\log|h(re^{i\theta})|d\theta .$$

Now by Shea's representation [4] we have

$$m(r, 0, g)\geqq\frac{1}{8\pi}\int_0^\infty\frac{N(sr, 0, g)}{s^2}L(s, \varepsilon)ds ,$$

$$L(s, \varepsilon)=\frac{2s\sin 2\varepsilon+2\sin\varepsilon}{s^2+2s\cos\varepsilon+1}+\frac{s\sin 2\varepsilon+\cos\varepsilon}{s^2+2s\sin\varepsilon+1}-\frac{s\sin 3\varepsilon+\cos\frac{3}{2}\varepsilon}{s^2+2s\sin\frac{3}{2}\varepsilon+1}$$
$$+\frac{2s+\sqrt{2}}{s^2+\sqrt{2}s+1}-\frac{2s\cos 2\varepsilon+\sin\left(\frac{\pi}{4}-\varepsilon\right)}{s^2+2s\cos\left(\frac{\pi}{4}-\varepsilon\right)+1} .$$

Since $s^2+\sqrt{2}s+1\leqq s^2+2s\cos(\pi/4-\varepsilon)+1$ and $\sin(\pi/4-\varepsilon)<\sqrt{2}$, the sum of the last two terms is positive. Since $s^2+2\sin\varepsilon+1\leqq s^2+2s\sin 3\varepsilon/2+1$ and $\cos\varepsilon>\cos 3\varepsilon/2$,

$$\cos\varepsilon\left(s^2+2s\sin\frac{3\varepsilon}{2}+1\right)>\cos\frac{3}{2}\varepsilon(s^2+2s\sin\varepsilon+1) .$$

Further using Lemma 5 we finally have

$$L(s, \varepsilon)>0$$

for $s\geqq 0, 0<\varepsilon<\pi/4$. Hence

$$\begin{aligned} m(r, 0, g)&\geqq\frac{1}{8\pi}\int_1^\infty\frac{N(sr, 0, g)}{s^2}L(s, \varepsilon)ds \\ &=\left\{\frac{1}{8\pi}\int_1^\infty\frac{L(s, \varepsilon)}{s^2}ds\right\}N(r, 0, g) \\ &\equiv A(\varepsilon)N(r, 0, g) . \end{aligned}$$

Therefore we have

$$\delta(0, g)\geqq\frac{A(\varepsilon)}{1+A(\varepsilon)} .$$

This completes the proof of Theorem 3.

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