Rational approximation with $C(\partial K)=R(\partial K)$

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1. Introduction. Recently many suthors have studied rational approximations by various methods [3, 7, 8, etc.].

Let K be a compact subset of the complex plane $\boldsymbol{C}$, let U be the interior of K, $\mathring{K}=U$, and let ∂K be the topological boundary of K. Let $C(K)$ be the algebra of all complex-valued continuous functions on K, let $A(K)$ be the algebra of all continuous functions on K, analytic in U, and let $R(K)$ be the uniform closure of rational functions with poles off K. By $H^\infty(U)$ we denote the algebra of all bounded analytic functions on U, and by H a subalgebra of $H^\infty(U)$ which is pointwise boundedly closed on U. In this paper, we will consider rational approximations under the condition: $C(\partial K)=R(\partial K)$. First, we will give a sufficent condition under which $H\cap C(K)=R(K)$. This result is an extension of Theorem 4 in [9] in some sense. Next, by proving that the set of non-peak points for $R(K)$ is included in $\bar{U}$, the closure of U, we will obtain two conditions each of which is equivalent to the coincidence of $R(K)$ and $A(K)$. Our main tool is A. M. Davie's theorem [2] on pointwise bounded closure.

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2. Notations. All norms will be supremum norms. By measures we mean finite regular complex Borel measures and all measures considered will be supported on compact plane sets. By an annihilating measures for $R(K)$ we mean a measure τ on K satisfying

$$\int_K f\,d\tau=0 \quad \text{for all} \quad f\in R(K).$$

We denote the set of annihilating measures by $R(K)^\perp$. If $w\in K$ we define a positive or complex representing measure of w for $R(K)$ to be a measure ν on K satisfying

$$f(w)=\int_K f\,d\nu \quad \text{for all} \quad f\in R(K).$$

Let σ be a positive measure on K. By $H^\infty(\sigma)$ we denote the weak-star closure of $R(K)$ in $L^\infty(\sigma)$. A point w in K is called a peak point if there exists a function f in $R(K)$ such that $f(w)=1$, while $|f(w')|<1$ for $w'\in K$, $w'\neq w$. Denote by Q the set of all non-peak points for $R(K)$. Let λ_Q and λ_U be the area measures $\lambda=dxdy$ restricted to Q and U respectively. Let μ be a measure on K, and set

$$\hat{\mu}(z)=\int_K \frac{1}{\xi-z}\,d\mu(\xi).$$

Then since $\tilde{\mu}(z)=\int_K|\xi-z|^{-1}\,d|\mu|(\zeta)<\infty$ almost everywhere (λ), $\hat{\mu}$ is defined a.e. (λ). It is well known that $\hat{\mu}=0$ a.e. (λ) off Q if and only if $\mu\in R(K)^{\perp}$ (cf. [1. Corollary 3. 3. 2.]). Let $A(U)$ and $R(U)$ be the restrictions of $A(K)$ and $R(K)$ to U respectively. If $h\in C(\bar{U})$, the distance from h to $R(U)$ (over U) is defined by

$$d\big(h, R(U)\big)=\inf\left\{\|h-f\|_U : f\in R(U)\right\}$$

and the distances $d(h, A(U))$ and $d(h, H^{\infty}(U))$ are defined similarly. Then it is clear that $d(h, H^{\infty}(U))\leqq d(h, A(U))\leqq d(h, R(U))$ for all $h\in C(\bar{U})$. Let H be a subalgebra of $H^{\infty}(U)$. We say that H is pointwise boundedly closed if every pointwise limit on U of a bounded sequeuce in H also belongs to H. We notice that $R(U)$ is provided with the norm $\|\cdot\|_U$ (not with the norm $\|\cdot\|_K$). We say that $R(U)$ is pointwise boundedly dense in H if for each $f\in H$ there exists a sequence $\{f_n\}$ in $R(U)$ such that $\sup\|f_n\|_U<\infty$ and $f_n(z)\to f(z)$ pointwise on U. Two points w_1 and w_2 in K (as functionals on $R(K)$) are said to be in the same part if

$$\|w_1-w_2\|=\sup\left\{\big|f(w_1)-f(w_2)\big| : \|f\|\leqq 1,\ f\in R(K)\right\}<2$$

(cf. [5], p. 143). If w_1 and w_2 in K are in the same part for $R(K)$, we put $w_1\sim w_2$, otherwise $w_1\not\sim w_2$, For a point w in K, we define the part through w by Q_w, that is,

$$Q_w=\left\{w'\in K ;\ \|w-w'\|<2\right\}.$$

3. Some lemmas. In this section, we will show some lemmas necessary to prove the main results.

The following lemma is due to Wilken [10].

LEMMA 1. *If m is a positive representing measure of a point w in K for $R(K)$ then the measure $\mu=(z-w)\cdot m$ is supported on $\bar{Q}_w$ and $\mu\in R(\bar{Q}_w)^{\perp}$.*

PROOF. It is known [1 ; Theorem 3. 2. 1.] that μ is supported on $\bar{Q}_w$ and $\mu\in R(\bar{Q}_w)^{\perp}$ if and only if the set

$$\left\{y ;\ \tilde{\mu}(y)<\infty \ \text{ and } \ \hat{\mu}(y)\neq 0\right\}$$

is contained in $\bar{Q}_w$. Since m represents the point w for $R(K)$, the function $\hat{\mu}$ vanishes outsides K. Now take any point $y\in K$ such that $\tilde{\mu}(y)<\infty$ and $\hat{\mu}(y)\neq 0$. Since the measure $\nu=\dfrac{\mu}{\hat{\mu}(y)(z-y)}$ represents the point y for $R(K)$ it is known [1 ; Theorem 2. 2. 1] that there is a positive representing measure m_y of y for $R(K)$ such that $m_y \ll |\nu|$, that is, m_y is absolutely continuous with respect to ν. This implies $m_y \ll m$. Hence $y\sim w$, that is, $y\in Q_w$ by a well known theorem [1, p. 131]. This completes the proof.

The following lemma is used by Gamelin [8 ; heorem 4. 1]. We give a proof for completeness.

LEMMA 2. *Under the condition* $C(\partial K)=R(\partial K)$, *for each nonpeak point* w *of* $R(K)$ *there is a point* w_0 *in* U *with* $w\sim w_0$.

PROOF. Suppose that $w\not\sim z$ for any point $z\in U$. Then the part Q_w through w is contained in ∂K and so is its closure $\bar{Q}_w$. Since w is not a peak point, by a well-known theorem (cf. [5, p. 56]) there is a positive representing measure m of w with $m(\{w\})=0$. By Lemma 1 the measure $\mu=(z-w)\cdot m$ is supported on $\bar{Q}_w$ and $\mu\in R(\bar{Q}_w)^{\perp}$. On the other hand, since $C(\partial K)=R(\partial K)$ by hypothsis and $\bar{Q}_w\subseteqq\partial K$, it follows that $C(\bar{Q}_w)=R(\bar{Q}_w)$. Therefore $\mu\in C(Q_w)^{\perp}$ and consequently $m=0$ because of $m(\{w\})=0$. This contradicts $m(K)=1$.

This lemma means that each point w in K, for which w w' all $w'\in U$, is a peak point for $R(K)$.

LEMMA 3. *If* $C(\partial K)=R(\partial K)$ *then* Q *is contained in* $\bar{U}$.

PROOF. Suppose that $Q\backslash\bar{U}$ contains a point w. Since w is a non-peak point, by a well-known theorem (cf. [1, p. 87]) there is a representing measure m with $m(\{w\})=0$. Consider the measure $\mu=m-\delta_w$ where δ_w is the Dirac measure concentrated at w. Then obviously $\mu\in R(K)^{\perp}$ and $\mu(\{w\})\neq 0$. Choose an open disc Δ with center w which is disjoint from $\bar{U}$, and a smooth function h which is identically equal to 1 on some neighborhood of w and vanishes outsides Δ. It is known (cf. [1 ; Lemma 3. 1. 8]) that the measure ν defined by

$$\nu=h\mu-\frac{1}{\pi}\frac{\partial h}{\partial\bar{z}}\cdot\hat{\mu}\lambda \qquad (\lambda \text{ is the Lebesgne measure})$$

has the prorerty: $\hat{\nu}=h\hat{\mu}$. This implies, just as in the proof of Lemma 1, that ν is supported on $\bar{\Delta}\cap K$ and $\nu\in R(\bar{\Delta}\cap K)^{\perp}$. Since $R(\partial K)=C(\partial K)$ by hypothesis and $\bar{\Delta}\cap K\subseteqq\partial K$, it follows that $C(\bar{\Delta}\cap K)=R(\bar{\Delta}\cap K)$, and $\nu\in C(\bar{\Delta}\cap K)^{\perp}$, that is, $\nu=0$. This is a contradiction because $\nu(\{w\})=\mu(\{w\})\neq 0$.

This lemma means that any point w in $\partial K-\bar{U}$ is a peak point for $R(K)$.

LEMMA 4. *Under the condition $C(\partial K)=R(\partial K)$, each function in $H^\infty(\lambda_Q)$, which vanishes on U, vanishes on K a.e. (λ_Q).*

PROOF. Suppose that $f\in H^\infty(\lambda_Q)$ and $f=0$ on U. Since $R(K)$ is pointwise boundedly dense in $H^\infty(\lambda_Q)$ by Davie's theorem [3; Theorem 2] there is a sequence $\{f_n\}$ in $R(K)$ such that $\|f_n\|_K\leqq\|f\|$ and $f_n\to f$ a.e. (λ_Q). Now since $f=0$ on U by assumption, $f_n\to 0$ a.e. (λ_U). Therefore there is a dense subset D of U such that $f_n(w)\to 0$ for $w\in D$. For each fixed $w\in D$ the sequence $\{(z-w)^{-1}(f_n(z)-f_n(w))\}$ is norm-bounded in $R(K)$ and converges to $(z-w)^{-1}f(z)$ a.e. (λ_Q). This implies that $f(z)(z-w)^{-1}$ belongs to the weak-star closure of $R(K)$, that is, $f(z)(z-w)^{-1}\in H^\infty(\lambda_Q)$. Since f vanishes on U and w can run over the set D, dense in U, it follows that for every $w\in U$ the function $(z-w)^{-1}f(z)$ belongs to $H^\infty(\lambda_Q)$. Repetition of this methods shows that $g\cdot f$ belonge to $H^\infty(\lambda_Q)$ for every rational function g with poles in U. Hence $g\cdot f$ belongs to $H^\infty(\lambda_Q)$ for every rational function g with poles off ∂K. Take any function h in $C(K)$. Since $C(\partial K)=R(\partial K)$ by hypothesis, there is a sequence $\{g_n\}$ of rational functions with poles off ∂K such that $\|h-g_n\|_K\to 0$. Since f vanishes on U,

$$\|h\cdot f-g_n\cdot f\|_K=\|h\cdot f-g_n\cdot f\|_{\partial K}\leqq\|f\|_K\cdot\|h-g_n\|_{\partial K}.$$

Each $g_n\cdot f$ belongs to $H^\infty(\lambda_Q)$, hence so does $h\cdot f$. Since $C(K)$ is weak-star dense in $L^\infty(\lambda_Q)$, it follows that $h\cdot f\in H^\infty(\lambda_Q)$ for every $h\in L^\infty(\lambda_Q)$. For each $\varepsilon>0$, consider the well-defined function $f^{-1}\cdot\chi_E$ where E is the set $\{z\,;\,|f(z)|\geqq\varepsilon\}$ and χ_E is the characteristic fuction of E. Since $\chi_E=(f^{-1}\cdot\chi_E)\cdot f$ belongs to $H^\infty(\lambda_Q)$, by Davie's theorem cited above there is a sequence $\{g_n\}$ in $R(K)$ such that $\|g_n\|\leqq\|\chi_E\|\leqq 1$ and $g_n\to\chi_E$ *a.e.* (λ_Q). Since χ_E vanishes on U as f and since g_n is a normal family on U, $g_n\to 0$ everywhere on U while $g_n\to 1$ on E a.e. (λ_Q). Then by a well-known theorem [1; Lemma 2.6.2] any z in U and any w in E with $g_n(w)\to 1$ do not belong to one and the same part. Then Lemma 2 shows that almost every point of E lies outside Q, hence $\lambda_Q(E)=0$. Since $\varepsilon>0$ is arbitrary, this implies that f vanishes on K a.e. (λ_Q).

LEMMA 5. *Under the condition $C(\partial K)=R(\partial K)$, for each closed subalgebra H of $H^\infty(U)$, which is contained in the pointwise bounded closure of $R(U)$, there is a closed subalgebra H_Q of $H^\infty(\lambda_Q)$ such that the restriction map $g\to g|U$ gives an isometric isomorphism of H_Q onto H.*

PROOF. Lemma 4 shows that the restriction T: $g\to g|U$ maps $H^\infty(\lambda_Q)$ injectively into $H^\infty(U)$. Take any f in H. By hypothesis there is a bounded sequence $\{f_n\}$ in $R(U)$ such that $f_n\to f$ on U. Choose any weak-star limiting function g of $\{f_n\}$ in $H^\infty(\lambda_Q)$. Obviously g coincides with f on U, that is, $Tg=f$. This means that H is contained in the image of $H^\infty(\lambda_Q)$ under T. Since H is closed and T is continuous, $H_Q=T^{-1}(H)$ is a closed subalgebra of $H^\infty(\lambda_Q)$ and T gives rise an isomorphism between the uniform algebras H_Q and H, hence T is isometric.

4. The algebra H which is pointwise boundedly closed. In this section, we will give our first main theorem. First we define a localization operator: Given a smooth function g with compact support, let us define the linear operator T_g on $H^\infty(U)$ by

$$(T_g f)(w)=\frac{1}{\pi}\int\frac{f(z)-f(w)}{z-w}\frac{\partial g}{\partial \bar{z}}\,d\lambda(z)$$

where λ is the Lebesgue planar measure. For properties of T_g, see [5, VIII, 7.1]. Let $w\in\mathbb{C}$ and $\delta>0$. By $\Delta(w;\delta)$ we denote an open disc centered at w with radius δ. We begin with the following lemma.

LEMMA 6. *Let H be a subalgebra of $H^\infty(U)$ which is pointwise boundedly closed and satisfies the following conditions:*

(1) $H\supseteqq R(U)$

(2) *$R(U)$ is pointwise boundedly dense in H.*

If $f\in H$ is analytic at $w\in\mathbb{C}$, then $(f(z)-f(w))(z-w)^{-1}$ is contained in H.

REMARK. By "analyticity of f at w" we mean that f is analytic on a neighborhood of w.

PROOF. If $w\notin K$, then $(z-w)^{-1}$ is contained in $R(U)$. So the assertion is evident. If $w\in U$, and f is the pointwise bounded limit of $\{f_n\}$ in $R(U)$, then $(f_n(z)-f_n(w))(z-w)^{-1}$ is norm-bounded in $R(U)$ and is a normal family on U. Thus $(f(z)-f(w))(z-w)^{-1}$ is the pointwise bounded limit of some subsequence of $(f_n(z)-f_n(w))(z-w)^{-1}$, and is contained in H. If $w\in\partial K$, there exist $\delta>0$ and $M>0$ such that $|(f(z)-f(z'))(z-z')^{-1}|\leqq M$ for every z and z' in $\Delta(w;\delta)$, $z\neq z'$. Then for every sequence $\{z_n\}$ in $\Delta(w;\delta)$ such that $z_n\neq w$ and $z_n\to w$, $(f(z)-f(z_n)(z-z_n)^{-1}$ is contained in H, norm-bounded and thus a normal family on U. Hence some subsequence of $(f(z)-f(z_n))(z-z_n)^{-1}$ converges pointwise on U to $(f(z)-f(w))(z-w)^{-1}$. Now $(f(z)-f(w))(z-w)^{-1}$ is contained in H because H is pointwise boundedly closed. This completes the proof.

Let $M(H)$ be the maximal ideal space of H. Then the coordinate function Z projcets $M(H)$ onto $\bar{U}$. Let $\hat{Z}$ be the Gelfand transform of Z in H. Then the following lemma is valid.

LEMMA 7. *Let H be a subalgebra of $H^\infty(U)$ which is pointwise boundedly closed. If H has the following properties:*

(1) $H \supseteqq R(U)$

(2) *H is invariant under T_g for every smooth function g with compact support.*

(3) *$R(U)$ is pointwise boundedly dense in H.*

Let $f \in H$ and $w_0 \in \partial U$. Then

$$\lim_{U \ni z \to w_0} \sup \left|f(z)\right| = \sup \left\{ \left|\varphi(f)\right| ; \varphi \in M(H) \quad with \quad \hat{Z}(\varphi) = w_0 \right\}$$

PROOF. Set $M_0(H) = \{\varphi \in M(H) : \hat{Z}(\varphi) = w_0\}$ and $\|f\|_{M_0(H)} = \sup \{|\varphi(f)| ; \varphi \in M(H)$ *with* $\hat{Z}(\varphi) = w_0\}$. For each sequence $\{z_n\}$ in U such that $z_n \to w_0$, there exists a element φ_0 in $M(H)$ satisfying $\varphi_0(f) = \lim_{n \to \infty} f(z_n)$ for fixed $f \in H$. Then clearly $\varphi_0(Z) = \hat{Z}(\varphi_0) = w_0$. Hence

$$\|f\|_{M_0(H)} \geqq \lim_{U \ni z \to w_0} \sup \left|f(z)\right| .$$

Suppose there exists positive unmber $\delta > 0$ such that $|f| \leqq 1$ on $U \cap \Delta(w_0 ; \delta)$. We must show that $\|f\|_{M_0(H)} \leqq 1$. Let g be a smooth function with compact support which is identically equal to 1 on some neighborhood of w_0, vanishes outsides $\Delta(w_0 ; \delta)$, $\left\| \frac{\partial g}{\partial \bar{z}} \right\| \leqq \frac{4}{\delta}$ and $0 \leqq g \leqq 1$. Then $T_g f \in H$ by hypothesis and $\|T_g f\| \leqq 16$. Since $f - T_g f$ is analytic at w_0, then $f - T_g f$ assumes a constant on $M_0(H)$ from Lemma 6. That is, $\varphi(f - T_g f) = (f - T_g f)(w_0)$ for all $\varphi \in M_0(H)$. Then $\|f - T_g f\|_{M_0(H)} \leqq 8$. Thus $\|f\|_{M_0(H)} \leqq 8 + \|T_g f\|_{M_0(H)} \leqq 24$. Then the same argument shows that $\|f^n\|_{M_0(H)} \leqq 24$ for every positive integer n. Taking nth roots and letting n tend to ∞, we obtain $\|f\|_{M_0(H)} \leqq 1$. Then the lemma is proved.

THEOREM 1. *Suppose $C(\partial K) = R(\partial K)$. Let H be a subalgebra of $H^\infty(U)$ which is pointwise boundedly closed. If H satisfies the conditions:*

(1) $H \supseteqq R(U)$.

(2) *H is invariant under T_g for every smooth function with compact support.*

(3) *$R(U)$ is pointwise boundedly dense in H.*

Then $H \cap C(K) = R(K)$, that is, every continuous functions on K whose restriction to U belongs to H lies in $R(K)$.

PROOF. Since H satisfies the condition in Lemma 5, by (3), there is an isometric-isomorphism T of H into $H^\infty(\lambda_Q)$ such that the restriction of Tf to U coincides with f. Let M denote the maximal ideal space of the uniform algebra $L^\infty(\lambda_Q)$. Then the adjoint of T induces a map S of M to $M(H)$ such that

$$\widehat{Tf}(\Phi) = \hat{f}(S\Phi) \quad \text{for} \quad \Phi \in M \quad \text{and} \quad f \in H.$$

It follows that

$$\widehat{Tf}\left((\hat{Z}\cdot S)^{-1}(w)\right)\subseteqq \hat{f}\left(\hat{Z}^{-1}(w)\right) \quad \text{for} \quad w\in \bar{U}.$$

Now take any function g in $H\cap C(K)$, that is g is continuously extended over K. Let us denote the continuous extension by the same letter g. Since $\lim_{U\ni z\to w} g(z)=g(w)$ for each $w\in\bar{U}$ by continuity, g assumes the constant $g(w)$ on $\hat{Z}^{-1}(w)$ form the above lemma. It follows that from the above that

$$\widehat{Tg}(\Phi)\subseteqq \hat{g}\left(\hat{Z}^{-1}\left(\hat{Z}\cdot S(\Phi)\right)\right)=g\left(\hat{Z}\cdot S(\Phi)\right) \quad \text{for} \quad \Phi\in M.$$

That is, $\widehat{Tg}(\Phi)$ assumes the value $g(\hat{Z}\cdot S(\Phi))$. While g, as a function in $L^\infty(\lambda_Q)$, assumes the value $g(w)$ on $(\hat{Z}\cdot S)^{-1}(w)$ for each $w\in\bar{U}=\bar{Q}$. This shows that

$$g(\Phi)=g\left(\hat{Z}\cdot S(\Phi)\right) \quad \text{for} \quad \Phi\in M.$$

Then it is proved that

$$\widehat{Tg}(\Phi)=\hat{g}(\Phi) \quad \text{for every} \quad \Phi\in M.$$

where g is considered as a function in $L^\infty(\lambda_Q)$. Since M separates functions of $L^\infty(\lambda_Q)$, the function Tg and g coincide a.e. (λ_Q), hence g belongs to $H^\infty(\lambda_Q)$. Now the assertion of the theorem follows from a result of Gamelin-Garnett [8; Theorem 1.1]: $H^\infty(\lambda_Q)\cap C(K)=R(K)$.

Remark. The weak-star closure of $A(U)$ in $L^\infty(\lambda_U)$ satisfies the condition (1) and (2) of Theorem 1, and so does $H^\infty(U)$ itself.

5. The algebra $A(U)$. In this section, we will give our second main result.

The following lemma is due to Davie-Gamelin-Garnett [4].

Lemma 8. *Let H be a closed subalgebra of $H^\infty(U)$ satisfying the following conditions:*

(1) $H\supseteqq A(U)$

(2) *If $f\in H$ and $w\in U$, then $(f(z)-f(w))(z-w)^{-1}\in H$. Then if $f\in H$ and $w_0\in\bar{U}$,*

$$\lim_{U\ni z\to w_0}\sup\left|f(z)\right|=\sup\left\{\left|f(\varphi)\right|:\ \varphi\in\hat{Z}^{-1}(w_0)\right\}.$$

Proof. See [4; Theorem 3.3].

Theorem 2. *Suppose $C(\partial K)=R(\partial K)$. Let H be a closed subalgebra of $H^\infty(U)$ which satisfies the conditions:*

(1) $H\supseteqq A(U)$

(2) *If $f\in H$ and $w\in U$, then $(f(z)-f(w))(z-w)^{-1}\in H$.*

(3) *$R(U)$ is pointwise boundedly dense in H.*
Then $H\cap C(K)=R(K)$.

PROOF. Since in view of (3) the algebra H satisfies the condition in Lemma 5, there is an isometric-isomorphism T of H into $H^{\infty}(\lambda_Q)$ such that the restriction Tf to U coincides with f. Moreover, since in view of (1) and (2) the algebra H satisfies the condition in Lemma 8,

$$\lim_{U\ni z\to w} \sup |f(z)| = \sup\left\{|\hat{f}(\varphi)| : \varphi\in \hat{Z}^{-1}(w)\right\}$$

for $w\in \bar{U}$ and $f\in H$. Especially, if g is continuously extended to $w\in \bar{U}$ (over U), g assumes tne constant value $\lim_{U\ni z\to w} g(z)$ on $\hat{Z}^{-1}(w)$. Then the rest of the theorem is proved as in the proof of Theorem 1.

We will show that $A(U)$ is a closed subalgebra of $H^{\infty}(U)$. Let $f\in H^{\infty}(U)$ be the uniform limit on U of $\{f_n\}$ in $A(K)$. That is, $\|f_n-f\|_U\to 0$. Then $\{f_n\}$ is a Cauchy sequence on $\bar{U}$ and thus there is a function g in $A(U)$ such that $\|f_n-g\|_U\to 0$ and $g=f$ on U. Since g can be continuously extended to K, we can assume $g\in A(K)$. Thus $f=g|U\in A(U)$ and $A(U)$ is a closed subalgebra of $H^{\infty}(U)$.

Now we can obtain the following second main theorem.

THEOREM 3. *Suppose $C(\partial K)=R(\partial K)$. Then the following conditions are mutually equivalent:*

(1) *$R(K)=A(K)$*

(2) *$d(h, R(U))=d(h, A(U))$ for all $h\in C(\bar{U})$.*

(3) *$R(U)$ is boundedly pointwise dense in $A(U)$.*

PROOF. (1)$\Rightarrow$(2) is evident. To see (2)$\Rightarrow$(3), it suffices to show that $A(K)$ is contained in $H^{\infty}(\lambda_Q)$ by Davie's Theorem [8, p. 129]. Take any f in $A(K)$. Then by (2) there is a sequence $\{f_n\}$ in $R(K)$ such that $\|f_n-f\|_U\to 0$. Since Q is contained in $\bar{U}$ by Lemma 3, it follows that

$$\|f_n-f\|_Q \leqq \|f_n-f\|_{\bar{U}} = \|f_n-f\|_U \longrightarrow 0,$$

hence f belongs to $H^{\infty}(\lambda_Q)$.

Suppose that (3) is valid. Then $A(U)$ satisfies all the condition (imposed on H) in Theorem 2, hence

$$A(K)=A(U)\cap C(K)=R(K),$$

that is, (3)$\Rightarrow$(1). Therefore (1), (2) and (3) are mutually equivalent.

COROLLARY. *Suppose $C(\partial K)=R(\partial K)$. Tnen the following conditions are equivalent:*

(1) *$d(h, R(U))=d(h, H^{\infty}(U))$ for all $h\in C(\bar{U})$.*

(2) *$R(U)$ is boundedly pointwise dense in $H^{\infty}(U)$.*

PROOF. (2) implies boundedly pointwise density of $A(U)$ in $H^{\infty}(U)$. Davie-Gamelin-Garnett [4, Theorem 1. 1] showed that this last condition is equivalent to that

$$d\left(h, A(U)\right)=d\left(h, H^{\infty}(U)\right) \quad \text{for all} \quad h \in C(\bar{U}).$$

Therefore (2) implies (1). Finally (1) implies boundedly pointwise density of $R(U)$ in $A(U)$ by Theorem 3 as well as in that of $A(U)$ in $H^{\infty}(U)$ as mentioned above. Hence (1) implies (2).

REMARK. We already know that the condition $A(K)=R(K)$ implies $C(\partial K)=R(\partial K)$. (cf. Gamelin [5, p. 227]). Practically, we can construct an example such that $C(\partial K)\neq R(\partial K)$, $R(U)$ is pointwise boundedly dense in $A(U)$ (also in $H^{\infty}(U)$), and thus $A(K)\neq R(K)$. Let E_1 be the Swiss cheese. So $\mathring{E}_1=\phi$. Hence $A(E_1)=C(E_1)\neq R(E_1)$. Let E_2 be a closed disc such that $E_1\cap E_2=\phi$. Hence $A(\mathring{E}_2)=R(\mathring{E}_2)$ is pointwise boundedly dense in $H^{\infty}(\mathring{E}_2)$. Moreover, let E_3 be a closed interval which intersects only one point with E_1 and E_2 respectively. We put $K=\bigcup_{i=1}^{3} E_i$. Then K is a compact set and $\mathring{K}=\mathring{E}_2$. Since each point of E_3 is clearly a peak point for $R(K)$, $R(E_3)=C(E_3)$. Then every point w in E_1 is not contained in the same part containing each point of E_2 because of the connectivity of the closure of a part [10]. That is, $w \notin Q_z$, for all $z\in E_2$. Hence

$R(K)=\{f\in C(K) : f|E_i\in R(E_i),\ i=1, 2, 3\}$ and $R(\mathring{K})$ is pointwise boundedly dense in $H^{\infty}(\mathring{K})=H^{\infty}(\mathring{E}_2)$. On the other hand, $R(E_1)\neq A(E_1)=C(E_1)$, we can obtain that $R(\partial K)\neq C(\partial K)$ and $R(K)\neq A(K)$.

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