Analytic Functions on Some Hyperbolic Riemann Surfaces

By Hiroshi TANAKA*)

Introduction

Z. Kuramochi gave in his paper [4] a very interesting theorem, which can be stated as follows.

Theorem of Kuramochi. Let R be a hyperbolic Riemann surface of the class O_{HB} (resp. O_{HD}). Then, for any compact subset K of R such that $R-K$ is connected, $R-K$ as an open Riemann surface belongs to the class O_{AB} (resp. O_{AD}).

The theorem was proved by using the existence of points of positive harmonic measure on Martin or Kuramochi boundary. It is known that the existence of points of positive harmonic measure on the Martin or the Kuramochi boundary is equivalent to the existence of those points on the Wiener or the Royden boundary. Then there were questions whether there exists a hyperbolic Riemann surface, which has no boundary points with positive harmonic measure on the Royden or the Wiener boundary and has yet the same property as stated in the theorem of Kuramochi. To these questions N. Toda and K. Matsumoto [17] and K. Matsumoto [11] gave answers in the positive and proved that $R\in O_{A^0_X}\cap U_S$ implies $R-K\in O_{AX}$ ($X=B$ or D) for every compact subset K of R with connected complement.

In this paper we shall deal with a problem similar to the above by considering the class O_{AN} and capacities instead of harmonic measures on the Royden boundary. The main pourpose of this paper is to show a theorem similar to the above: $R\in O_{A^0_N}\cap U_N$ implies $R-K\in O_{AN}$ for any compact subset K of R with connected complement.

1. Capacity on the Royden boundary (cf. [14, 15])

Let R be a hyperbolic Riemann surface. For a subset A of R, we denote by ∂A the (relative) boundary of A in R. We call a closed or open subset A of R is regular if ∂A is non-empty and consists of at most a countable number of analytic arcs clustering nowhere in R. We fix a closed disk K_0 in R once for all and let $R_0=R-K_0$.

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*) 田　中　　博

We denote by Δ_N (resp. Δ_D) the Kuramochi boundary (resp. the Royden boundary) of R. Let $\tilde{C}$ (resp. C) be the Kuramochi capacity on Δ_N (resp. the capacity on Δ_D) (cf. [14, 15]). Let μ be the harmonic measure on Δ_N. We set $\Delta_S=\{b\in\Delta_N;\ \tilde{C}(\{b\})>0\}$ and $\Delta_S^D=\{\xi\in\Delta_D;\ C(\{\xi\})>0\}$. Furthemore let $\Delta_{SS}=\{b\in\Delta_N;\ \mu(\{b\})>0\}$. Then $\Delta_{SS}\subset\Delta_S\subset\Delta_1$. A point in $\Delta_S-\Delta_{SS}$ (resp. Δ_{SS}) is called a singular point of first kind (resp. a singular point of second kind) by Z. Kuramochi [5]. We denote by U_{HN} (resp. U_{HD}) the class of all Riemann surfaces such that $\Delta_S\neq\emptyset$ (resp. $\Delta_{SS}\neq\emptyset$). The harmonic boundary of R_D^* is denoted by Γ_D ([2]).

LEMMA. *If U is an open subset of R_D^* with $U\cap\Gamma_D\neq\emptyset$, then there is a regular region G on R such that $G\subset U\cap R$ and $G\notin SO_{HD}$*[1].

PROOF. There is an open subset U_1 of R_D^* such that $U_1\subset U$, $U_1\cap\Gamma_D\neq\emptyset$ and $U_1\cap R$ is a regular open set in R, Since $U_1\cap\Gamma_D\neq\emptyset$, we can find a compact subset K of $U_1\cap\Gamma_D$ with $C(K)>0$. By the aid of Lemma 7 in [15], we see that there exists a bounded continuous Dirichlet function f on R such that $0\leqq f\leqq 1$, $f\not\equiv 0$, $f=0$ on $R-U_1\cap R$ and f is harmonic in each component of $U_1\cap R$. Then there is a component G of $U_1\cap R$ with $f>0$ on G. Since the restriction of f to G belongs to $HBD(G)$ and $=0$ on ∂G $(\subset\partial(U_1\cap R))$ we see that $G\notin SO_{HD}$.

Let G be a regular region on R. Set $Q=\{f|G;\ f\in BCD(R)\}$[2]. Then it can be seen that there is a topological mapping of $\bar{G}^D$ onto the Q-compactification G_Q^* (cf. [2]) whose restriction to G is the identity. Thus we may identify $\bar{G}^D$ with G_Q^*. Since $Q\subset BCD(G)$, G_Q^* is a quotient space of the Royden compactification G_D^* of G. Hence there exists a canonical mapping[3] η of G_D^* onto $\bar{G}^D$. We set $b_G=(\bar{G}^D-\overline{\partial G}^D)\cap\Delta_D$. Then it is known ([2, 13]) that η is a homeomorphism of $G\cup\eta^{-1}(b_G)$ onto $G\cup b_G$.

By the aid of the proof of the Proposition 1 in [16] and Lemma 7 in [15], we can prove

PROPOSITION 1. *Let K be a compact subset of b_G. Then $C(K)=0$ if and only if $C^G(\eta^{-1}(K))=0$, where C^G is the capacity on G_D^*-G with respect to a fixed closed disk in G.*

COROLLARY. *If $(\bar{G}^D-\overline{\partial G}^D)\cap\Delta_S^D\neq\emptyset$, then $G\in U_{HN}$.*

According to M. Nakai [12], we say that a Riemann surface R is said to be *almost finite genus* if there exists a finite or infinite countably sequence $\{A_n\}$ of relatively compact annuli in R such that

1) See p. 107 in [2] for the definition of SO_{HD}.
2) $f|G$ means the restriction of f to G.
3) Cf. [14].

(α) $\quad \bar{A}_n \cap \bar{A}_m = \emptyset \quad (n \neq m)$,

(β) $\quad R - \bigcup_{n=1}^{\infty} \bar{A}_n$ is a planar subregion of R,

(γ) $\quad \sum_{n=1}^{\infty} 1/\log \operatorname{mod} A_n < \infty$.

By an annulus on a Riemann surface R we mean a region which is conformally equivalent to a doubly connected plane region.

PROPOSITION 2 ([8]). *Any Riemann surface of almost finite genus does not belong to* U_{HN}.

2. *AN*-functions

Let f be an analytic function on R with values in a complex plane $\boldsymbol{C}$. Let a' be any number in $\boldsymbol{C}$. We denote by G_r the covering surface generated by f over $\{|w-a'|<r\}$ $(r>0)$. Let $A(r)$ be the area of G_r. If $\underline{\lim}_{r\to 0} \frac{A(r)}{r^2} < \infty$, then a' is called an ordinary point with respect to f in the sense of A. Beurling [1] (cf. [7]).

DEFINITION 1. A function f in $AD(R)$ is said to be an *AN-function* if either it is a constant function or it is not a constant function and any complex number in $\boldsymbol{C}$ is an ordinary point with respect to f.

We denote by $AN = AN(R)$ the family of all AN-functions on R. By definition, we have $AN(R) \subset AD(R)$. Thus for the corresponding null classes, we have $O_{AD} \subset O_{AN}$.

The following properties are easy to see:

(i) If G is a region on R and f is a function in $AN(R)$, then the restriction $f|G$ of f to G belongs to $AN(G)$.

(ii) If $f \in AD(R)$ is finitely sheeted[4)], then it belongs to $AN(R)$.

3. Functions with Iversen's property

Consider a noncompact bordered Riemann surface (R, α) with compact border α which may be empty.

DEFINITION 2 (cf. [13]). A function in the class $M(R \cup \alpha)$ of (single-valued) meromorphic functions on $R \cup \alpha$ is said to have Iversen's property with respect to the ideal boundary β in the sense of Kerékjártó-Stoïlow of R if the following conditions are satisfied:

4) We say that f is finitely sheeted if it is finitely sheeted as a mapping of R into $\boldsymbol{C}$, cf. [16].

(a) f is not constant,

(b) for an arbitrary disk U on $\{|w|\leqq\infty\}$ with $f(\alpha)\cap U=\emptyset$ and $f(R)\cap U\neq\emptyset$, and for every component V of $f^{-1}(U)$, the set $U-f(V)$ is totally disconnected in U, i.e., $U-f(V)$ does not contain nondegenerate continua.

Let R' be a subregion of R such that (R', α'), $\alpha'=\partial R'$, is a noncompact bordered surface with compact border α', and the ideal boundary β' of R' is a subset of β. For $f\in M(R\cup\alpha)$, we denote by f' the restriction of f to $R'\cup\alpha'$, i.e., $f'\in M(R'\cup\alpha')$.

DEFINITION 3 (cf. [13]). We shall say that $f\in M(R\cup\alpha)$ has the localizable Iversen property with respect to β if the following is true:

(c) Not only does f have Iversen's property with respect to β but f' as well has this property with respect to β' for every R'.

Let (R, α) be a non-compact bordered Riemann surface with ideal boundary β in the sense of Kerékjártó-Stoïlow. Let f be a function in $M(R\cup\alpha)$. Let p be a point of β and $\{G_n\}_{n=1}^{\infty}$ be a determining sequence of p. Since $\bigcap_{n=1}^{\infty}\overline{f(G_n)}$ (in $R\cup\beta$) does not depend on the choice of such a sequence, we denote it by $C_R(f, p)$. The set $C_R(f, p)$ is called the cluster set of f at p.

THEOREM 1 (cf. [13]). *If $f\in M(R\cup\alpha)$ has the localizable Iversen property, then either the cluster set $C_R(f, p)$ of f at a Kerékjártó-Stoïlow's ideal boundary point p consists of a single point or $C_R(f, p)=\{|w|\leqq\infty\}$.*

4. AN-function on O_{A^0N}-surfaces

We consider the family SO_{AD} (resp. SO_{AN}) of bordered Riemann surfaces (R, γ) with boundary γ such that every AD-function (resp. AN-function) of R with Re $f=0$ on γ reduces to a constant. We denote by O_{A^0D} (resp. O_{A^0N}) the class of all open Riemann surfaces, any subregion of which belongs to the class SO_{AD} (resp. SO_{AN}). The class O_{A^0D} was introduced by T. Kuroda [9]. By definition, we see that $O_{A^0D}\subset O_{A^0N}$.

Let R be a Riemann surface in the class O_{A^0N}. Consider a non-compact subregion (G, α) of R with compact border α and non-empty Kerékjártó-Stoïlow's ideal boundary $\beta=\beta(G)$. We allow the case $R=G$ and $\beta=\emptyset$.

By modifying the proof of Theorem VI, 2B, in [12], we shall prove

THEOREM 2. *Let $R\in O_{A^0N}$. Then every function f in $AN(G\cup\alpha)$, $G\cup\alpha\subset R$, has the localizable Iversen property.*

PROOF. Since G is arbitrary, it suffices to prove that $f\in AN(G\cup\alpha)$ has Iversen property. For this purpose take a disk U in $\boldsymbol{C}$ with $f(\alpha)\cap U=\emptyset$

and $f(G)\cap U=\emptyset$ and choose a component V of $f^{-1}(U)$. We must show that $U-f(V)$ is totally disconnected.

First we shall show that $f(V)$ is dense in U. If this were not the case, then we could find two concentric disks W_1 and W_2 such that $U-f(V)\supset W_1\supset \overline{W}_2\supset W_2$. We assume that $U=\{|w-w_0|<r\}$ $(r>0)$. Set $\tilde{g}(w)=\frac{(w-w_0)}{r}-r/(w-w_0)$ for $w\in U-\overline{W}_2$. Then $\tilde{g}\in AN(U-\overline{W}_2)$ and Re $\tilde{g}=0$ on ∂U. Let $g=\tilde{g}|(U-\overline{W}_1)$ and consider $h=g\circ f$ on $V\cup\partial V$. Since g is finitely sheeted, it can be seen that $h\in AN(V\cup\partial V)$ and Re $h=0$ on ∂V. Therefore $(V,\partial V)\notin SO_{AN}$ which contradicts $R\in O_{A^0N}$.

Next suppose $U-f(V)$ contains a proper continuum K. Then we can choose a disk U_1 with $\bar{U}_1\subset U$ such that U_1-K consists of at least two components. Thus there exists a component V_1 of $f^{-1}(U_1)$ contained in V. Clearly $f(V_1)$ belongs to a component of U_1-K and hence $f(V_1)$ cannot be dense in U_1. This is a contradiction.

THEOREM 3 (cf. [13]). *Let $R\in O_{A^0N}$. Then every $f\in AN(G\cup\alpha)$, $G\cup\alpha\subset R$, is bounded continuous on the relative Kerékjártó-Stoïlow compactification $G^*_{KS,\alpha}$ of (G,α).*

PROOF. By a discussion similar to that in the proof of Theorem VI, 2C, in [13], we can prove the theorem.

4. The classes $U_N\cap O_{A^0N}$ and U_{HN}

Let p be a point of $\beta=\beta(R)$ and $\{G_n\}_{n=1}^{\infty}$ be a determining sequence of p. If $\lim_{n\to\infty} 1_{G_n\cup\partial G_n}>0$ (resp. $\lim_{n\to\infty} 1_{\widetilde{G_n\cup\partial G_n}}>0$)[5], then we say that p is of positive harmonic measure (resp. of positive capacity). For the above $\{G_n\}_{n=1}^{\infty}$, let $A(p)=\bigcap_{n=1}^{\infty}\overline{G_n\cup\partial G_n^N}$ (the closures are taken in R_N^*). It is easy to see that p is of positive harmonic measure (resp. of positive capacity) if and only if $\mu(A(p))>0$ (resp. $\tilde{C}(A(p))>0$).

DEFINITION 4. We denote by U_S (resp. U_N) the class of all hyperbolic Riemann surfaces such that there exists a point p in $\beta(R)$ with positive harmonic measure (resp. positive capacity).

By definition, we have $U_S\subset U_N$.

THEOREM 4. *Suppose $R\in U_N\cap O_{A^0N}$ and K be an arbitrary compact set in R with connected complement. Then $R-K$ belongs to O_{AN}.*

PROOF. Take a regular subregion $R_0\subset R$ such that $R_0\supset K$ and $R-\bar{R}_0$ is connected. Theorem 3 shows that $AN(R-R_0)\subset ABD(R-R_0)$ and every

5) See p. 43 and p. 164 in [2].

$f\in AN(R-R_0)$ can be continuously extended over $R^*_{KS,\alpha}$, $\alpha=\partial R_0$. We must prove that f is a constant. Suppose the contrary is true. Since $R\in U_N$, there is a point p in β with positive capacity. We denote by $A(p)$ the set of all points of Δ_N lying over p. Then $\tilde{C}(A(p))>0$ and $\tilde{C}(A(p)\cap\Delta_1)>0$ since $\tilde{C}(\Delta_0)=0$. By Theorem 3, we see that $\lim_{z\to p} f(z)$ exists. We denote by a' the limit. Since $\lim_{z\to b} f(z)=\lim_{z\to p} f(z)=a'$ for all $b\in A(p)$, we see that $A(p)\cap \Delta_1\subset\{b\in\Delta_1\,;\, f^\vee(b)=a'\}$[6]. It follows form Theorem 9 in [7] that $0\leqq\tilde{C}(A(p)\cap\Delta_1)\leqq\tilde{C}(b\in\Delta_1\,;\, f^\vee(b)=a'\})=0$. This is a contradiction.

THEOREM 5. $U_N\cap O_{A^0N}\subset|\supset U_{HN}$.

PROOF. Remove a closed disk K from an $R'\in O_{HD}-O_G$. Then $R=R'-K\in U_{HN}\subset U_{HN}$ but $R\notin O_{A^0N}$, and therefore $U_{HN}\not\subset U_N\cap O_{A^0N}$. Next we shall show that there exists an $R\in U_N\cap O_{A^0N}-U_{HN}$, i.e.

$$U_N\cap O_{A^0N}\not\subset U_{HN}\,.$$

Let R be the Riemann surface constructed by K. Matsumoto in [11]. He showed that $R\in U_S\cap O_{A^0D}-U_{HD}$ $(\subset U_N\cap O_{A^0N}-U_{HD})$. Since R is almost finite genus, it follows from Proposition 2 that R does not belong to U_{HN}. Thus $R\in U_N\cap O_{A^0N}-U_{HN}$.

5. Some classification theorems

THEOREM 6. $O_{A^0N}<O_{AN}$.

PROOF. First we shall prove $O_{A^0N}\subset O_{AN}$. Let $R\in O_{A^0N}$. Suppose there is a non-constant function f in $AN(R)$. Let z_0 be a fixed point in R. Let G be a component of $\{z\in R\,;\, \mathrm{Re}\, f(z)>\mathrm{Re}\, f(z_0)\}$. Then $f-\mathrm{Re}\, f(z_0)$ is non-constant and belongs to $AN(G)$. Since $\mathrm{Re}\,(f-\mathrm{Re}\, f(z_0))=0$ on ∂G, we see that $G\notin SO_{AN}$ and $R\notin O_{A^0N}$. This is a contradiction. Thus we have $O_{A^0N}\subset O_{AN}$. Secondly we shall prove that there exists a Riemann surface belonging to $O_{AN}-O_{A^0N}$. Let $R'\in O_{HD}-O_G$ and K be a closed disk in R'. Then $R=R'-K\in O_{AD}-O_{A^0N}\subset O_{AN}-O_{A^0N}$.

COROLLARY. $O_{AD}-O_{A^0N}\neq\emptyset$.

DEFINITION 4 (cf. [3]). We denote by O^∞_{HD} (resp. O^∞_{HN}) the family of all hyperbolic Riemann surfaces R such that the closure of Δ^D_{SS} (resp. Δ^D_S) in R^*_D equals Γ_D.

By definition, we see that $O^\infty_{HD}\subset O^\infty_{HN}\subset U_{HN}$.

THEOREM 7. *If a Riemann surface R belongs to O^∞_{HN}, then any regular*

6) Cf. [16].

subregion of R belongs to $SO_{HD}\cup U_{HN}$.

PROOF. Let G be any subregion of R. Then $(\overline{G}^D-\overline{\partial G^D})\cap\Gamma_D=\emptyset$ or $\neq\emptyset$. In the first case we see that $G\in SO_{HD}$. Next suppose $(\overline{G}^D-\overline{\partial G^D})\cap\Gamma_D\neq\emptyset$. Since $\overline{G}^D-\overline{\partial G}^D$ is open in R_D^* (Satz 9. 9 in [2]), by assumption, we have that $(\overline{G}^D-\overline{\partial G}^D)\cap\Delta_S^D\neq\emptyset$. Thus $G\in U_{HN}$ by the Corollary to Proposition 1.

COROLLARY. *Let* $R\in O_{HN}^\infty$ *and G be any subregion of R. If* $G\notin SO_{HD}$, *then* $G\in O_{AN}$.

THEOREM 8. $O_{HN}^\infty<O_{A^0N}$.

PROOF. Let $R\in O_{HN}^\infty$. Let G be any regular subregion of R. Then it follows from Theorem 7 that $G\in SO_{HD}\cup U_{HN}\subset SO_{AN}$. Thus $R\in O_{A^0N}$. It is known (cf. [13]) that there exists a hyperbolic plane region R such that $R\in O_{A^0D}$ $(\subset O_{A^0N})$. Then it follows from Theorem 12, c), in [5] that $R\in O_{A^0N}-O_{HN}^\infty$.

COROLLARY. $O_{A^0N}-O_{HN}^\infty\neq\emptyset$.

6. A covering property of analytic functions

The definition of a mapping of type B1 is due to M. Heins [3]. Let ϕ be an analytic mapping of a hyperbolic Riemann surface R into another R'. For a mapping ϕ of type B1, K. Matsumoto [10] proved that ϕ is of type B1 if and only if each component of $\phi^{-1}(G')$ belongs to SO_{HB}[7] for any relatively compact regular subregion G' of R' (cf. [2]).

By a modification of the proof of Theorem 8 in [10], we shall prove.

THEOREM 9. *Let* $R\in O_{HN}^\infty$ *and f be a non-constant analytic function on R. Let U be any open disk in* $\{|w|<\infty\}$ *and V be a component of* $f^{-1}(U)$. *If* $f|V\in AN(V)$, *then* $f|V$ *is of type Bl. Hence V covers each point of U the same number of times except for at most a closed set of capacity zero and* $f|V$ *is finitely sheeted.*

PROOF. Suppose $f|V$ is not of type B1. Then there exist an open disk in U with $\bar U_0\subset U$ and a component V_0 of $(f|V)^{-1}(U_0)$ such that $V_0\subset V$ and $V_0\notin SO_{HB}$ in V. Let s be a continuous superharmonic function on U such that $s=0$ on ∂U_0, $s=1$ on U_0 and s is harmonic in $U-U_0$. On the other hand, since $V_0\notin SO_{HB}$ in V, there is a non-constant function u in $HB(V_0)$ such that $u=0$ in ∂V_0 and $0\leqq u\leqq 1$. We set $u^*(z)=u(z)$ for $z\in V_0$ and $=0$ for $z\in V-V_0$. Then u^* is subharmonic in V and $u^*\leqq s\circ f$ in V. Let $s\circ f=h+p$ be the Riesz decomposition of $s\circ f$ in V, where h is the

7) See p. 31 in [2] for the definition of SO_{HB}.

harmonic part of $s\circ f$. Since $\|s\circ f\|_U^2 \leqq \max_U |\text{grad } s|^2 \|f\|_V^2 < \infty$, it follows from the Dirichlet principle that $h \in HBD(V)$. Since $h=0$ on ∂V and u^* is non-constant, h is non-constant. Thus $V \notin SO_{HD}$. Hence it follows from the Corollary to Theorem 7 that $V \in O_{AN}$. This is a contradiction.

Department of Mathematics
Hokkaido University

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