Nonmaximal weak-*Dirichlet algebras

By Takahiko NAKAZI

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0. Introduction

Let A be a weak-*Dirichlet algebra of $L^{\infty}(m)$ which was introduced by Srinivasan and Wang [7]. Let $H^{\infty}(m)$ denote the weak-*closure of A in $L^{\infty}(m)$. Suppose there exists at least one positive nonconstant function v in $L^{1}(m)$ such that the measure vdm is multiplicative on A. Then Merrill [4] characterizes the classical space $H^{\infty}(d\theta)$ by invariant subspaces of $H^{\infty}(m)$ or the maximality of $H^{\infty}(m)$ as a weak-*closed subalgebra of $L^{\infty}(m)$. In section 1 we characterize $H^{\infty}(d\theta d\phi)$, which is certain weak-* Dirichlet algebra on the torus, by invariant subspaces of $H^{\infty}(m)$. We need not assume the existence of the above v. Then, in some special case, Muhly [6] shows that $H^{\infty}(m)$ is a maximal weak-*closed subalgebra of $L^{\infty}(m)$. But in general, $H^{\infty}(m)$ is not maximal and so there exist weak-* closed subalgebras of $L^{\infty}(m)$ which contain $H^{\infty}(m)$ properly. In section 2 we construct some typical subalgebra in such subalgebras and we determine forms of all weak-*closed subalgebras which contain this subalgebra. This is applied to determine forms of all subalgebras which contain $H^{\infty}(d\theta d\theta)$.

Recall that by definition a weak-*Dirichlet algebra is an algebra A of essentially bounded measurable functions on a probability measure space $(X, \mathfrak{M}, m)$ such that (i) the constant functions lie in A; (ii) $A+\bar{A}$ is weak-* dense in $L^{\infty}(m)$ (the bar denotes conjugation, here and always); (iii) for all f and g in A, $\int fg\,dm=(\int f dm)(\int g\,dm)$. The abstract Hardy spaces $H^{p}(m)$, $1\leq p\leq\infty$, associated with A are defined as follows. For $1\leq p<\infty$, $H^{p}(m)$ is the $L^{p}(m)$-closure of A, while $H^{\infty}(m)$ is defined to the weak-*closure of A in $L^{\infty}(m)$. For $1\leq p\leq\infty$, $H_{0}^{p}=\{f\in H^{p}(m): \int f dm=0\}$. For any subset $M\subseteq L^{\infty}(m)$, denote by $[M]_{2}$ the $L^{2}(m)$-closure of M. A closed subspace M of $L^{p}(m)$ is called B-invariant if $f\in M$ and $g\in B$ imply that $fg\in M$, where B is a subalgebra of $L^{\infty}(m)$. In particular, if $B=L^{\infty}(m)$, M is called doubly-invariant. For any measurable subset E of X, the function χ_{E} is the characteristic function of E. If $f\in L^{p}(m)$, write E_{f} for the support set of f and write χ_{f} for the characteristic function of E_{f}.

We use the following result.

(a) *If M is a weak-*closed A-invariant subspace of $L^{\infty}(m)$, then $M=$*

$[M]_2 \cap L^\infty(m)$.

For weak-*Dirichlet algebras this has never published, but the proof is easy if we use the logmodularity of $H^\infty(m)$.

1. Characterization of $H^\infty(d\theta d\phi)$

Let A be the algebra of continuous, complex-valued functions on the torus $T^2=\{(z, w): |z|=|w|=1\}$ which are uniform limits of polynomials in $z^n w^m$ where

$$(n, m)\in \Gamma=\{(n, m): m>0\} \cup \{(n, 0): n\geq 0\}.$$

Denoting the normalized Haar measure on T^2 by $d\theta d\phi$, then A is a weak-* Dirichlet algebra of $L^\infty(d\theta d\phi)$. Recall $H^\infty(d\theta d\phi)$ is the weak-*closure of A in $L^\infty(d\theta d\phi)$.

In general, let A be a weak-*Dirichlet algebra of $L^\infty(m)$. Suppose there exists at least one positive nonconstant function v in $L^1(m)$ such that for all f and g in A, $\int fgv\,dm=(\int fv\,dm)(\int gv\,dm)$. Then by the logmodularity of $H^\infty(m)$, $H_0^\infty=ZH^\infty(m)$ for some inner function Z in $H^\infty(m)$, where a function $f\in H^\infty(m)$ is called inner if $|f|=1$ a.e..

In [4] Merrill obtains the following result for the characterization of the classical space $H^\infty(d\theta)$.

(b) *The following properties for $H^\infty(m)$ are equivalent.*

(1) *$H^\infty(m)$ is isomorphic to the classical space $H^\infty(d\theta)$.*

(2) *Every nonzero weak-*closed A-invariant subspace of $H^\infty(m)$ has the form*

$$M=FH^\infty(m)$$

where F is an inner function in M.

(3) *$H^\infty(m)$ is a maximal weak-*closed subalgebra of $L^\infty(m)$.*

In this section we characterize $H^\infty(d\theta d\phi)$ which is not a maximal weak-*Dirichlet algebra [4]. Let J^∞ be the weak-*closure of $\bigcup_{n=0}^{\infty}\bar{Z}^n H^\infty(m)$ and let I^∞ be $\bigcap_{n=0}^{\infty} Z^n H_0^\infty$.

THEOREM 1. (1) *J^∞ is the minimum weak-*closed subalgebra of $L^\infty(m)$ which contains $H^\infty(m)$ properly.* (2) *I^∞ is the maximal weak-*closed ideal of J^∞ in $H^\infty(m)$.*

PROOF. First, we shall show that if B is a weak-*closed subalgebra of $L^\infty(m)$ such that $B\supsetneq H^\infty(m)$, then $B\supseteq J^\infty$. If m is multiplicative on B, then $\bar{B}$ is orthogonal to H_0^∞ and hence $B\subseteq H^2(m)$ [7, p 226] and hence $B\subseteq H^2(m)\cap L^\infty(m)=H^\infty(m)$ by (a) in Introduction. This contradicts to

$B \supsetneq H^\infty(m)$. If m is not multiplicative on B, the function Z has the inverse in B. For, if not, there exists a complex homomorphism ϕ on B such that $\phi(Z)=0$. Then ker $\phi \supseteq H_0^\infty = ZH^\infty(m)$. If ϕ is restricted to $H^\infty(m)$, then ker $\phi = H_0^\infty$, so by the logmodularity, the unique representing measure of ϕ is m. This contradicts that m is not multiplicative on B. Thus B is the weak-*closed subalgebra of $L^\infty(m)$ that contains $\bar{Z}$ and $H^\infty(m)$, so $B \supseteq J^\infty$. This proves (1).

Now if K is the weak-*closed ideal of J^∞ such that $I^\infty \subseteq K \subseteq H^\infty(m)$, since both Z and $\bar{Z}$ is in J^∞, the subalgebra $K = ZK$. Thus $K \subseteq \bigcap_{n=1}^{\infty} Z^n H^\infty(m) = I^\infty$. It is known [5] that I^∞ is the ideal of J^∞. This proves (2).

Denote by $\mathscr{H}^p$ $(1 \leq p \leq \infty)$ the closure in $L^p(m)$ (weak-*closure for $p=\infty$) of polynomials in Z. Denote by $\mathscr{L}^p$ $(1 \leq p \leq \infty)$ the closure in $L^p(m)$ (weak-* closure for $p=\infty$) of polynomials in Z and $\bar{Z}$. Let I^p be the closure of I^∞ in $L^p(m)$ and let $\mathscr{I}^p$ be the closure of $I^p + \bar{I}^p$ in $L^p(m)$ and let J^p be the closure of J^∞ in $L^p(m)$. The following result is known [4, Lemma 5].

(c) *If* $1 \leq p \leq \infty$, *then*

$$H^p(m) = \mathscr{H}^p + I^p, \quad L^p(m) = \mathscr{L}^p + \mathscr{I}^p$$
$$J^p = \mathscr{L}^p + I^p$$

where + denotes algebraic direct sum and if $p=2$, *each decomposition is orthogonal.*

If $1 < p < \infty$, we can show easily that $L^p(m) = J^p + \bar{I}^p$.

The following result is known, too [3].

(d) *For* $1 \leq p \leq \infty$, *there exists an isometric-*isomorphism (i.e., taking complex conjugates into complex conjugates) between* $L^p(d\theta)$ *of the disc and* $\mathscr{L}^p$ *in* $L^p(m)$, *whcih maps the classical space* $H^p(d\theta)$ *onto* $\mathscr{H}^p$ *in* $H^p(m)$.

We can prove the following results (e) and (f). The proofs are almost parallel to those of (c) and (d). Suppose there exists a nontrivial inner function W in I^∞. Denote by $\boldsymbol{H}^p$ $(1 \leq p \leq \infty)$ the closure in $L^p(m)$ (weak-* closure for $p=\infty$) of polynomials in $Z^n W^m$ where $(n, m) \in \Gamma$. Then $\boldsymbol{H}^p$ is a subspace (subalgebra for $p=\infty$) of $H^p(m)$ by $ZI^p = I^p$ which (2) in theorem 1 shows. Denote by $\boldsymbol{L}^p$ $(1 < p < \infty)$ the closure in $L^p(dm)$ (weak-* closure for $p=\infty$) of polynomials in Z, $\bar{Z}$, W and $\bar{W}$. Let

$$S^p = \left\{ f \in H^p(m) : \int Z^n \bar{W}^m f dm = 0, (n, m) \in \Gamma \right\}.$$

Denote by $\mathscr{S}^p$ the closure of $S^p + S^p$ in $L^p(m)$ (weak-*closure for $p=\infty$).

(e) *For* $1 \leq p \leq \infty$, *there exists an isometirc*-isomorphism between* $L^p(d\theta d\phi)$ *of the torus and* $\boldsymbol{L}^p$, *which map* $H^p(d\theta d\phi)$ *onto* $\boldsymbol{H}^p$.

(f) *If* $1 \leq p \leq \infty$, *then*

$$H^p(dm) = \boldsymbol{H}^p + S^p, \qquad L^p(dm) = \boldsymbol{L}^p + \mathcal{S}^p$$

where + denotes algebraic direct sum and if $p=2$, *each decomposition is orthogonal.*

LEMMA 1. *Suppose* $I^\infty = WJ^\infty$ *for some inner function* W *in* I^∞. *Then* S^∞ *is a weak-*closed* J^∞*-invariant subspace of* $H^\infty(m)$ *such that* $S^\infty = WS^\infty$.

PROOF. By the above remark (c) and $I^\infty = WJ^\infty$, $I^2 \ominus WI^2 = W\mathcal{L}^2$, where $\ominus$ is orthogonal complement. Denote $S = I^2 \ominus \sum_{j=1}^{\infty} W^j \mathcal{L}^2$, then $S = \bigcap_{j=1}^{\infty} W^j I^2$ and $I^2 = S + \sum_{j=1}^{\infty} W^j \mathcal{L}^2$. The proof of $WS=S$ is the same as [1, p 109] and S is a J^∞-invariant subspace of $H^2(m)$ by that $S = \bigcap_{j=1}^{\infty} W^j I^2$ and I^2 is a J^∞-invariant subspace by (2) of theorem 1. The proof of $S = S^2$ is trivial. By the definition, $S^\infty = S^2 \cap L^\infty(m)$ and hence $S^\infty = WS^\infty$ and S^∞ is a J^∞-invariant subspace.

THEOREM 2. *The following properties for* $H^\infty(m)$ *are equivalent.*

(1) $H^\infty(m)$ *is isomorphic to* $H^\infty(d\theta\, d\phi)$.

(2) (a) J^∞ *has no doubly invariant subspace,* (b) *every nonzero weak-* closed* J^∞*-invariant subspace* M *of* $H^\infty(m)$ *has the form*

$$M = \chi_E F J^\infty$$

where χ_E *is a characteristic function in* J^∞ *and* F *is a unimodular function.*

PROOF. (1)$\Rightarrow$(2). If M is a J^∞-invariant subspace, then $\bar{Z}M \subset M$ and so M is a sesqui-invariant subspace [5]. So by [5, p 473], $M = \chi_E F J^\infty$ where χ_E is a characteristic function in $\mathcal{L}^\infty$ and F is a unimodular function. But we can show easily that for any characteristic function χ_E, $\chi_E \in J^\infty$ if and only if $\chi_E \in \mathcal{L}^\infty$. This proves (2). (2)$\Rightarrow$(1). By the hypothesis of (b) in (2), we can write $I^\infty = \chi_E W J^\infty$, where $\chi_E \in J^\infty$ and W is a unimodular function. If $m(E) < 1$, by the remark (c), J^∞ must have some doubly invariant subspace. So we can write $I^\infty = WJ^\infty$ with W in I^∞. For this inner function W, S^∞ is a weak-*closed J^∞-invariant subspace of $H^\infty(m)$ and $S^\infty = WS^\infty$ by Lemma 1. If $S^\infty \neq \{0\}$, by the hypothesis of (b) in (2), we can write $S^\infty = \chi_E F J^\infty$ where $\chi_E \in J^\infty$ and F is a unimodular function. By that $\chi_E F \in \chi_E F J^\infty$ and $\overline{W} S^\infty = S^\infty$, the function $\chi_E F \overline{W}$ is in $\chi_E F J^\infty$ and hence there exists some function g in J^∞ such that $\chi_E = \chi_E W g$. From $W \in I^\infty$, it follows that $\chi_E W g \in I^\infty$ and hence $\chi_E \in I^\infty$. This shows that $\chi_E = 0$ by $\chi_E \in J^\infty$ and hence $S^\infty = \{0\}$. By the remark (f), $H^\infty(m) = \boldsymbol{H}^\infty$ and by the remark (e), this proves (1).

$H^\infty(d\theta d\phi)$ is not maximal as a weak-*closed subalgebra of $L^\infty(d\theta d\phi)$. So it is impossible to characterize $H^\infty(d\theta d\phi)$ by the maximality. One question that arises is: is it possible to characterize $H^\infty(d\theta d\phi)$ by subalgebras of $L^\infty(m)$ which contain it? In the next section we shall answer this.

2. Subalgebras which contain $H^\infty(m)$

Let A be a weak-*Dirichlet algebra of $L^\infty(m)$. We need not always the assumption such that there exists a positive nonconstant function v in $L^1(m)$ such that the measure $v\,dm$ is multiplicative on A.

Muhly [6] show that $H^\infty(m)$ is a maximal weak-*closed subalgebra of $L^\infty(m)$ if and only if no nonzero function in $H^\infty(m)$ can vanish on a set of positive measure. If V is a weak-*closed subalgebra which is generated by $H^\infty(m)$ and χ_f for all $f\in H^\infty(m)$, then the subalgebra V contains $H^\infty(m)$ and $\chi_f\in V$ for every $f\in H^\infty(m)$. We determine forms of all subalgebras which contain V.

THEOREM 3. *Let V be a weak-*closed subalgebra of $L^\infty(m)$ which contains $H^\infty(m)$. The following are equivalent.*

(1) *$\chi_f\in V$ for every $f\in V$.*

(2) *$\chi_f\in V$ for every $f\in H^\infty(m)$.*

(3) *Each weak-*closed subalgebra B of $L^\infty(m)$ that contains V has the form*

$$B=\chi_E V+\chi_E^c L^\infty(m)$$

for some $\chi_E\in V$.

PROOF. (1)$\Rightarrow$(2) trivial.

(2)$\Rightarrow$(3). Let K be an orthogonal complement of B in $L^2(m)$. We may assume $K\neq\{0\}$. Let E be the support set of K, then $\chi_E\in V$. For since B contains $H^\infty(m)$, the set $K\subseteq\bar{H}_0^2$ [7, p 226]. For each f in $H^2(m)$, there exists a function g in $H^\infty(m)$ such that $\chi_f=\chi_g$ [6]. So if f in K, then $\chi_f=\chi_{\bar f}\in V$ by the hypothesis of (2). If f and g in K, let $F=E_f\setminus E_g$, then $\chi_F\in V$. Since $\chi_F B\subseteq B$, we can show $\chi_F K\subseteq K$ and hence $h=g+\chi_F f$ is in K. So if $f, g\in K$, there exists $h\in K$ with $E_h=E_f\cup E_g$. This shows that there exists a function f in K such that $E_f=E$ and hence $\chi_E\in V$. Since $\chi_E^c K=\{0\}$ and $\chi_E\in V$, we can get $B\supseteq\chi_E V+\chi_E^c L^\infty(m)$ and $\chi_E V+\chi_E^c L^\infty(m)$ is a weak-*closed subalgebra.

We shall show $B=\chi_E V+\chi_E^c L^\infty(m)$. Suppose $B\neq\chi_E V+\chi_E^c L^\infty(m)$. Just as Muhly [6], there exists a nonconstant unimodular function q and $\bar q$ in B such that $\bar q\notin\chi_E V+\chi_E^c L^\infty(m)$. Then $\chi_E\bar q\notin\chi_E V$. Let N be the weak-*

closure of polynomials of q, $\bar{q}$ and all characteristic functions in V. Then N is a commutative von Neumann algebra as an algebra of operators on $L^2(m)$. By $\chi_E\bar{q}\notin V$, V can not contain the whole $\chi_E N$. There exists χ_{E_0} in N such that $\chi_{E_0\cap E}\neq 0$ and for any nonzero χ_F in V

$$\chi_{E_0\cap E}\chi_F\neq\chi_F.$$

For suppose there exists a nonzero χ_F in V such that $\chi_{H\cap E}\chi_F=\chi_F$ for any χ_H in N such that $\chi_{H\cap E}\neq 0$. Then $H\cap E\supset F$ for the nonzero χ_F in V. If $H\cap E\neq F$, since $\chi_H\chi_F^c\in N$ and $\chi_H\chi_F^c\neq 0$, there exists a nonzero $\chi_{F'}$ in V such that $H\cap F^c\cap E\supset F'$ arguing as above. This leads to that $\chi_{H\cap E}\in V$ for any χ_H in N, i.e. $\chi_E N\subseteq V$ by that N is a commutative von Neuman algebra. This contradiction shows that there exists such a χ_{E_0} in N. By $\chi_{E_0\cap E}\in B$, it follows that $\chi_{E_0\cap E}K\subseteq K$. If $\chi_{E_0\cap E}K\neq\{0\}$, we can show that there exists some nonzero χ_{F_0} in V such that $\chi_{E_0\cap E}\chi_{F_0}=\chi_{F_0}$. $\chi_{E_0\cap E}K=\{0\}$. Since $m(E_0\cap E)>0$, this contradicts that E is the support set of K. Thus $B=\chi_E V+\chi_E^c L^\infty(m)$.

(3)$\Rightarrow$(1). Suppose f in any function in V. We can assume that $0<\chi_f<1$. Let $D=D(f)$ be the weak-*closure of $\{fg: g\in V\}$, then $D\subseteq V$ and the support set of D coincides with the support set of f. Let $B=\{v\in L^\infty(m): vD\subseteq D\}$. Then $V\subseteq B$. From the hypothesis of (3), we can write $B=\chi_E V+\chi_E^c L^\infty(m)$ for some $\chi_E\in V$. Then we can choose χ_E in V such that $\chi_E V$ has no doubly invariant subspace. If $m(E)=0$, then $B=L^\infty(m)$ which means that D is doubly-invariant and hence $\chi_f L^\infty(m)\subseteq V$. So $\chi_f\in V$. Suppose $m(E)>0$. Since $(1-\chi_f)\ L^\infty(m)\subseteq B$ and $\chi_E^c L^\infty(m)$ is the maximum doubly-invariant subspace of B, we have $E_f\supseteq E$. If $E_f\neq E$, define $g=\chi_E^c f$, then the function g is in V and $g\neq 0$. Arguing as above, there exist a nonzero χ_F in V such that $E_f\cap E^c\supseteq F$. This shows that $\chi_f\in V$.

COROLLARY 1. (*Muhly* [6]) *The following properties for* $H^\infty(m)$ *are equivalent.*

(1) *no nonzero function in* $H^\infty(m)$ *can vanish on a set of positive measure.*

(2) $H^\infty(m)$ *is a maximal weak-*closed subalgebra of* $L^\infty(m)$.

PROOF. (1)$\Rightarrow$(2). If f is any function in $H^\infty(m)$, then $\chi_f\equiv 0$ or $\chi_f\equiv 1$ and hence $\chi_f\in H^\infty(m)$. Apply theorem 3 with $V=H^\infty(m)$. (2)$\Rightarrow$(1). The condition (3) in theorem 3 is satisfied with $V=H^\infty(m)$ because of the maximality of $H^\infty(m)$. Therefore $\chi_f\in H^\infty(m)$ for every $f\in H^\infty(m)$. But the only real valued functions in $H^\infty(m)$ are constants, hence $\chi_f\equiv 0$ or $\chi_f\equiv 1$.

If there exists a positive nonconstant function v in $L^1(m)$ such that the measure vdm is multiplicative on A, we can choose J^∞ as V. Here J^∞ is the minimum weak-*closed subalgebra of $L^\infty(m)$ which contains $H^\infty(m)$ properly.

THEOREM 4. *The following properties for $H^\infty(m)$ are equivalent.*

(1) *$\chi_f \in J^\infty$ for every $f \in H^\infty(m)$.*

(2) *Each weak-*closed subalgebra B of $L^\infty(m)$ that contains $H^\infty(m)$ properly has the form*

$$B=\chi_E J^\infty+\chi_E^c L^\infty(m)$$

for some $\chi_E \in J^\infty$.

PROOF. If $B \supsetneqq H^\infty(m)$, then $B \supseteq J^\infty$ since J^∞ is the minimum weak-* closed subalgebra. So by theorem 3, we can get this theorem.

In section 3 we shall show that $\chi_f \in J^\infty$ for every f in $H^\infty(d\theta d\phi)$, i.e. this algebra satisfies the condition (1) of theorem 4. Moreover we shall give an example (2) such that $H^\infty(m)$ satisfies the condition (1) of theorem 4 and it is not isomorphic to $H^\infty(d\theta d\phi)$. Now we can give the negative answer to the question raised at the end of section 1. For $H^\infty(m)$ in example (2) and $H^\infty(d\theta d\phi)$ have same subalgebras in the form which contain them by theorem 4.

COROLLARY 2. *Suppose $J^\infty \neq L^\infty(m)$ and $\chi_f \in J^\infty$ for every f in $H^\infty(m)$. Then there is no algebra which contains $H^\infty(m)$ and is maximal among the proper weak-*closed subalgebras of $L^\infty(m)$.*

PROOF. Suppose B is a maximal weak-*closed subalgebra of $L^\infty(m)$ such that $H^\infty(m) \subsetneqq B \subsetneqq L^\infty(m)$. By theorem 4 we can write $B=\chi_E J^\infty+\chi_E^c L^\infty(m)$ for some E such that $m(E)>0$ and $\chi_E \in J^\infty$. Then we can choose χ_E such that $\chi_E J^\infty$ has no doubly invariant subspace. But $\chi_E \in J^\infty$ if and only if $\chi_E \in \mathscr{L}^\infty$. By the remark (d) in section 1 $\mathscr{L}^\infty$ is isomorphic to $L^\infty(d\theta)$ of the disc. If F in $L^\infty(d\theta)$ corresponds to $f \in \mathscr{L}^\infty$, then $f(x)=F(Z(x))$ a.e. [5, Lemma 4]. Hence there is a measurable set E' such that $E' \subset E$ and $m(E) \neq m(E')>0$ and $\chi_{E'} \in \mathscr{L}^\infty$. If $B'=\chi_{E'} J^\infty+\chi_{E'^c} L^\infty(m)$, then $L^2(m) \supsetneqq [B']_2 \supsetneqq [B]_2$ by that $\chi_{E'} J^\infty$ has no doubly invariant subspace and hence $B \subsetneqq B' \subsetneqq L^\infty(m)$.

COROLLARY 3. *Suppose $\chi_f \in J^\infty$ for every f in $H^\infty(m)$. If B is a weak-* closed subalgebra of $L^\infty(m)$ which contains $H^\infty(m)$ and a function v such that $\chi_E v \in J^\infty$ for any nonzero $\chi_E \in J^\infty$, then $B=L^\infty(m)$.*

PROOF. By theorem 4, we can write $B=\chi_E J^\infty+\chi_E^c L^\infty(m)$ for some $\chi_E \in J^\infty$. Since B contains v, $\chi_E v \in \chi_E J^\infty \subseteq J^\infty$. If $m(E)>0$, then $\chi_E v \notin J^\infty$ by assumption. This implies $m(E)=0$, hence $B=L^\infty(m)$.

3. Example

(1) Let A be the weak-*Dirichlet algebra on the torus which was raised at the first of section 1. Then there exist positive nonconstant functions in $L^1(d\theta d\phi)$ which are multiplicative on A. $H_0^\infty(d\theta d\phi)=zH^\infty(d\theta d\phi)$ and J^∞ is the weak-*closure of $\bigcup_{n=0}^{\infty}\bar{z}^n H^\infty(d\theta d\phi)$. Then $\chi_f\in J^\infty$ for every $f=f(z,w)\in H^\infty(d\theta d\phi)$. In fact, there exist polynomials $p_n(w)$ such that for almost all points z_0 in T, as $n\to\infty$

$$\int_T \left|f(z_0,w)-p_n(w)\right|^2 d\phi \longrightarrow 0.$$

Then it follows that $f(z_0,w)=0$ a.e. $d\phi$ or $|f(z_0,w)|>0$ a.e. $d\phi$. Let $E_1=\{z_0\in T: |f(z_0,w)|>0 \text{ a.e. } d\phi\}$. Then the set $E_1\times T$ is a support set of f. For every (n,m) with $m>0$

$$\iint_{T^2}\chi_{E_1\times T}z^n w^m\, d\theta\, d\phi$$
$$=\int_{E_1}d\theta\int_T z^n w^m\, d\phi=0.$$

Hence $\chi_f=\chi_{E_1\times T}\in J^\infty$ by that $L^2(d\theta d\phi)=\bar{J}^2+I^2$ and the remark (a) in Introduction. Thus by theorem 4, each weak-*closed subalgebra B of $L^\infty(d\theta d\phi)$ that contains $H^\infty(d\theta d\phi)$ properly has the form $B=\chi_{E_1\times T}J^\infty+\chi_{F_1\times T}L^\infty(d\theta d\phi)$ where E_1 is some measurable set of T and $F_1=T\backslash E_1$. It is known [2] that there exists a maximal uniform closed subalgebra of $C(T^2)$ the set of all complex-valued continuous functions on T^2, which contains A. But by corollary 2, there is no algebra which contains $H^\infty(d\theta d\phi)$ and is maximal among the proper weak-*closed subalgebras of $L^\infty(d\theta d\phi)$. Moreover as v in corollary 3, we can take $u\bar{w}^r$ (r is a positive real number and $u\in\mathscr{L}^\infty$ and $|u|>0$), $\chi_E(E=T\times E_2, d\phi(E_2)<1)$, etc.

(2) Let K be the Bohr compactification of the real line. Let A be the algebra of continuous, complex-valued functions on $T\times K$ which are uniform limits of polynomials in $z^n\chi_\tau$ where

$$(n,\tau)\in\Gamma=\{(n,\tau):\tau>0\}\cup\{(n,0):n\geq 0\}$$

and denote by χ_τ the characters on K, where τ in the real line. Denote by m the normalized Haar measure on $T\times K$, then A is the weak-*Dirichlet algebra of $L^\infty(m)$ [5]. There exist positive nonconstant functions in $L^1(m)$ which are multiplicative on A. $H_0^\infty=zH^\infty(m)$ and J^∞ is the weak-*closure of $\bigcup_{n=0}^{\infty}\bar{z}^n H^\infty(m)$. We can show that $\chi_f\in J^\infty$ for every $f\in H^\infty(m)$ as in (1).

(3) Let A be the algebra of continuous, complex-valued functions on $K\times K$ which are uniform limits of polynomials is $\chi_{\tau_1}\chi_{\tau_2}$ where

$$(\tau_1, \tau_2)\in\Gamma=\{(\tau_1, \tau_2): \tau_2>0\}\cup\{(\tau_1, 0): \tau_1\geq 0\}$$

and denote by χ_{τ_i} the characters on K, where τ_i in the real line. Denote by m the normalized Haar measure on $K\times K$, then A is the weak-* Dirichlet algebra of $L^\infty(m)$. Then there exist no positive nonconstant functions in $L^1(m)$ which are multiplicative on A. Let V be the weak-* closure of $\bigcup_{\tau_1\geq 0}\bar{\chi}_{\tau_1}H^\infty(m)$, then $H^\infty(m)\subsetneqq V\subsetneqq L^\infty(m)$ and V is a weak-*closed subalgebra. We can show that $\chi_f\in V$ for every $f\in H^\infty(m)$ as in (1). By theorem 3, we can know the form of weak-*closed subalgebras of $L^\infty(m)$ which contains V properly.

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T. Nakazi
Research Institute of Applied Electricity
Hokkaido University
Sapporo, Japan