# Notes on Green lines

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## 1. Introduction

Let $R$ be a hyperbolic Riemann surface and $g(z)=g(z, z_0)$ be the Green function on $R$ with a fixed pole $z_0$ in $R$. For the following definitions and properties of Green lines and compactifications of $R$, we refer to Sario-Nakai [7] and Constantinescu-Cornea [2] respectively. We consider the Green lines issuing from the fixed point $z_0$. The set $L$ of all Green lines admits the Green measure $m$. A Green line $l$ for which $\inf_{z\in l} g(z)=0$ is called a regular Green line. Any regular Green line tends to the ideal boundary of $R$ as $g(z)\to 0$. The set of all regular Green lines is denoted by $L_r$. It is known (Brelot-Choquet [1]) that $m(L-L_r)=0$.

Let $R^*$ be a resolutive compactification of $R$ and $\mu_z$ be the harmonic measure on the ideal boundary $\Delta=R^*-R$ with respect to $z\in R$. We are interested in the behavior of $l\in L_r$ in $R^*$. We set $e(l)=\bar{l}-l\cup\{z_0\}$ with $\bar{l}$ the closure of $l$ in $R^*$. We call $e(l)$ the end part of $l$ in $R^*$. Given a subset $S\subset\Delta$ we write $\tilde{S}=\{l\in L_r | e(l)\cap S\neq\phi\}$ and $\check{S}=\{l\in L_r | e(l)\subset S\}$. Let $C(\Delta)$ be the set of all bounded continuous functions on $\Delta$. We set $C_D(\Delta)=\{f\in C(\Delta) | H_f^{R,R^*}\in HD(R)\}$. If $C_D(\Delta)$ is dense in $C(\Delta)$ with respect to the uniform convergence topology, then $R^*$ is said to be a regular compactification of $R$ (Maeda [4]).

In this paper we shall prove the following theorems:

THEOREM 1. *Let $R^*$ be a resolutive compactification of $R$. For every compact set $K$ (resp. open set $U$) in $\Delta$,*

$$\overline{m}(\check{K})\leq\mu_{z_0}(K), \quad \underline{m}(\tilde{U})\geq\mu_{z_0}(U),$$

*where $\overline{m}$ and $\underline{m}$ are the outer and inner measures induced by $m$. For every Baire set $S$ in $\Delta$, $\overline{m}(\check{S})\leq\mu_{z_0}(S)\leq\underline{m}(\tilde{S})$.*

COROLLARY 1. *Let $R^*$ be resolutive. If $R^*$ is metrizable, then for every Borel set $S$ in $\Delta$, $\overline{m}(\check{S})\leq\mu_{z_0}(S)\leq\underline{m}(\tilde{S})$.*

COROLLARY 2. (*i*) *Let $R^*$ be resolutive and $\Gamma$ be the harmonic boundary of $R^*$. If $R^*$ is metrizable, then $m(\tilde{\Gamma})=1$.*

(*ii*) *Let $R_M^*$ be the Martin compactification of $R$ and $\Delta_1$ be the set of all minimal points of $\Delta_M=R_M^*-R$. Then $m(\check{\Delta}_1)=1$.*

THEOREM 2. *Let $R^*$ be a regular compactification of $R$ and $\Gamma_r$ be the set of all regular points for Dirichlet problem with respect to $R^*$. If $R^*$ is metrizable, then $e(l)\cap\Gamma_r$ consists of at most a single point for m-almost every $l\in L_r$.*

COROLLARY. *Let $R^*$ be regular and metrizable. If $\Gamma_r$ is of $\mu_{z_0}(\Gamma_r)=1$, then $e(l)\cap\Gamma_r$ consists of exactly a single piont for m-almost every $l\in L_r$.*

Although the next Theorem 3 follows from Theorem 8 of Maeda [3], we shall give an alternative proof.

THEOREM 3. ([3]) *Let $R_N^*$ be the Kuramochi compactification of $R$ and $\mu_z^N$ be the harmonic measure on $\Delta_N=R_N^*-R$ with respect to $z\in R$ and $e_N(l)$ be the end part of $l\in L_r$ in $R_N^*$. For every compact set $K$ in $\Delta_N$ we set $K^*=\{l\in L_r|e_N(l)$ is a single point and $e_N(l)\in K\}$. Then $K^*$ is m-measurable and $m(K^*)=\mu_{z_0}^N(K)$.*

REMARK. For the case of the Royden compactification, the following Nakai's theorem is much better than Theorem 1.

Nakai's theorem. *Let $R_D^*$ be the Royden compactification of $R$ and $\mu_z^D$ be the harmonic measure on $\Delta_D=R_D^*-R$. For every $F_\sigma$-set $K$ (resp. $G_\delta$-set $U$) in $\Delta_D$,*

$$\overline{m}(\check{K})\leq\mu_{z_0}^D(K),\quad \underline{m}(\check{U})\geq\mu_{z_0}^D(U).$$

## 2. The proof of Theorem 1.

We consider two kinds of Dirichlet problems:

(a) Let $\psi$ be a bounded function on $L_r$. We consider the following classes:

$$\bar{\mathfrak{F}}_\psi=\left\{s\,\middle|\,\begin{array}{l}\text{superharmonic, bounded below on } R,\\ \varliminf_{z\in l,g(z)\to 0} s(z)\geq\psi(l) \text{ for } m\text{-almost every } l\in L_r\end{array}\right\}$$

and $\underline{\mathfrak{F}}_\psi=\{-s|s\in\bar{\mathfrak{F}}_{-\psi}\}$. We set $\bar{G}_\psi(z)=\inf\ \{s(z)|s\in\bar{\mathfrak{F}}_\psi\}$ and $\underline{G}_\psi(z)=\sup\ \{s(z)|s\in\underline{\mathfrak{F}}_\psi\}$ $(z\in R)$. It is known ([1]) that $\underline{G}_\psi$ and $\bar{G}_\psi$ are harmonic on $R$ and that

$$\underline{G}_\psi(z_0)\leq\underline{\int}\psi\,dm\leq\overline{\int}\psi\,dm\leq\bar{G}_\psi(z_0)\,. \tag{1}$$

(b) Let $R^*$ be a compactification. Let $\phi$ be a bounded function on $\Delta=R^*-R$. We consider the following classes:

$$\mathcal{S}_\phi^{R,R^*}=\mathcal{S}_\phi=\left\{s\,\middle|\,\begin{array}{l}\text{superharmonic, bounded below on } R,\\ \varliminf_{z\to b} s(z)\geq\phi(b) \text{ for every } b\in\Delta\end{array}\right\}$$

and $\underline{\mathscr{S}}_{\phi}^{R,R*}=\underline{\mathscr{S}}_{\phi}=\{-s|s\in\overline{\mathscr{S}}_{-\phi}\}$.

We set $\overline{H}_{\phi}^{R,R*}(z)=\overline{H}_{\phi}(z)=\inf\ \{s(z)|s\in\overline{\mathscr{S}}_{\phi}\}$ and $\underline{H}_{\phi}^{R,R*}(z)=\underline{H}_{\phi}(z)=\sup\ \{s(z)|s\in\underline{\mathscr{S}}_{\phi}\}$ $(z\in R)$. We know that $\underline{H}_{\phi}$ and $\overline{H}_{\phi}$ are harmonic on $R$. We make use of the next result.

LEMMA 1. (cf. Hilfssatz 8. 3 and Satz 8. 3 in [2]). Let $R^*$ be a resolutive compatification. For any bounded function $\phi$,

$$\underline{H}_{\phi}(z)\leq\underline{\int}\phi d\mu_z\leq\overline{\int}\phi d\mu_z\leq\overline{H}_{\phi}(z)$$

for every $z\in R$. If $\phi$ is bounded lower semicontinuous (resp. bounded upper semicotinuous), then $\underline{H}_{\phi}(z)=\int\phi d\mu_z$ $\left(\text{resp. } \overline{H}_{\phi}(z)=\int\phi d\mu_z\right)$. If $\phi$ is a bounded Baire function on $\Delta$, then $\underline{H}_{\phi}(z)=\overline{H}_{\phi}(z)=\int\phi d\mu_z$.

The proof of Theorem 1.

Let $E$ be any subset of $\Delta$. We denote by $\chi_E$ and $\chi_{\tilde{E}}$ (or $\chi_{\check{E}}$) be the characteristic funtion of the set $E$ and $\tilde{E}$ (or $\check{E}$) on $\Delta$ and $L_r$ respectively. We note $\sup_{b\in e(l)}\chi_E(b)=\chi_{\tilde{E}}(l)$ for every $l\in L_r$. Let $s\in\underline{\mathscr{S}}_{\chi_E}$. Since $\overline{\lim}_{z\to b}\ s(z)\leq\chi_E(b)$ for every $b\in\Delta$, we have

$$\overline{\lim_{z\in l,g(z)\to 0}}\ s(z)\leq\sup_{b\in e(l)}\ \overline{\lim_{z\to b}}\ s(z)\leq\sup_{b\in e(l)}\ \chi_E(b)=\chi_{\tilde{E}}(b)\,.$$

Hence $s\in\underline{\mathscr{F}}_{\chi_{\tilde{E}}}$. Then $\underline{\mathscr{S}}_{\chi_E}\subset\underline{\mathscr{F}}_{\chi_{\tilde{E}}}$ and by (1) we have

$$\underline{H}_{\chi_E}(z_0)\leq\underline{m}(\tilde{E})\,. \tag{2}$$

Let $U$, $K$ and $S$ be an open set, a compact set and a Baire set in $\Delta$ respectively. Then $\chi_U$ and $\chi_S$ are a lower semicontinuous function and a Baire function respectively. Hence by Lemma 1 and by (2) we have

$$\mu_{z_0}(U)\leq\underline{m}(\tilde{U})\ \text{and}\ \mu_{z_0}(S)\leq\underline{m}(\tilde{S})\,. \tag{3}$$

We note $\check{E}=L_r-\widetilde{(\Delta-E)}$ for any subset $E$ of $\Delta$. Since $\Delta-K$ and $\Delta-S$ are an open set and a Baire set in $\Delta$, by (3) we have

$$\overline{m}(\check{K})=m(L_r)-\underline{m}(\widetilde{\Delta-K})\leq 1-\mu_{z_0}(\Delta-K)=\mu_{z_0}(K)$$

and similarly $\overline{m}(\check{S})\leq\mu_{z_0}(S)$. Thus we have the theorem.

The proof of Corollary 1 is obvious.

The proof of Corollary 2. We know (cf. [2]) the next facts: (i) $\Gamma$ is a compact set in $\Delta$ and the support of $\mu_z$ is equal to $\Gamma$, (ii) $R_M^*$ is metrizable and $\Delta_1$ is a $G_\delta-$set and $\mu_z(\Delta_1)=1$. Hence Corollary 2 follows from Corollary 1.

## 3. The proof of Theorem 2.

For the following definitions and properties of $Q$-compactifications we refer to Abschnitt 9 of [2].

Let $R^*$ be regular. Then there exists a subfamily $Q$ of the vector sum $HBD(R)+BCW_0(R)$ such that $R^*=R_Q^*$ (Proposition 9 in Tanaka [8]). We use the same notation $f$ as the continuous extention of any $f\in Q$ to $R_Q^*$. We set

$$Q_1=\{H_f^{R,R*}|f\in Q\} \text{ and } Q_0=\{f-H_f^{R,R*}|f\in Q\}.$$

Then $Q_1\subset HBD(R)$ and $Q_0\subset BCW_0(R)$. We consider two compactifications $R_{Q_1\cup Q_0}^*$ and $R_{Q_1}^*$ besides $R^*=R_Q^*$. We denote by $\Gamma_{Q_1\cup Q_0}$ and $\Gamma_{Q_1}$ the harmonic boundary of $\Delta_{Q_1\cup Q_0}=R_{Q_1\cup Q_0}^*-R$ and $\Delta_{Q_1}=R_{Q_1}^*-R$ respectively. We note that every $f\in Q$ can be continuously extended over $R_{Q_1\cup Q_0}^*$ and that $Q_1\cup Q_0\supset Q_1$. Hence there exists the canonical mapping $\pi$ (resp. $\pi_1$) of $R_{Q_1\cup Q_0}^*$ onto $R^*$ (resp. $R_{Q_1}^*$) (cf. Satz 9. 4 in [2]).

By a discussion similar to that in the proof of Satz 9. 4 in [2], we can prove

LEMMA 2. *If* $b\in\Gamma_r$, *then* $\pi^{-1}(b)$ *is a single point and* $\pi^{-1}(b)\in\Gamma_{Q_1\cup Q_0}$.

Let $e_{Q_1\cup Q_0}(l)$ and $e_{Q_1}(l)$ be the end part of $l\in L_r$ in $R_{Q_1\cup Q_0}^*$ and $R_{Q_1}^*$ respectively. We set

$$A=\left\{l\in L_r|e(l)\cap\Gamma_r \text{ contains at least two distinct points}\right\}.$$

Let $l\in A$. Since $\pi(e_{Q_1\cup Q_0}(l))=e(l)$, by Lemma 2 we see that $e_{Q_1\cap Q_0}(l)\cap\Gamma_{Q_1\cup Q_0}$ contains at least two distinct points. On the other hand it follows from Satz 9.4 in [2] that $\pi_1$: $\Gamma_{Q_1\cup Q_0}\to\Gamma_{Q_1}$ is a homeomorphism. Hence we obtain that $e_{Q_1}(l)\cap\Gamma_{Q_1}$ contains at least two distinct points for every $l\in A$. Since $R_{Q_1}^*$ is metrizable and $Q_1\subset HBD(R)$, by the aid of Theorem 2 in Maeda ]3], we see that $m$-almost every Green line tends only one point of $\Delta_{Q_1}$. Hence $m(A)=0$. Thus we have the theorem.

The corollary follows from Corollary 2 of Theorem 1 and Theorem 2.

## 4. The proof of Theorem 3.

We set $L_N=\{l\in L_r|e_N(l)$ is a single point$\}$. Maeda (Theorem 2 in [3]) proved that

$$m(L_N)=1. \qquad (4)$$

Let $S$ be any subset of $\Delta_N$. We set $S^*=\{l\in L_N|e_N(l)\in S\}$. Let $\pi$ be the canonincal mapping from $R_D^*$ onto $R_N^*$. By an easy computation we see

$S^*=\overset{\smile}{\pi^{-1}(S)}\cap L_N=\widetilde{\pi^{-1}(S)}\cap L_N$. Hence by (4) we heve

(5) $$\overline{m}(S^*)=\overline{m}\left(\widetilde{\pi^{-1}(S)}\right) \text{ and } \underline{m}(S^*)=\underline{m}\left(\overset{\smile}{\pi^{-1}(S)}\right)$$

On the other hand we know

(6) $$\mu_z^D\left(\pi^{-1}(S)\right)=\mu_z^N(S)$$

for every Borel set in $\Delta_N$.

Let $K$ and $U$ be a compact set and an open set in $\Delta_N$ respectively. By (5), (6) and Nakai's theorem we have

(7) $$\overline{m}(K^*)=\overline{m}\left(\widetilde{\pi^{-1}(K)}\right)\leq\mu_{z_0}^D\left(\pi^{-1}(K)\right)=\mu_{z_0}^N(K).$$

$$\underline{m}(U^*)=\underline{m}\left(\overset{\smile}{\pi^{-1}(U)}\right)\geq\mu_{z_0}^D\left(\pi^{-1}(U)\right)=\mu_{z_0}^N(U).$$

Take a sequence $\{U_n\}_{n=1}^{\infty}$ of open sets in $\Delta_N$ with

$$U_{n+1}\subset\overline{U}_{n+1}\subset U_n \text{ and } \bigcap_{n=1}^{\infty} U_n=K. \text{ Then } U_{n+1}^*\subset U_n^* \text{ and } \bigcap_{n=1}^{\infty} U_n^*=K^*.$$

By the decreasing monotone continuity of $\underline{m}$ and the continuity of $\mu$ and by (7),

$$\underline{m}(K^*)=\lim_{n\to\infty}\underline{m}(U_n^*)\geq\lim_{n\to\infty}\mu_{z_0}^N(U_n)=\mu_{z_0}^N(K).$$

Hence by (7) we see that $K^*$ is $m$-measurable and $m(K^*)=\mu_{z_0}^N(K)$. Thus we have the theorem.

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