Quasi-invariant measures on linear topological spaces

By Yasuji TAKAHASHI

§ 1. Introduction

In [1], Dao-Xing has shown that the following:

THEOREM *Let H be a separable Hilbert space, and let $\mathfrak{F}$ be the totality of weak Borel sets in H. Let Φ be a linear subspace of H, and suppose that Φ itself is a complete σ-Hilbert space with respect to the sequence of inner products $(\varphi, \psi)_n$, $n=1, 2, 3, \cdots$*
where $(\varphi, \varphi)_1 \leqq (\varphi, \varphi)_2 \leqq \cdots$.
Also, suppose that the inclusion mapping T from Φ into H is continuous. For each n, let Φ_n denote the completion of Φ with respect to the inner product $(\varphi, \psi)_n$. Then, the following conditions are equivalent.

(1) *There exists a Φ-quasi-invariant finite measure (non-trivial) μ on $(H, \mathfrak{F})$.*

(2) *There exists n such that the inclusion mapping T can be extended to a Hilbert-Schmidt operator from Φ_n into H.*

In the Dao-Xing's original Theorem, it is necessary that μ is regular. In this paper, we shall show that this assumption can be taken off, furthermore this theorem can be extended to complete σ-normed spaces.

Throughout this paper (except for § 2.1°. and § 5.), we shall assume that linear spaces are with real coefficients.

§ 2. Basic definitions and well known results

1°. p-absolutely summing operators ($1 \leqq p < \infty$)

Let E and F be Banach spaces.

DEFINITION 2. 1. 1. *Let $\{x_n\}$ be a sequence from a Banach space E. $\{x_n\}$ is called scalarly l_p if for each continuous linear functional $x^* \in E^*$, we have the inequality*

$$\sum_{n=1}^{\infty} |x^*(x_n)|^p < \infty .$$

$\{x_n\}$ is called absolutely l_p if $\sum_{n=1}^{\infty} \|x_n\|^p < \infty$.

DEFINITION 2. 1. 2. *A linear operator T from E into F is called p-*

absolutely summing if for each $\{x_n\}\subset E$ *which is scalarly* l_p, $\{T(x_n)\}\subset F$ *is absolutely* l_p.

We shall say "absolutely summing" instead of "1-absolutely summing".

PROPOSITION 2. 1. 1. (c.f. [2])

A linear operator T from E into F is p-absolutely summing iff there is a constant C such that for every finite set in E $x_1, x_2, \cdots, x_n$, *the following*

$$\sum_{k=1}^{n} \|T(x_k)\|^p \leqq C \sup_{\|x^*\|\leqq 1} \left(\sum_{k=1}^{n} |x^*(x_k)|^p\right)$$

holds.

PROPOSITION 2. 1. 2. (c.f. [3])

Let a linear operator T from E into F be p-absolutely summing. If $1\leqq p\leqq q<\infty$, *then T is q-absolutely summing.*

PROPOSITION 2. 1. 3. (c.f. [3])

A linear operator T from E into F is p-absolutely summing iff there is a probability measure μ *on the compact set* $K^*=$ *the* w^**-closure of the set of all extreme points of the unit ball of* E^*, *and a constant C such that*

$$\|T(x)\| \leqq C\left(\int_{K^*} |x^*(x)|^p d\mu(x^*)\right)^{\frac{1}{p}}, \quad \text{for } x\in E.$$

COROLLARY 2. 1. 1. (c.f. [3])

Let T be a 2-absolutely summing operator. Then there is a probability measure μ *on* K^* *and an operator* $S:\ L_2(\mu)\to F$ *such that*

(*a*) *S is a continuous linear operator.*

(*b*) $T=S\circ J\circ I$, *where* $I:\ E\to C(K^*)$ *is the canonical isometry* $x\to x(x^*)$ *and* $J:\ C(K^*)\to L_2(\mu)$ *is the identity operator.*

PROPOSITION 2. 1. 4. (c.f. [3])

Let K be a compact Hausdorff space and μ *be a probability measure on K. Then the identity operator* $I:\ C(K)\to L_p(\mu)$ *is p-absolutely summing.*

PROPOSITION 2. 1. 5. (c.f. [3], [4])

Let H_1 *and* H_2 *be Hilbert spaces and let T be a linear operator from* H_1 *into* H_2. *Then the followings are equvalent.*

(*a*) *T is p-absolutely summing.*

(*b*) *T is a Hilbert-Schmidt operator.*

By Corollary 2.1.1., Proposition 2.1.4. and Proposition 2.1.5., we have the following Proposition.

PROPOSITION 2.1.6. (c.f. [3])

Let H be a Hilbert space and E be a Banach space. Then the followings are equivalent.

(*a*) *T is 2-absolutely summing.*

(*b*) *There exists a Hilbert space H_1 such that*

$$H \xrightarrow[U]{} H_1 \xrightarrow[V]{} E$$

$T = V \circ U$ where U is a Hilbert-Schmidt operator and V is a continuous linear operator.

EXAMPLE 1. identity operator I: $l_1 \to l_2$ is absolutely summing.

EXAMPLE 2. identity operator I: $l_2 \to l_\infty$ is not p-absolutely summing, for $1 \leqq p < \infty$.

REMARK. Generally, p-absolutely summing operator is not necessarily compact. (c.f. Ex. 1.)

But a p-absolutely summing operator T from a Hilbert space H into a Banach space E is compact.

2°. Cylinder sets and Cylinder measures

In this subsection, we describe certain σ-algebras which will often be used in the ensuing discussion, and examine the relations between them.

DEFINITION 2. 2. 1. *Let E be a real linear topological space and E^* be a adjoint space of E. If A is a Borel set in real n-dimensional space R_n, and $x_1, x_2, \cdots, x_n \in E$, the set*

$$\left\{x^* \middle| (x^*(x_1), \cdots, x^*(x_n)) \in A,\ x^* \in E^*\right\}$$

will be called the Borel cylinder with base A corresponding to $x_1, \cdots, x_n$. If the elements $x_1, \cdots, x_n$ generate the linear subspace M of E, then we also call the above set a Borel cylinder corresponding to M, or a Borel M-cylinder.

The totality of Borel cylinders corresponding to a fixed M form a σ-algebra, which we denote by $S(M)$, and the totality of all Borel cylinders forms an algebra S. Let $\mathfrak{F}$ denote the smallest σ-algebra containing S; we call the elements of $\mathfrak{F}$ weak Borel sets.

Similarly, let $\mathfrak{F}$ be the smallest σ-algebra of subsets of E which contains all sets of the form

$$\left\{x \middle| x^*(x) < a\right\} \quad -\infty < a < \infty,\ x^* \in E^*.$$

The elements of $\mathfrak{F}$ will be called weak Borel sets.

The following lemma shows that the weak Borel sets constitute a sufficiently wide class of sets.

LEMMA 2.2.1. (c.f. [6])

If E is a separable σ-normed space, then every open (or clased) subset of E is a weak Borel set.

LEMMA 2.2.2. (c.f. [6])

Let E be a separable σ-normed space, with the norm sequence $\{\|x_n\|\}$. Then, $S_{-n}(R)=\{\|x^*\|_{-n}\leqq R\}$ is a weak Borel set in E^*.

By this lemma, we can conclude that E_n^* is a weak Borel set in E^*.

DEFINITION 2.2.2. *Let E be a linear topological space, and let S be the algebra of all Borel cylinders in E^*. Suppose that P is a set function on S having the following property: if M is any finite dimensional linear subspace of E, and $S(M)$ is the σ-algebra of Borel cylinders corresponding to M, then the restriction of P to $S(M)$ is a probability measure. Then we call P a cylinder measure on E^*. Clearly, any cylinder measure P also has the following properties:*

(1) $o\leqq P(Z)\leqq 1$ *for all* $Z\in S$

(2) $P(E^*)=1$

(3) *P is finitely additive.*

However, P is not, in general, σ-additive.

But if it happens that P is σ-additive, then, using well-known technique, we can extend P to a probability measure on the σ-algebra $\mathfrak{F}$ generated by S.

Next, we shall show the continuity of cylinder measures.

DEFINITION 2.2.3. *Let E be a linear topological space, and let P be a cylinder measure on E^*. Suppose that, given any positive number ε, there exists a neighborhood V of zero in E such that*

$$P\left(\left\{x^*\mid\ |x^*(x)|>1,\ x^*\in E^*\right\}\right)<\varepsilon$$

whenever $x\in V$. Then we say that P is continuous.

LEMMA 2.2.3. (c.f. [1], [5])

Let E be a linear topological space and let P be a cylinder measure on E^*. Then the function

$$L(x)=\int_{E^*}e^{ix^*(x)}dP(x^*)$$

is continuous iff P is continuous.

LEMMA 2.2.4. (c.f. [1], [5])

Let E be a linear topological space and $L(x)$ be a continuous positive *definite function on E [with $L(0)=1$]. Then, there is a unique continuous cylinder measure P on (E^*, S), such that*

$$L(x)=\int_{E^*} e^{ix^*(x)}dP(x^*) \qquad \text{for } \ x\in E.$$

REMARK. In Lemma 2.2.4., if E is a nuclear space, then P is a probability measure on $(E^*, \mathfrak{F})$.

If E is a σ-Hilbert space and $L(x)$ is continuous relative to the nuclear topology, then also P is a probability measure.

(For details, c.f. [1], [5], [6], [7])

3°. The existence of quasi-invariant measures

DEFINITION 2. 3. 1. *Let E be a linear space, F be a linear subspace of E, and let $\mathfrak{B}$ be a σ-algebra in E, which is invariant under translations. A measure μ on $(E, \mathfrak{B})$ is called F-quasi-invariant if*

$$\mu(B)=0 \text{ implies } \mu(B+x)=0 \qquad \text{for every } x\in F,\ B\in\mathfrak{B}.$$

DEFINITION 2. 3. 2. *Let E be a linear topological space, E^* be a adjoint space of E, let $\|x\|_H$ be a continuous Hilbertian norm on E.*

It is easily seen that the following $L(x)$ is continuous positive-definite function on E.

$$L(x)=e^{-\frac{\|x\|^2}{2}}$$

The corresponding measure on E^* (by Lemma 2.2.4.) is called a Gaussian measure. (mean zero, variance 1)

PROPOSITION 2. 3. 1. (c.f. [5[, [8])

Let E be a nuclear space, and $\|x\|_H$ be a continuous Hilbertian norm on E. Then, the corresponding Gaussian measure μ_H on E^* is σ-additive and E-quasi-invariant.

$$(E\subset H\cong H^*\subset E^*)$$

PROPOSITION 2. 3. 2. (c.f. [1], [5], [8])

Let H, G be separable Hilbert spaces such that G is a linear subspace of H and the natural imbedding $(G\rightarrow H)$ is a Hilbert-Schmidt operator.

Let $\mathfrak{F}$ be the σ-algebra of weak Borel sets in H. Then, there exists a (Gaussian) probability measure μ on $(H, \mathfrak{F})$ which is G-quasi-invariant.

4°. Dao-Xing's inequality

In this subsection, we shall show the most important lemma for our purpose.

LEMMA 2. 4. 1. (Dao-Xing's inequality) (c.f. [1], [9])

Let E be a linear topological space, F a linear subspace of E, and let $\mathfrak{B}$ be a σ-algebra in E which is invariant under translations and contains all cylinder sets.

Let $\mathfrak{T}$ be the topology on F such that $(F, \mathfrak{T})$ is a linear topological space of the second category, satisfying the first axiom of countability, and suppose that $\mathfrak{T}$ is stronger than the topology on F induced by E.

If there exists a F-quasi-invariant measure μ on $(E, \mathfrak{B})$, then there exists a neighborhood V of zero in $(F, \mathfrak{T})$ and positive number C, such that

$$\sup_{x\in V} |x^*(x)| \leqq C\int_E |x^*(x)| d\mu(x) \qquad \textit{for every} \quad x^* \in E^*.$$

REMARK. In original Dao-Xing's inequality, it was necessary that μ was locally finite and regular. But in [9], these assumptions were omitted.

§ 3. Main theorems and other results

In this section, we shall prove the following main theorems.

Throughout this section, we assume that linear topological spaces are with real coefficients.

THEOREM A. *Let H be a separable Hilbert space, with the inner product (x, y), and let $\mathfrak{F}$ be the totality of weak Borel sets in H.*

Let E be a linear subspace of H, and suppoe that E itself is a complete σ-normed space with respect to the sequence of norms $\|x\|_n$ $n=1, 2, \cdots$, where $\|x\|_1 \leqq \|x_2\| \leqq \cdots$.

Also, suppose that the inclusion mapping T from E into H is continuous. For each n, let E_n denote the completion of E with respect to the norm $\|x\|_n$. Then the following conditions are equivalent.

(1) *There exists a E-quasi-invariant finite measure (non-trivial) μ on $(H, \mathfrak{F})$.*

(2) *There exists n such that the adjoint operator T^* from H^* into E_n^* is absolutely summing.*

(3) *There exists a separable Hilbert space H_1 such that*

$$E \underset{J}{\subset} H_1 \underset{K}{\subset} H$$

$T=K\circ J$ where injection map J is continuous and K is a Hilbert-Schmidt operator respectively.

THEOREM B. *Let $1\leqq p\leqq 2$, let $\{a_n\}$ be a sequence of positive numbers, and let $l^p(a_n)$ denote the totality of real number sequences $\xi=\{\xi_n\}$ which satisfy the condition*

$$\|\xi\| = \left(\sum_{n=1}^{\infty} a_n|\xi_n|^p\right)^{\frac{1}{p}} < \infty$$

$l^p(a_n)$ forms a Banach space with respect to the usual coordinatewise linear operations and the norm $\|\xi\|$. In particular, we write l^p instead of $l^p(1)$. Let $\mathfrak{F}$ be the σ-algebra in $l^p(a_n)$ generated by the totality of Borel cylinders

$$\left\{\xi | (\xi_1, \xi_2, \cdots, \xi_n) \in B\right\}$$

(where B represents an arbitrary Borel sets in n-dimensional space).

Let Φ be a linear subspace of $l^p(a_n)$, and suppose that Φ itself is a complete σ-Hilbert space with respect to the sequence of inner products $(\varphi, \psi)_n$, $n=1, 2, \cdots$ where $(\varphi, \varphi)_1 \leqq (\varphi, \varphi)_2 \leqq \cdots$.

Also, suppose that the inclusion mapping T from Φ into $l^p(a_n)$ is continuous. For each n, let Φ_n denote the completion of Φ with respect to the inner product $(\varphi, \psi)_n$. Then the following conditions are equivalent.

(1) *There exists a Φ-quasi-invariant finite measure (non-trivial) on $(l^p(a_n), \mathfrak{F})$.*

(2) *There exists n_0 such that the adjoint operator T^* from $l^p(a_n)^*$ into $\Phi_{n_0}^*$ is absolutely summing.*

(3) *There exists separable Hilbert spaces H_1 and H_2 such that*

$$\Phi \underset{I}{\subset} H_1 \underset{J}{\subset} H_2 \underset{K}{\subset} l^p(a_n)$$

$T = K \circ J \circ I$ where injection map I and K are continuous, J is a Hilbert-Schmidt operator.

In order to prove these two theorems, the following proposition is necessary.

PROPOSITION 3.1. *Let F be a Banach space, E be a linear subspace of F, and suppose that E itself is a complete σ-normed space. Also, suppose that the inclusion mapping T from E into F is continuous.*

Then, the existence of a E-quasi-invariant finite measure (non-trivial) μ on $(F, \mathfrak{F})$ implies that, there exists n_0 such that

(1) *T^* is absolutely summing* $(T^*: F^* \to E_{n_0}^*)$

(2) *T^* is compact* $(T^*: F^* \to E_{n_0}^*)$

PROOF. (1): We may assume that μ satisfies the condition

$$\int_F \|x\| d\mu(x) < \infty$$

for otherwise, we can replace μ by the equivalent measure

$$\mu_1(A)=\int_A e^{-\|x\|}d\mu(x) \qquad A\in\mathfrak{F}$$

which certainly satisfies this condition.

Now, by Dao-Xing's inequality, there exists a positive number C_1 and n_0 such that

$$\|T^*(x^*)\|_{E^*_{n_0}} \leqq C_1\int_F |x^*(x)|d\mu(x) \qquad x^*\in F^* \qquad (*)$$

Let $\{x^*_n\}\subset F^*$ be scalarly l_1, then we may assume that

$$\sum_{n=1}^{\infty}|x^*_n(x)|<\infty \qquad x\in F$$

Putting

$$p(x)=\sum_{n=1}^{\infty}|x^*_n(x)| \qquad x\in F,$$

obviously $p(x)$ is a lower semicontinuous seminorm on F.

Since F is a Banach space, using Gelfand's theorem, $p(x)$ is continuous. Therefore, there exists a positive number C_2 such that

$$p(x)\leqq C_2\|x\| \qquad x\in F$$

Using $(*)$, we get

$$\sum_{n=1}^{\infty}\|T^*(x^*_n)\|_{E^*_{n_0}} \leqq C_1\int\sum_{n=1}^{\infty}|x^*_n(x)|d\mu(x)=C_1\int p(x)\,d\mu(x)$$
$$\leqq C_1C_2\int\|x\|d\mu(x)<\infty$$

Q. E. D.

that is the assertion.

(2): Let $\{x^*_n\}\subset F^*$ be a sequence which converges weakly to zero. Using $(*)$ and Fatou's lemma, we have

$$\overline{\lim_{n\to\infty}}\ \|T^*(x^*_n)\|_{E^*_{n_0}} \leqq \overline{\lim_{n\to\infty}}\int_F |x^*_n(x)|d\mu(x)$$
$$=\int_F \overline{\lim_{n\to\infty}}\ |x^*_n(x)|d\mu(x)=0\,.$$

Namely, $\{T^*(x^*_n)\}$ converges strongly to zero in $E^*_{n_0}$. Therefore, T^* is compact.

Q. E. D.

Example. Identity operator I: $l^1\to l^\infty$ is absolutely summing, but it is not compact. Therefore, there is not a l^1-quasi-invariant finite measure on $(l^\infty, \mathfrak{F})$.

COROLLARY 3. 1. *Let E be a Banach space, $\mathfrak{B}$ be a σ-algebra in E which is invariant under translations and contains all cylinder sets.*

Then, the following conditions are equivalent.

(1) *There exists a E-quasi-invariant finite measure* (*non-trivial*) *on* $(E, \mathfrak{B})$.

(2) *E is finite dimensional.*

PROOF of THEOREM *A*.

(1)$\Longrightarrow$(2): Using Proposition 3.1., it is obvious.

(2)$\Longrightarrow$(3): By assumption, there exists n such that the adjoint operator T^* from H^* into E_n^* is absolutely summing.

Using Proposition 2.1.2. and Proposition 2.1.6., there exists a Hilbert space G such that

$$H^* \xrightarrow[U]{} G \xrightarrow[V]{} E_n^*$$

$T^* = V \circ U$ where U is a Hilbert-Schmidt operator and V is a continuous linear operator respectively.

Here, we may assume that G is separable and $U(H^*)$ is dense in G. For otherwise, we could replace G by $\overline{U(H^*)}$ (closure of $U(H^*)$ in G, which certainly satisfies this condition.

Considering the adjoint operator of $T^* = V \circ U$, we get $T^{**} = U^* \circ V^*$

$$E_n^* \xrightarrow[V^*]{} G^* \xrightarrow[U^*]{} H$$

Since $U(H^*)$ is dense in G, U^* is a injection. Also, since $T = T^{**}{}_{|E}$, it is easily seen that we have the assetion.

(3)$\Longrightarrow$(1): Using Proposition 2.3.2., it is obvious.

PROOF of THEOREM *B*.

(1)$\Longrightarrow$(2): Using Proposition 3.1., it is obvious.

(2)$\Longrightarrow$(3): We shall define

$$\varphi(\xi) = (a_n^{\frac{1}{p}} \xi_n) \qquad \text{for} \quad \xi = (\xi_n) \in l^p(a_n).$$

Then, $\varphi(\xi)$ is a linear isometry from $l^p(a_n)$ onto l^p. Therefore, we may assume that $a_n = 1$ $n = 1, 2, \cdots,$.

By assumption, there exists n_0 such that the adjoint operator T^* from $(l^p)^*$ into $\Phi_{n_0}^*$ is absolutely summing. Therefore, by Proposition 2.1.2., it is p-absolutely summing. Let $\{e_n^*\}$ be a sequence from $(l^p)^*$, $\left(\text{where } e_n^* = (\delta_{nk})_k\right.$ $\left.\delta_{nk} = \begin{cases} 1 & \text{if } k = n \\ 0 & \text{if } k \neq n \end{cases}\right)$ then it is easily seen that $\{e_n^*\}$ is scalarly l_p, therefore we get

$$\sum_{n=1}^{\infty} \|T^*(e_n^*)\|_{\Phi_{n_0}^*}^p < \infty .$$

Next, we shall show that a linear operator T from Φ_{n_0} into l^p is p-absolutely summing.

Let $\{x_k\} \subset \Phi_{n_0}$ be scalarly l_p, namely

$$\sum_{k=1}^{\infty} |\langle x_k, x^* \rangle|^p < \infty \qquad x^* \in \Phi_{n_0}^*,$$

then, it is easily seen that the following inequality holds

$$\sup_{\|x^*\| \leqq 1} \sum_{k=1}^{\infty} |\langle x_k, x^* \rangle|^p < \infty .$$

Then,

$$\sum_{k=1}^{\infty} \|T(x_k)\|^p = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} |\langle T(x_k), e_n^* \rangle|^p = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} |\langle x_k, T^*(e_n^*) \rangle|^p$$

$$\leqq \left(\sup_{\|x^*\| \leqq 1} \sum_{k=1}^{\infty} |\langle x_k, x^* \rangle|^p \right) \sum_{n=1}^{\infty} \|T^*(e_n^*)\|_{\Phi_{n_0}^*}^p < \infty .$$

Thus, T is p-absolutely summing, and also T is 2-absolutely summing. By similar discussions for Theorem A, we have the assertion.

(3)$\Longrightarrow$(1): Using Proposition 2.3.2., it is obvious.

Q. E. D.

REMARK (1): Theorem A and Theorem B is the generalization of the Dao-Xing's theorem for complete σ-normed spaces and $l^p(a_n)$-spaces, and also, in Proposition 3.1., if E is a complete σ-Hilbert space and F is a separable Hilbert space, using Proposition 2.1.5., we get the Dao-Xing's theorem.

REMARK (2): In Proposition 3.1., Theorem A and Theorem B, if a quasi-invariant measure μ is σ-finite, then these results are valid.

Next, we shall show the relations between nuclear spaces and quasi-invariant measures.

LEMMA 3.1. (c. f. [3])

Let E, F and G be Banach spaces respectively, and $T(E \to F)$, $S(F \to G)$ be absolutely summing operators respectively.

Then, $S \circ T(E \to G)$ is nuclear.

PROPOSITION 3.2. *Let $E = \bigcap_{n=1}^{\infty} E_n$ be a complete σ-normed space. Then the following conditions are equivalent.*

(1) *For any n, there exists a E-quasi-invariant finite measure μ_n on $(E_n, \mathfrak{B}_n)$.*

(2) *E is a nuclear space.*

PROOF.

(1)$\Longrightarrow$(2): Using Proposition 3.1., for any m, there exists n such that T^* (adjoint operator of the natural injection T) is an absolutely summing operator from E_m^* into E_n^*. Furthermore, for n, there exists s such that T^* is an absolutely summing operator from E_n^* into E_s^*. Using Proposition 2.1.2., Corollary 2.1.1., Proposition 2.1.4. and Proposition 2.1.5., it is easily seen that the natural injection T from E_s into E_m is absolutely summing.

Similarly, there exists t such that the natural injection T from E_t into E_s is absolutely summing.

Using Lemma 3.1., we have the assertion.

(2)$\Longrightarrow$(1): Since E is nuclear, hence E is a nuclear σ-Hilbert space. Thus, we get $E=\bigcap_{n=1}^{\infty} E_n=\bigcap_{n=1}^{\infty} \Phi_n$ (where Φ_n is a separable Hilbert space).

For any n, there exists s and t such that

$$\Phi_t \xrightarrow[T]{} \Phi_s \xrightarrow[S]{} E_n$$

(where natural injection T is a Hilbert-Schmidt operator and S is a continuous linear operator respectively.)

Using the similar technique for the proof of Theorem A, we have the assertion.

Q. E. D.

PROPOSITION 3. 3. *Let E be a separable σ-normed space, and $\mathfrak{F}$ be the totality of weak Borel sets in E^*. Then the following conditions are equivalent.*

(1) *For any n, there exists a E_n^*-quasi-invariant finite measure μ_n on $(E^*, \mathfrak{F})$.*

(2) *E is a nuclear space.*

PROOF.

(1)$\Longrightarrow$(2): Since $E_s^*\in\mathfrak{F}$ (see Lemma 2.2.2.), the sequence of sets E_s^* is monotonic increasing, and $E^*=\bigcup_{s=1}^{\infty} E_s^*$, there exists s such that $\mu_n(E_s^*)>0$ [with $s\geqq n$]. Restricting $\mathfrak{F}$ and μ_n to E_s^*, we obtain a E_n^*-quasi-invariant finite measure μ_n on $(E_s^*, \mathfrak{F}_{|E_s^*})$.

Using Proposition 3.1., natural map $(E_s^{**}\to E_n^{**})$ is absolutely summing. Furthermore there exists t such that natural map $(E_t^{**}\to E_s^{**})$ is absolutely summing. Using the same method for the proof of Proposition 3. 2., we have the assertion.

(2)$\Longrightarrow$(1): By the similar discussions for the proof of Proposition 3.2., we have the assertion.

Q. E. D.

EXAMPLE. $L=\bigcap_{n=1}^{\infty} l^p(a_{m,n})$ $1\leqq p<\infty$, $0<a_{m,n}\leqq a_{m,n+1}<\infty$ $(m, n=1, 2, 3, \cdots)$

L is a complete σ-normed space with respect to the following norms

$$\|\xi\|_n=\left(\sum_{m=1}^{\infty} a_{m,n}|\xi_m|^p\right)^{\frac{1}{p}} \qquad \textit{for any} \quad \xi=(\xi_m)\in L.$$

Then, the following conditions are equivalent.

(1) *For any n, there exists a L-quasi-invariant finite measure μ_n on $(l^p(a_{m,n}), \mathfrak{B}_n)$.*

(2) *L is nuclear.*

(3) *For any n, there exists s such that*

$$\sum_{m=1}^{\infty}\frac{a_{m,n}}{a_{m,s}}<\infty .$$

PROOF.

(1)$\Longleftrightarrow$(2): By Proposition 3.2., it is obvious.

(1)$\Longrightarrow$(3): By Proposition 3.1., there exists s such that the adjoint operator $(l^p(a_{m,n}))^*\to(l^p(a_{m,s}))^*$ is absolutely summing. By easy calculations, we have the assertion.

(3)$\Longrightarrow$(2): It is easily seen that if for any n, there exists s such that

$$\sum_{m=1}^{\infty}\left(\frac{a_{m,n}}{a_{m,s}}\right)^{\frac{1}{p}}<\infty ,$$

then, L is nuclear.

But by assumption, we obtain the followings;
there exists $t, u, \cdots$ such that

$$\sum_{m=1}^{\infty}\frac{a_{m,s}}{a_{m,t}}<\infty , \qquad \sum_{m=1}^{\infty}\frac{a_{m,t}}{a_{m,u}}<\infty \cdots .$$

Using Hölder's inequality, we have the assertion.

Q. E. D.

Finally, we shall show the relations between Bochner's theorem and quasi-invariant measures.

PROPOSITION 3. 4. (c.f. [7])

Let Φ be a separable σ-Hilbert space, with the inner products $(\varphi_1, \varphi_2)_n^{\Phi}$, and let Ψ be a linear subspace of Φ, and suppese that Ψ itself is a complete separable σ-Hilbert space with respect to the inner products $(\psi_1, \psi_2)_n^{\Psi}$. Also, suppose that the inclusion mapping T from Ψ into Φ is continuous. For *each n, let Φ_n, (Ψ_n) denote the completion of Φ, (Ψ) with respect to the inner products $(\varphi_1, \varphi_2)_n^{\Phi}$, $((\psi_1, \psi_2)_n^{\Psi})$ respectively. Then, the following conditions are equivalent.*

(1) *T is a Hilbert-Schmidt operator from Ψ into Φ in σ-Hilbert spaces. Namely, for any m, there exists n such that T is a Hilbert-Schmidt operator from Ψ_n into Φ_m.*

(2) *For any n, there exists a Ψ-quasi-invariant finite measure μ_n on $(\Phi_n, \mathfrak{B}_n)$.*

(3) *For any continuous cylinder set measure μ in Φ^*, the cylinder set measure $T^*\mu$ in Ψ^* induced by T and μ is σ-additive.*

(4) *Let μ_n be the Gaussian measure, defined in Φ^* by $(\varphi_1, \varphi_2)_n^{\Phi}$, then for any n, the measure $T^*\mu_n$ in Ψ^* induced by T and μ_n is σ-additive.*

(5) *For any positive definite continuous function $L(\varphi)$ on Φ with $L(0)=1$, there exists a unique probability measure μ on $(\Psi^*, \mathfrak{F})$ such that*

$$L(\psi)=\int_{\Psi^*} e^{iF(\psi)}\, d(F)$$

PROOF.

(1)$\Longrightarrow$(2): For any n, there exists s such that T is a Hilbert-Schmidt operator from Ψ_s into Φ_n.

Using the similar method for the proof of Theorem A, there exists a separable Hilbert space H such that

$$\Psi \underset{S}{\subsetneqq} H \underset{U}{\subsetneqq} \Phi_n$$

$T=U\circ S$ where the injection map S is a continuous linear operatar and U is a Hilbert-Schmidt operator respectively.

Using Proposition 2.2.3., we have the assertion.

(2)$\Longrightarrow$(1): Using Proposition 3.1. and Proposition 3.1.5., it is obvious.

Finally, by [7], (1), (3) and (4) are equivalent, and by Lemma 2.2.3. and Lemma 2.2.4., (3) and (5) are equivalent. Thus we have the conclusion.

Q. E. D.

REMARK (1) In the above Proposition, σ-algebra $\mathfrak{B}_n(\mathfrak{F})$ are weak Borel sets in $\Phi_n(\Psi^*)$ respectively.

REMARK (2) In the above Proposition, let Φ and Ψ be separable Hilbert spaces, then the Dao-Xing's results is that (1), (3) and (4) are equivalent.

§ 4. Further discussions for $l^p(a_n)$ and $L^p(X, \mu)$

Throughout this section, we assume that linear spaces are with real coefficients.

1°. Let $1 \leqq q < \infty$, let $\{b_{m,n}\}$ be a double sequence of positive numbers with

$$b_{m,n} \leqq b_{m+1,n} \qquad m, n = 1, 2, \cdots$$

and let $\bigcap_{m=1}^{\infty} l^q(b_{m,n})$ denote the totality of real number sequences $\xi = (\xi_n)$ which satisfies the condition

$$\|\xi\|_m = \left(\sum_{n=1}^{\infty} b_{m,n} |\xi_n|^q\right)^{\frac{1}{q}} < \infty \qquad m = 1, 2, \cdots$$

then, $\bigcap_{m=1}^{\infty} l^q(b_{m,n})$ forms a complete σ-normed space with respect to the usual coordinatewise linear operations and norms $\|\xi\|_m$. Let $l^p(a_n)$ and $\mathfrak{F}$ be the same notations of Theorem B.

Then, we have the followings.

Proposition 4. 1. 1. *Let $\bigcap_{m=1}^{\infty} l^q(b_{m,n})$ be a linear subspace of $l^p(a_n)$ and let the injection map T: $\bigcap_{m=1}^{\infty} l^q(b_{m,n}) \to l^p(a_n)$ be continuous. If we also assume that $1 \leqq p \leqq 2$, $1 \leqq q \leqq 2$, then the following conditions are equivalent.*

(1) *There exists a $\bigcap_{m=1}^{\infty} l^q(b_{m,n})$-quasi-invariant finite measure on $(l^p(a_n), \mathfrak{F})$.*

(2) *There exists m_0 such that the adjoint operator $T^*: l^p(a_n)^* \to l^q(b_{m_0,n})^*$ is absolutely summing.*

(3) *There exists m_0 such that the adjoint operator $T^*: l^p(a_n)^* \to l^q(b_{m_0,n})^*$ is p-absolutely summing.*

(4) *There exists m_0 such that*

$$\sum_{n=1}^{\infty} a_n / b_{m_0,n}^{\frac{p}{q}} < \infty .$$

(5) *There exist separable Hilbert space H_1 and H_2 such that*

$$\bigcap_{m=1}^{\infty} l^q(b_{m,n}) \underset{I}{\subsetneqq} H_1 \underset{J}{\subsetneqq} H_2 \underset{K}{\subsetneqq} l^p(a_n)$$

$T = K \circ J \circ I$ where injection map I and K are continuous, and J is a Hilbert-Schmidt operator.

Proof. Using Proposition 2.1.2., Proposition 2.3.2. and Proposition 3.1., (1)$\Longrightarrow$(2)$\Longrightarrow$(3) and (5)$\Longrightarrow$(1) are obvious.

(4)$\Longrightarrow$(5): We assume that there exists m_0 such that

$$\sum_{n=1}^{\infty} a_n / b_{m_0,n}^{\frac{p}{q}} < \infty ,$$

then we have

$$l^q \underset{I}{\subsetneqq} l^2 \underset{J}{\subsetneqq} l^2(a_n/b_{m_0,n}^{\frac{p}{q}}) \underset{K}{\subsetneqq} l^p(a_n/b_{m_0,n}^{\frac{p}{q}})$$

where natural injections I and K are continuous, and J is a Hilbert-Schmidt operator.

Next, we shall consider $\varphi(\xi)=(b_{m_0,n}^{\frac{1}{q}}\xi_n)$ for $\xi=(\xi_n)\in l^p(a_n)$, then it is clear that $l_p(a_n)$ and $l^p(a_n/b_{m_0,n}^{\frac{p}{q}})$ are linearly isometric by $\varphi(\xi)$. Thus we have

$$\begin{aligned}\bigcap_{m=1}^{\infty} l^q(b_{m,n}) \subset l^q(b_{m_0,n}) &= \varphi^{-1}(l^q) \\ &\subset \varphi^{-1}(l^2) \subset \varphi^{-1}\left(l^2(a_n/b_{m_0,n}^{\frac{p}{q}})\right) \\ &\subset \varphi^{-1}\left(l^p(a_n/b_{m_0,n}^{\frac{p}{q}})\right) = l^p(a_n).\end{aligned}$$

Putting $H_1=\varphi^{-1}(l^2)$, $H_2=\varphi^{-1}\left(l^2(a_n/b_{m_0,n}^{\frac{p}{q}})\right)$, we have the assertion.

Finally, for (3)$\Longrightarrow$(4), we shall prove more general case in § 4. 2°. Proposition 4.2.1..

Q. E. D.

REMARK. If $1\leqq p<\infty$, $1\leqq q\leqq 2$, then we can show that (1), (2), (3) and (4) are equivalent, and (5)$\Longrightarrow$(1) is valid, however, (4)$\Longrightarrow$(5) is not valid. Furthermore, if $1\leqq p<\infty$, $1\leqq q<\infty$, then, (1)$\Longrightarrow$(2)$\Longrightarrow$(3)$\Longrightarrow$(4) are valid, however, (4)$\Longrightarrow$(1) is not valid. (c.f. Proposition 4.1.2., Proposition 4.1.3.)

PROPOSITION 4. 1. 2. *Let* $2\leqq p<\infty$, $1<q<\infty$, *and* $l^q\subset l^p(a_n)$, $\left[\textit{with } \sum_{n=1}^{\infty} a_n<\infty\right]$. *If there exists a* l^q*-quasi-invariant finite measure* μ *on* $(l^p(a_n), \mathfrak{F})$.

Then, we have

$$\sum_{n=1}^{\infty} a_n^{\frac{q^*}{2}}<\infty \qquad \left(\frac{1}{q}+\frac{1}{q^*}=1\right)$$

PROOF. Since $l^p(a_n)\subset l^2(a_n)$, by assumption, we can easily show that there exists l^q-quasi-invariant finite measure μ on $(l^2(a_n), \mathfrak{F})$.

Then, by Theorem A, injection map $(l^q\rightarrow l^2(a_n))$ is a Hilbert-Schmidt operator, theorefore it is q^*-absolutely summing.

From this, we have easily the assertion.

Q. E. D.

PROPOSITION 4. 1. 3. *Let* $1\leqq p<\infty$, $1<q<\infty$, *and* $l^q\subset l^p(a_n)$, *where the injection map is continuous. If the injection map is a Hilbert-Schmidt operator, then we have*

$$\sum_{n=1}^{\infty} a_n^{\frac{q^*}{p}} < \infty .$$

PROOF is easy.

2°. Throughout this subsection, let X be a set and $\mathfrak{B}$ be a σ-algebra in X, and let μ and ν be non-trivial positive measure on $(X, \mathfrak{B})$.

PROPOSITION 4.2.1. *Let $L^q(X, \nu) \subset L^p(X, \mu)$ $(1 \leqq p < \infty,\ 1 \leqq q < \infty)$ be usual Banach spaces and let the injection map T $(L^q(X, \nu) \to L^p(X, \mu))$ be continuous. Let $\mathfrak{B}_p$ be a σ-algebra in $L^p(X, \mu)$ which is invariant under translations and contains all cylinder sets.*

Then, the following implications (1)$\Longrightarrow$(2)$\Longrightarrow$(3)$\Longrightarrow$(4) holds.

(1) *There exists a $L^q(X, \nu)$-quasi-invariant finite measure on $(L^p(X, \mu), \mathfrak{B}_p)$.*

(2) *The adjoint operator T^*: $L^p(X, \mu)^* \to L^q(X, \nu)^*$ is absolutely summing.*

(3) *The adjoint operator T^*: $L^p(X, \mu)^* \to L^q(X, \nu)^*$ is p-absolutely summing.*

(4) *For any $\{X_n\} \subset X$ which is measurable and pairwise disjoint with $0 < \mu(X_n) < \infty$, $0 < \nu(X_n) < \infty$, we have*

$$\sum_{n=1}^{\infty} \frac{\mu(X_n)}{\nu(X_n)^{\frac{p}{q}}} < \infty .$$

PROOF. The implicatios (1)$\Longrightarrow$(2)$\Longrightarrow$(3) are valid by Proposition 3.1. and Proposition 2.1.2..

(3)$\Longrightarrow$(4): We shall define

$$f_n(x) = \begin{cases} \mu(X_n)^{\frac{1}{p}-1} & \text{for } x \in X_n \\ 0 & \text{for } x \in X_n^c \end{cases},$$

then $\{f_n\} \subset L^p(X, \mu)^*$ is scalarly l_p by the following $(*)$.

$(*)$ If $p = 1$, then for any $g \in L^\infty(X, \mu)^*$, there exist complex sequence $\{\alpha_n\}$ such that $|\alpha_n| = 1$, $|\langle f_n, g \rangle| = \alpha_n \langle f_n, g \rangle$.

Therefore, for any positive integer N, we have

$$\begin{aligned} \sum_{n=1}^{N} |\langle f_n, g \rangle| &= \left\langle \sum_{n=1}^{N} \alpha_n f_n, g \right\rangle \\ &\leqq \left\| \sum_{n=1}^{N} \alpha_n f_n \right\|_{L_\infty} \|g\|_{L_\infty^*} \leqq \|g\|_{L_\infty^*} . \end{aligned}$$

Thus, we have the assertion.

If $p > 1$, then for any $g \in L^p(X, \mu)$, we have

$$|\langle f_n, g\rangle| = \left|\int_{X_n} f_n(x)\, g(x)\, d\mu(x)\right|$$
$$\leqq \left[\int_{X_n} |f_n(x)|^{p^*} d\mu(x)\right]^{\frac{1}{p^*}} \left[\int_{X_n} |g(x)|^p\, d\mu(x)\right]^{\frac{1}{p}}$$
$$= \left[\int_{X_n} |g(x)|^p\, d\mu(x)\right]^{\frac{1}{p}}.$$

From this, we have

$$\sum_{n=1}^{\infty} |\langle f_n, g\rangle|^p \leqq \sum_{n=1}^{\infty} \int_{X_n} |g(x)|^p\, d\mu(x)$$
$$\leqq \int_X |g(x)|^p\, d\mu(x) < \infty.$$

That is the assertion.

Next, by assumption, the adjoint operator T^*: $L^p(X, \mu)^* \rightarrow L^q(x, \nu)^*$ is p-absolutely summing, namely

$$\sum_{n=1}^{\infty} \|T^*(f_n)\|^p_{L^{(q)*}} < \infty$$

On the other hand,

$$\|T^*(f_n)\|_{(L^q)^*} = \sup_{\|g\|_{L^q} \leqq 1} |\langle T^*(f_n), g\rangle|$$
$$= \sup_{\|g\|_{L^q} \leqq 1} |\langle f_n, T(g)\rangle| = \sup_{\|g\|_{L^q} \leqq 1} \left|\int f_n g\, d\mu\right|$$
$$\geqq \mu(X_n)\, \mu(X_n)^{\frac{1}{p}-1} \left[\frac{1}{\nu(X_n)}\right]^{\frac{1}{q}} = \frac{\mu(X_n)^{\frac{1}{p}}}{\nu(X_n)^{\frac{1}{q}}}.$$

Thus, we have

$$\sum_{n=1}^{\infty} \frac{\mu(X_n)}{\nu(X_n)^{\frac{p}{q}}} \leqq \sum_{n=1}^{\infty} \|T^*(f_n)\|^p_{(L^q)^*} < \infty.$$

That is the conclusion.

Q. E. D.

Let $(X, \mathfrak{B}, \mu)$ be a measure space. The μ-measurable set E of positive measure is called an atom whenever for any μ-measurable subset E_1 of E we have either $\mu(E_1)=0$ or $\mu(E-E_1)=0$.

If $(X, \mathfrak{B}, \mu)$ be a σ-finite measure space, then we may show that $X = X_1 + X_2$ uniquely, where neither X_1 nor any of its measurable subsets is an atom, and X_2 is a union of an at most countable number of atoms of finite measure. When this, we shall say X_1 non atomic part of μ.

THEOREM 4.2.1. *Let $(X, \mathfrak{B}, \mu)$ be a non-trivial finite measure space, and let $\mathfrak{B}_p$ be a σ-algebra in $L^p(X, \mu)$ which is invariant under translations and contains all cylinder sets. If $1 \leqq p \leqq q \leqq 2$, then the following condition are equivalent.*

(1) *There exists a $L^q(X, \mu)$-quasi-invariant finite measure (non-trivial on $(L^p(X, \mu), \mathfrak{B}_p)$.*

(2) *The adjoint operator $(L^p(X, \mu)^* \to L^q(X, \mu)^*)$ is absolutely summing.*

(3) *The adjoint operator $(L^p(X, \mu)^* \to L^q(X, \mu)^*)$ is p-absolutely summing.*

(4) *For any $\{X_n\} \subset X$ which is measurablt and pairwise disjoint, we have $\sum_{n=1}^{\infty} \mu(X_n)^{1-\frac{p}{q}} < \infty$.*

PROOF. The implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$ are valid by Proposition 3.1., Proposition 2.1.2. and Proposition 4.2.1..

$(4) \Rightarrow (1)$: Suppose that the condition (4), then it is easily seen that the non-atomic part of μ has zero measure.

Since $\mu(X) < \infty$, μ is concentrated on at most countable sets.

In this case, using Proposition 4.1.1., we have the assertion.

Q.E.D.

REMARK. If $1 \leqq p \leqq q < \infty$, the implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$ are valid. However, $(4) \Rightarrow (1)$ is not valid.

COROLLARY 4.2.1. *Let $1 \leqq p \leqq q < \infty$, and let $(X, \mathfrak{B}, \mu)$ be a finite measure space. If the non-atomic part of μ has a positive measure, then there exist no $L^q(X, \mu)$-quasi-invariant finite measure on $(L^p(X, \mu), \mathfrak{B}_p)$.*

EXAMPLE. *Let μ be a Lebesgue measure on $([a, b], \mathfrak{B})$, and let $1 \leqq p \leqq q < \infty$. Then there exist no $L^q(X, \mu)$-quasi-invariant finite measure on $(L^p(X, \mu), \mathfrak{B}_p)$.*

PROPOSITION 4.2.2. *Let $(X, \mathfrak{B}, \mu)$ be a σ-finite measure space, and $\mathfrak{B}_p$ be a σ-algebra in $L^p(X, \mu)$ which is invariant under translations and contains all cylinder sets. If the non-atomic part of μ has a positive measure, then there exist no $L^p(X, \mu) \cap L^q(X, \mu)$-quasi-invariant finite measure on $(L^p(X, \mu), \mathfrak{B}_p)$, for any $1 \leqq p \leqq \infty$, $1 \leqq q \leqq \infty$.*

PROOF. Let X_1 be a non-atomic part of μ.

Case 1. $0 < \mu(X_1) < \infty$.

Assume the contrary, then it is easily seen that there exists $L^p(X_1, \mu) \cap L^q(X_1, \mu)$-quasi-invariant finite measure on $(L^p(X_1, \mu), \mathfrak{B}_p^1)$, but by Corollary 4.2.1., it is a contradiction.

Case 2. $\mu(X_1) = \infty$.

Then, there exists $\{X^{(n)}\} \subset X_1$ which is measurable and pairwise disjoint with

$$\mu(X^{(n)})=1 \qquad n=1,2,\cdots.$$

We shall define

$$f_n(x)=\begin{cases}1 & \text{for} \quad x\in X^{(n)} \\ 0 & \text{for} \quad x\in X^{(n)c} \qquad \text{(complement of } X^{(n)}).\end{cases}$$

Then, we have $\{f_n\}\subset L^p(X,\mu)\cap L^q(X,\mu)$.

Since $L^p(X,\mu)\cap L^q(X,\mu)$ is a Banach space with the norm $\|f\|=\|f\|_p+\|f\|_q$ for $f\in L^p(X,\mu)\cap L^q(X,\mu)$, and natural injection $(L^p(X,\mu)\cap L^q(X,\mu)\to L^p(X,\mu))$ is continuous, if we assume the contrary, by Proposition 3. 1., natural injection $(L^p(X,\mu)\cap L^q(X,\mu)\to L^p(X,\mu))$ must be compact.

However, $\{f_n\}$ is bounded in $L^p(X,\mu)\cap L^q(X,\mu)$ and

$$\|f_n-f_m\|_p=\begin{cases}2^{\frac{1}{p}} & \text{for} \quad 1\leqq p<\infty \\ 1 & \text{for} \quad p=\infty\end{cases} \qquad \text{for any } n\neq m,$$

therefore, any subsequence of $\{f_n\}$ is not convergent in $L^p(X,\mu)$.

That is a contradiction.

Q. E. D.

Remark. In the above Proposition, we may take the following condition (*) instead of the above condition.

(*) *There exist positive number C_1, C_2 and $\{X_n\}\subset X$ which is measurable and pairwise disjoint such that*

$$C_1\leqq\mu(X_n)\leqq C_2 \qquad n=1,2,\cdots.$$

Example. *Let μ be a Lebesgue measure on $(R^N,\mathfrak{B})$, then there exist no $L^p(R^N,\mu)\cap L^q(R^N,\mu)$-quasi-invariant finite measure on $(L^p(R^N,\mu),\mathfrak{B}_p)$, for* $1\leqq p\leqq\infty$, $1\leqq q\leqq\infty$.

§ 5. Appendix

Let H_1 and H_2 be Hilbert spaces. The class of all Hilbert-Schmidt operators from H_1 into H_2 (resp. of all absolutely p-summing operators) will be denote by $\mathfrak{S}_2(H_1,H_2)$ (resp. $\Pi_p(H_1,H_2)$).

Then, using a quasi-invariant measure, we shall give another proof of the following Pietsch's Theorem.

Theorem. $\mathfrak{S}_2(H_1,H_2)=\Pi_p(H_1,H_2)$ *for* $1\leqq p\leqq 2$.

First, we shall prove the following Lemma.

Lemma 5.1. (c. f. [3])

Let E_1 and E_2 be normed sprces. If $1\leqq p\leqq q<\infty$, then

$$\Pi_p(E_1, E_2) \subset \Pi_q(E_1, E_2).$$

PROOF. We may assume that $1 \leqq p < q < \infty$.

If $T \in \Pi_p(E_1, E_2)$, then for each $\{x_n\} \subset E_1$ which is scalarly l_q, and for each $\{\lambda_n\} \in l_{q/q-p}$ and $x^* \in E_1^*$ (dual of E_1), we have

$$\sum_{n=1}^{\infty} |\langle |\lambda_n|^{\frac{1}{p}} x_n, x^* \rangle|^p = \sum_{n=1}^{\infty} |\lambda_n| |\langle x_n, x^* \rangle|^p$$
$$\leqq \left(\sum_{n=1}^{\infty} |\lambda_n|^{q/q-p} \right)^{q-p/q} \left(\sum_{n=1}^{\infty} |\langle x_n, x^* \rangle|^q \right)^{\frac{p}{q}} < \infty.$$

Namely, $\{|\lambda_n|^{\frac{1}{p}} x_n\} \subset E_1$ is scalarly l_p, therefore we have

$$\sum_{n=1}^{\infty} \|T(|\lambda_n|^{\frac{1}{p}} x_n)\|^p = \sum_{n=1}^{\infty} |\lambda_n| \, \|T(x_n)\|^p < \infty.$$

Thus, we have

$$\left\{ \|T(x_n)\|^p \right\} \in l_{q/p} \qquad \text{(dual of } l_{q/q-p}).$$

From this, we have easily the conclusion.

Q. E. D.

LEMMA 5. 2. $\mathfrak{S}_2(H_1, H_2) \subset \Pi_1(H_1, H_2)$

PROOF.

Case 1. Let H_1 and H_2 be separable Hilbert spaces with real coefficients.

If $T \in \mathfrak{S}_2(H_1, H_2)$, then we have the following decomposition:

$$H_1 \xrightarrow{T_1} \overline{T(H_1)} \xrightarrow{I} H_2 \qquad T = I \circ T_1$$

where $\overline{T(H_1)}$ is the closure of $T(H_1)$ in H_2, $T_1(=T)$ is the Hilbert-Schmidt operator from H_1 into $\overline{T(H_1)}$ and I is the identity map from $\overline{T(H_1)}$ into H_2.

Since the image of H_1 by T_1 is dense in $\overline{T(H_1)}$, T_1^* (adjoint operator of T_1) is a injection map from $\overline{T(H_1)}^*$ (dual of $\overline{T(H_1)}$) into H_1^* and also a Hilbert-Schmidt operator.

Then, we may consider that $\overline{T(H_1)}^*$ is a linear subspace of H_1^* and natural injection is a Hilbert-Schmidt operator.

Therefore, by Proposition 2.3.2., there exists a $\overline{T(H_1)}^*$-quasi-invariant Gaussian measure μ on $(H_1^*, \mathfrak{F})$.

Hence, by Proposition 3.1., T_1^{**} (adjoint operator of T_1^*) is a absolutely summing operator from H_1 into $\overline{T(H_1)}$.

From this and $T_1^{**} = T_1$, we have easily the conclusion.

Case 2. Let H_1 and H_2 be separable Hilbert spaces with complex coefficients.

Since H_1 and H_2 are isomorphic to l^2 (usual Hilbert space with complex coefficient), we may assume that $H_1=H_2=l^2$.

For any $\xi=(\xi_n)\in l^2$, we shall denote (Re. ξ_n) [resp. (Im. (ξ_n)] by Re. ξ [resp. Im. ξ], then it is easily seen that Re. $\xi\in l^2_R$ (usual Hilbert space with real coefficient) and Im. $\xi\in l^2_R$.

Let $\{x_n\}\subset l^2$ be scalarly l_1, then by the following (*), {Re. $x_n\}\subset l^2_R$ and {Im. $x_n\}\subset l^2_R$ are scalarly l_1 respectively.

(*) By assumptions, for any $x=(\zeta_k)\in l^2_R$ we have

$$\sum_{n=1}^{\infty}|\langle x_n, x\rangle|<\infty .$$

Let Re. $x_n=(\xi_k^{(n)})$ and Im. $x_n=(\eta_k^{(n)})$, then

$$\sum_{n=1}^{\infty}\left|\sum_{k=1}^{\infty}\xi_k^{(n)}\zeta_k+i\sum_{k=1}^{\infty}\eta_k^{(n)}\zeta_k\right|<\infty ,$$

hence we have

$$\sum_{n=1}^{\infty}\left|\sum_{k=1}^{\infty}\xi_k^{(n)}\zeta_k\right|<\infty , \qquad \sum_{n=1}^{\infty}\left|\sum_{k=1}^{\infty}\eta_k^{(n)}\zeta_k\right|<\infty .$$

Namely,

$$\sum_{n=1}^{\infty}|\langle Re.\ x_n, x\rangle|<\infty , \qquad \sum_{n=1}^{\infty}|\langle Im.\ x_n, x\rangle|<\infty .$$

Now, let $T\in\mathfrak{S}_2(H_1, H_2)$, since

$$\begin{aligned} T(x_n) &= T(Re.\ x_n)+iT(Im.\ x_n) \\ &= Re.\ T(Re.\ x_n)+iIm.\ T(Re.\ x_n)+Re.\ iT(Im.\ x_n)+iIm.\ iT(Im.\ x_n) , \end{aligned}$$

it is sufficient to show that:

$$\sum_{n=1}^{\infty}\|Re.\ T(Re.\ x_n)\|<\infty , \qquad \sum_{n=1}^{\infty}\|Im.\ T(Re.\ x_n)\|<\infty ,$$

$$\sum_{n=1}^{\infty}\|Re.\ iT(Im.\ x_n)\|<\infty , \qquad \sum_{n=1}^{\infty}\|Im.\ iT(Im.\ x_n)\|<\infty .$$

Let J be a continuous linear mapping from l^2 into l^2_R such that $J(x)=Re.\ x$ for any $x\in l^2$, then $J\circ T$ is a Hilbert-Schmidt operator from l^2_R into l^2_R.

Using Case 1., $J\circ T$ is a absolutely summing operator from l^2_R into l^2_R, and therefore we have

$$\sum_{n=1}^{\infty}\|Re.\ T(Re.\ x_n)\|<\infty .$$

By similar arguments, we have the assertion.

Finally, we shall prove the general case.

Let $T\in\mathfrak{S}_2(H_1, H_2)$, for any $\{x_n\}\subset H_1$ which is scalarly l_1, we shall denote a closed linear subspace of H_1 generated by $\{x_n\}$ by M_1 and denote the closure of $T(M_1)$ in H_2 by M_2, then T is a Hilbert-Schmidt operator from M_1 into M_2, and $\{x_n\}\subset M_1$ is scalarly l_1, and therefore using Case 2., we have

$$\sum_{n=1}^{\infty}\|T(x_n)\|<\infty .$$

That is the assertion.

Q. E. D.

PROOF of THEOREM. By Lemma 5.1. and Lemma 5.2., we have

$$\mathfrak{S}_2(H_1, H_2)\subset\Pi_p(H_1, H_2), \qquad \Pi_p(H_1, H_2)\subset\Pi_2(H_1, H_2)$$
$$\text{for } 1\leqq p\leqq 2 .$$

Therefore, it is sufficient to show that $\Pi_2(H_1, H_2)\subset\mathfrak{S}_2(H_1, H_2)$.

If $T\in\Pi_2(H_1, H_2)$, for any orthonormal sequence $\{e_n\}\subset H_1$, since $\{e_n\}\subset H_1$ is scalarly l_2, we have

$$\sum_{n=1}^{\infty}\|T(e_n)\|^2<\infty .$$

Namely, T is a Hilbert-Schmidt operator from H_1 into H_2.

Q. E. D.

Department of Mathematics
Hokkaido University

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(Received on July 1, 1974)