The endomorphism ring of an indecomposable module with an Artinian projective cover

By Tsutomu TAKEUCHI

As is well known, each endomorphism of an Artinian, Noetherian and indecomposable module is either nilpotent or an automorphism. C. I. Vinsonhaler has proved that each endomorphism of an indecomposable module with a Noetherian injective hull is either nilpotent or an automorphism (see [6], [7] and [5 ; p. 75]). In this note we shall verify that each endomorphism of an indecomposable module with an Artinian projective cover is either nilpotent or an automorphism.

We will assume throughout that M denotes a nonzero unital left R-module, where R is a nonzero ring with identity.

THEOREM. *Let M be an indecomposable left R-module and let P be an Artinian left R-module which is a projective cover of M with a minimal epimorphism $\varphi: P \rightarrow M$. Then each endomorphism α of M is either nilpotent or an automorphism.*

REMARK 1. M is called a semiperfect module iff every factor module of M has a projective cover (cf. Mares [4]).

Let N be a submodule of M. A cocomplement N^c of N in M is a minimal submodule of M such that $N+N^c=M$. M is called a cocomplemented module iff every submodule of M has a cocomplement in M.

As is easily proven, semiperfect modules are cocomplemented. Conversely, projective cocomplemented modules are semiperfect (see Kasch-Mares [3]).

REMARK 2. In the following commutative diagram with left R-modules M_i and homomorphisms ψ_i $(i=1, 2, 3)$:

$$\begin{tikzcd} & M_1 \arrow[dl, "\psi_3"'] \arrow[d, "\psi_1"] \\ M_3 \arrow[r, "\psi_2"] & M_2, \end{tikzcd}$$

if ψ_1 is an epimorphism and if ψ_2 is a minimal epimorphism, then ψ_3 is necessarily an epimorphism.

PROOF of THEOREM. Since also M is Artinian, there exists a positive integer r such that $M\alpha^r = M\alpha^{r+1} = \cdots$. Evidently, every Artinian module is cocomplemented, so that P is semiperfect (Remark 1). Then, so is M since every factor module of a semiperfect module is semiperfect. Put $M_0 = \mathrm{Ker}\ \alpha^r$ and let P_0 be a projective cover of M/M_0 with a minimal epimorphism $\varphi_0 : P_0 \longrightarrow M/M_0$. If $\pi : M \longrightarrow M/M_0$ is the natural epimorphism, then there exists a homomorphism $\psi_0 : P \longrightarrow P_0$ such that $\psi_0\varphi_0 = \varphi\pi$, by the projectivity of P. But ψ_0 is an epimorphism (Remark 2), whence P_0 is Artinian. Since $M_0\alpha \subset M_0$, α induces the endomorphism $\bar{\alpha}$ of M/M_0 by $(a+M_0)\bar{\alpha} = a\alpha + M_0$ $(a \in M)$. Moreover, $M\alpha^r = M\alpha^{r+1}$ asserts that $\bar{\alpha}$ is an epimorphism. Then we have, by the projectivity of P_0, an endomorphism ψ of P_0 such that $\psi\varphi_0 = \varphi_0\bar{\alpha}$.

$$\begin{tikzcd} & P_0 \arrow[d, "\varphi_0"] \arrow[ddl, "\psi"] \\ & M/M_0 \arrow[d, "\bar{\alpha}"] \\ P_0 \arrow[r, "\varphi_0"] \arrow[uur, bend left, "\psi'"] & M/M_0 \end{tikzcd}$$

But ψ is an epimorphism (again by Remark 2), so that the projective P_0 has a splitting endomorphism ψ' ; $\psi'\psi$ is the identity mapping of P_0. Since P_0 is Artinian, there exists a positive integer s such that $P_0\psi'^s = P_0\psi'^{s+1}$. Since ψ' is monomorphic, we obtain $P_0 = P_0\psi'$, so that (ψ' or) ψ is an isomorphism.

Now, put $K = \mathrm{Ker}\ \varphi_0\bar{\alpha}$. Considering that $K\psi$ is included in K which is also Artinian, we know the existence of a positive integer t satisfying $K\psi^t = K\psi^{t+1}$. The monomorphism ψ deduces $K = K\psi$. Then, this yields that $\bar{\alpha}$ is a monomorphism. Next, let a be an arbitrary element of Ker α^{2r}. Then, $a\alpha^r$ is in M_0 and hence $(a+M_0)\bar{\alpha}^r = 0$. Since $\bar{\alpha}$ is monomorphic, a is contained in M_0, showing that $\mathrm{Ker}\,\alpha^r = \mathrm{Ker}\,\alpha^{2r}$. This implies, together with $M\alpha^r = M\alpha^{2r}$, that M is the direct sum of $M\alpha^r$ and $\mathrm{Ker}\,\alpha^r$. It follows from the assumption of M to be indecomposable that $M\alpha^r = 0$ or that $M\alpha^r = M$ and $\mathrm{Ker}\,\alpha^r = 0$. Consequently, α is nilpotent or α^r is, and hence, α is an automorphism, as required.

Department of Mathematics
Hokkaido University

References

[1] J. W. FISHER: Nil subrings of endomorphism rings of modules, *Proc. Amer. Math. Soc.* 34 (1972), 75–78.

[2] J. W. FISHER: Finiteness conditions for projective and injective modules, *Proc. Amer. Math. Soc.* 40 (1973), 389–394.

[3] FR. KASCH and E. A. MARES: Eine Kennzeichnung semi-perfekter Moduln, *Nagoya Math. J.* 27 (1966), 525–529.

[4] E. A. MARES: Semi-perfect modules, *Math. Zeitschr.* 82 (1963), 347–360.

[5] H. TACHIKAWA: *Quasi-Frobenius Rings and Generalizations*, Lecture Notes in Mathematics, No. 351, Springer (1973).

[6] C. I. VINSONHALER: Orders in QF-3 rings, *J. Algebra* 14 (1970), 83–90.

[7] C. I. VINSONHALER: Supplement to the paper, "Orders in QF-3 rings", *J. Algebra* 17 (1971), 149–151.

(Received June 18, 1974)