A characterization of A_7 and M_{11}, III

Dedicated to Professor Kiiti Morita on his 60th birthday

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1. Introduction

In this paper we shall prove the following theorem.

THEOREM 1. *Let G be a doubly transitive group on the set $\Omega=\{1, 2, \cdots, n\}$. If the stabilizer $G_{1,2}$ of points 1 and 2 is isomorphic to the Janko's simple group $J(11)$ of order $2^3\cdot 3\cdot 5\cdot 7\cdot 11\cdot 19$ or a group $R(q)$ of Ree type, then G has a regular normal subgroup.*

By Walter's theorem a simple group with abelian Sylow 2-subgroups is isomorphic to $J(11)$, $R(q)(q\neq 3)$, $PSL(2, 2^m)$ or $PLS(2, q)$ with $q\equiv 3$ or 5 (mod 8). Theorefore by Theorem 1 and theorems in [7] we have the following.

THEOREM 2. *Let G be a doubly transitive group on the set $\Omega=\{1, 2, \cdots, n\}$. If $G_{1,2}$ is isomorphic to a simple group with abelian Sylow 2-subgroups, then G is isomorphic to the alternating group A_7 of degree seven, the Mathieu group M_{11} of degree eleven or G has a regular normal subgroup.*

Let X be a subset of a permutation group. Let $F(X)$ denote the set of all fixed points of X and $\alpha(X)$ be the number of points in $F(X)$. $N_G(X)$ acts on $F(X)$.

Let $\chi_1(X)$ and $\chi(X)$ be the kernel of this representation and its image, respectively. The other notation is standard.

2. Preliminaries

Let G be a doubly transitive group on Ω not containing a regular normal subgroup such that $G_{1,2}$ is isomorphic to $J(11)$ or $R(q)$. Let K be a Sylow 2-subgroup of $G_{1,2}$. Then K is an elementary abelian 2-group of order 8. Let I be an involution of G with the cycle structure $(1, 2)\cdots$. Then I normalizes $G_{1,2}$. Since $\mathrm{Aut}\,(G_{1,2})/\mathrm{Inn}\,(G_{1,2})$ is of odd order, we may assume I centralizes $G_{1,2}$. Let τ be an involution of K. Let τ fix i points of Ω, say $1, 2, \cdots, i$. Since every involution of G is conjugate to an involution in $IG_{1,2}$, it is conjugate to I or $I\tau$.

Let d be the number of elements in $G_{1,2}$ inverted by I. Set $\gamma=[G_{1,2}: C_G(\tau)\cap G_{1,2}]$. Let β be the number of involutions with the cycle structures

(1, 2) $\cdots$ which are conjugate to τ. Let $g_1^*(2)$ and $g^*(2)$ be numbers of involutions which only the point 1 and which fix no point of Ω, respectively. Then $n=i(\beta i-\beta+\gamma)/\gamma$ and $d=\beta+g_1^*(2)$ if n is odd and $d=\beta+g^*(2)/(n-1)$ if n is even.

LEMMA 1. *G has two classes of involutions.*

PROOF. See [6, Lem. 5].

LEMMA 2. *$d=\gamma+1$ and $\beta=1$ or γ.*

PROOF. By Lemma 1 G has two classes of involutions. If $I\tau$ is conjugate to τ, $\beta=\gamma$ and if I is conjugate to τ, $\beta=1$.

3. The case n is odd

LEMMA 3. *$\beta=1$ and $g_1^*(2)=\gamma$*

PROOF. If $\beta=\gamma$, then $g_1^*(2)=1$. By [2] G must have a regular normal subgroup.

LEMMA 4. *$\chi(\tau)$ contains a regular normal subgroup and $\alpha(C_{G_{1,2}}(\tau))$ is odd*

PROOF. Assume the lemma is false. If $G_{1,2}=R(3)$, then $\chi(\tau)_{1,2}=1$, Z_3 or A_4 and if $G_{1,2}=J(11)$ or $R(q)$ with $q>3$, then $\chi(\tau)_{1,2}=1$ or $PSL(2, r)$ with $r\equiv\pm3 \pmod 8$. By [1], [7] and [9] $\chi(\tau)=PGL(2, 4)$ and $i=5$, or $\chi(\tau)=A_7$ and $i=15$ or 7. If $i\neq5$ or 15, then $n=i(i-1+7.9)/7.9$ and if $i=7$, then $n=7(6+\gamma)/\gamma$, which is a contradiction.

LEMMA 5. *$\alpha(G_{1,2})$ is odd*

PROOF. Since $\alpha(<I, C_{G_{1,2}}(\tau)>)=1$ by Lemma 4, let a be the point of $F(<I, C_{G_{1,2}}(\tau)>)$. Let Δ be a $G_{1,2}$-orbit containing a. If $|\Delta|=1$, then $\alpha(G_{1,2})$ is odd since $F(G_{1,2})^I=F(G_{1,2})$. Assume $|\Delta|>1$. Since I centralizes $G_{1,2}$, Δ is contained in $F(I)$. If $G_{1,2}=J(11)$ or $R(q)$ with $q>3$, then $C_{G_{1,2}}(\tau)$ is maximal in $G_{1,2}$ and hence $G_{1,2,a}=C_{G_{1,2}}(\tau)$. There exists an element x of $N_{G_{1,2}}(K)$ of order 7 not contained in $C_{G_{1,2}}(\tau)$. Since $G_{1,2,a}xK=G_{1,2,a}x$, $|F(K)\cap\Delta|\geq7$. Thus $\alpha(<I, K>)\geq7$, which is a contradiction. Next assume $G_{1,2}=R(3)$. $C_{G_{1,2}}(\tau)$ is not maximal in $G_{1,2}$. If $G_{1,2,a}$ does not contain $N_{G_{1,2}}(K)$, then we have a contradiction as above. If $G_{1,2,a}$ containes $N_{G_{1,2}}(K)$, then $|\Delta|=9$. Let H be a Sylow 7-subgroup of $N_{G_{1,2}}(K)$. Since $\alpha(<I, H>)\geq2$, $<I, H>$ is isomorphic to a subgroup of $G_{1,2}$. On the other hand a subgroup of$G_{1,2}$ of order 14 is not abelian, which is a contradiction.

By [8] and Lemma 1 $g_1^*(2)=1$. This contradicts Lemma 3.

4. The case n is even

1. Case $G_{1,2}=J(11)$. Since Aut $J(11)\cong J(11)$ and $R(q)$ does not involve $J(11)$ ([5, Lem. 7, 6], $G_1=J(11)\, O(G_1)$ by [12]. Thus $O(G_1)$ is regular on $\Omega-\{1\}$. By [3] G contains a normal complete Frobenius subgroup G'. Then KG' is a solvable 2-transitive group on Ω. By [4] K must be cyclic, which is a contradiction.

2. Case $G_{1,2}=R(3)$ $(=P\Gamma L(2, 8))$. If $|\chi(\tau)_{1,2}|$ is odd, then G contains a regular normal subgroup by [11]. Thus $\chi(\tau)_{1,2}=A_4$ and $\chi(\tau)=A_6$ $(i=6)$ or $AG(2, 4)$ $(i=16)$. Since $\gamma=63$, $\beta=1$ or 63 by Lemma 2. If $i=6$, then $\beta=63$, $n=36$ and $|G|=36\cdot 35\cdot 9\cdot 8\cdot 21$. If $i=16$, then $\beta=63$, $n=16^2$ and $|G|=16^2\cdot 15\cdot 17\cdot 9\cdot 8\cdot 21$. Thus G_1 does not involve $J(11)$ or $R(q)$ with $q>3$. By [12] $G_1/O(G_1)=P\Gamma L(2, 8)$. By [3] G contains a regular normal subgroup and K must be cyclic by [4], which is a contradiction.

3. Case $G_{1,2}=R(q)$, $q>3$. If $\chi(\tau)_{1,2}=1$, then G contains a regular normal subgroup by [11]. Thus $\chi(\tau)_{1,2}=PSL(2, q)$. By [7] $\chi(\tau)$ contains a regular normal subgroup. Let S be a normal subgroup containing $\chi_1(\tau)=\langle\tau\rangle$ such that $S/\langle\tau\rangle$ is a regular normal subgroup of $\chi(\tau)$. Then S is an elementary abelian 2-group of order $2i$.

LEMMA 6. *If an involution of S is conjugate to τ, then it is conjugate to τ under $N_G(S)$.* $|N_G(S)|=i^2(i-1)|C_{G_{1,2}}(\tau)|$.

PROOF. Assume $\eta^g=\tau$ is in $S\cap S^g$. KS is a Sylow 2-subgroup of $C_G(\tau)$. We shall prove that S is a unique elementary abelian subgroup of KS of order $2i$. Since $\chi(\tau)$ contains a regular normal subgroup and it has two classes involutions, an involution τ' of $C_G(\tau)$ not contained in S fixes at least two points of $F(\tau)$. By the argument in [7] $i=\alpha(\langle\tau, \tau'\rangle)^2$. Thus $|C_S(\tau')|=2\sqrt{i}$ and hence $|C_{KS}(\tau')|=8\sqrt{i}$. If $8\sqrt{i}\geq|S|=2i$, then $i=4$ or 16. Since $n=i(\beta(i-1)+\gamma)/\gamma$, $\beta=\gamma$, $n=i^2$ and $g^*(2)=n-1$. Thus the set T consisting of elements of $C_G(\tau)$ which fix no point of Ω and the identity element is a group and it is transitive on $F(\tau)$. $T^g=T$ since S^g is contained in $C_G(\tau)$. $F(\tau')=F(\tau)$ and $\tau=\tau'$, which is a contradiction. Thus g is in $N_G(S)$. The other part of Lemma 6 is trivial.

LEMMA 7. *$\beta=\gamma$ and $n=i^2$.*

PROOF. By lemma 2 $\beta=1$ or γ. By Lemma 6 $n=i(\beta(i-1)+\gamma)/\gamma$ must be divisible by i^2. Thus $n=i^2$.

LEMMA 8. *Every involution of G_1 acts trivially on $O(G_1)$.*

PROOF. Assume $O(G_1)\neq 1$. Let $K'=\langle\tau, \tau'\rangle$ is a four group contained $G_{1,2}$. Since every involution of G_1 is conjugate to each other, by a theorem of Brauer-Wielandt [13] $|O(G_1)|C_{O(G_1)}(K')|^2=|C_{O(G_1)}(\tau)|^3$. Since $O(G_1)\cap G_{1,2}=1$,

$|C_{O(G_1)}(\tau)|$ is a factor of $i-1$ and $|O(G_1)|$ is a factor of $n-1=i^2-1$. Thus $|O(G_1)|$ is a factor of $i-1$ and hence $O(G_1)$ is contained in $C_G(\tau)$.

By [12] there exists a normal subgroup G_1' of odd index containing $O(G_1)$ such that $G_1'/O(G_1)$ is isomorphic to $R(r)$ and $G_1/O(G_1)$ is isomorphic to a subgroup of Aut $R(r)$.

LEMMA 9. $R(r)\neq R(q)$

PROOF. Assume $R(r)=R(q)$. $G_1'=O(G_1)\ G_{1,2}$. By Lemma 8 $G_{1,2}$ is normal in G_1' and hence in G_1, which is a contradrction.

LEMMA 10. $i+1=(r^3+1)r^2(q+1)/(r+1)q^2(q^3+1)$, $i-1=|O(G_1)|\ |G_1/G_1'|\ |r(r^2-1)/q(q^2-1)|$ *and* $\sqrt{i}-1=|O(G_1)|\ |G_1/G_1'|\ (r+1)/(q+1)$.

PROOF. Since $R(r)$ has a doubly transitive permutation representation such that the stabilizer of two points is cyclic, $[C_{\mathrm{Aut}\ R(r)}(\eta):C_{R(r)}(\eta)]=[\mathrm{Aut}\ R(r):R(r)]$ for every involution η of $R(r)$. Thus $|C_{G_1}(\tau)|=|C_{R(r)}(\bar{\tau})|\ |G_1/G_1|\ |O(G_1)|$ by Lemma 8. Since $[G_1:C_{G_1}(\tau)]=(i+1)\ |G_{1,2}:G_{1,2}(\tau)|$, we get first two equalities in the lemma. Let $K'=<\tau,\tau'>$, be a subgroup of $G_{1,2}$ of order 4. By the argument in [7] $\alpha(K)-1=\sqrt{i}-1=|C_{G_1}(K'):C_{G_{1,2}}(K')|=|G_1:G_1'|\ |O(G_1)|\ |C_{R(r)}(\bar{K}'):C_{R(q)}(\bar{K}')|=|G_1:G_1'|\ |O(G_1)|\ (r+1)/(q+1)$.

By this lemma $\sqrt{i}+1=(i-1)/(\sqrt{i}-1)=r(r-1)/q(q-1)$. Thus $i+1=|\sqrt{i}|^2+1=(r(r-1)/q(q-1)-1)^2+1\equiv 2 \pmod 3$ since $r>q$ by Lemma 9. On the other hand $i+1\equiv 0 \pmod 3$ by Lemm 10, which is a contradiction.

This completes the proof of Theorem 1.

5. Corollaries

COROLLARY 1. *Let G be a 3-transitive group on $\Omega=\{1,2,\cdots,n\}$. If the stabilizer $G_{1,2,3}$ of points 1, 2 and 3 is isomorphic to a simple group with abelian Sylow 2-subgroup or $R(3)$, then $G=A_8$ and $n=8$.*

PROOF. If G_1 contains a normal subgroup which is regular on $\Omega-\{1\}$, then G contains a normal subgroup M such that $M\leq G\leq$ Aut M and M acts on Ω as one of the following groups in its usual 2-transitive representation: a sharply transitive group, $PSL(2,q)$, $S_z(q)$, $PSU(3,q)$ or a group of Ree type. If M is sharply transitive, then M_1 is a normul subgroup of 2-transitive group G_1 and elementary abelian. Thus $|M_1|$ is prime and G_1 is solvable, which is a contradiction. If $M=PSL(2,q)$ or $S_z(q)$, then $G_{1,1,3}$ must be cyclic since Aut M/M is cyclic. If $M=PSU(3,q)$or a group of Ree type, then $G_{1,2,3}$ must have a cyclic normal subgroup $M_{1,2,3}$, which is a contradiction. Thus by Theorem 2 $G_1=A_7$ with $n=8$ or $G_1=M_{11}$ with $n=13$. If $G_1=A_7$, then $G=A_8$. By [10] there exists no group such that $n=13$ and $G_1=M_{11}$.

Similary we have the following corollary of Theorem in [6].

COROLLARY 2. *Let G be a 3-transitive group on Ω. If $G_{1,2,3}$ is complete Frobenius group such that its kernel is a 2-group, then $G=A_7$ or G contains a regular normal subgroup, $G_1=A_7$ and $n=16$.*

PROOF. Let M be as in Corollary 1. If M is sharply transitive, then G is solvable. This contradicts [4]. Thus by [6.] $G_1=A_6$ with $n=7$ or $G_1=A_7$ with $n=16$. If $G_1=A_6$, then $G=A_7$. If $G_1=A_7$, then G is isomorphic to a subgroup of $AG(4, 2)$ (see [10]).

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