Some remarks on local solvability and hypoellipticity of second-order abstract evolution equations

By Junichi ARAMAKI

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0. Introduction

In this paper we shall consider local solvability and hypoellipticity of the following operator:

$$P=(\partial_t+at^kA)(\partial_t+bt^kA)-ct^{k-1}A\,. \tag{0.1}$$

Here A is a selfadjoint operator in a Hilbert space H, a, b, c are complex numbers and k a positive odd integer. Namely we shall prove the following

THEOREM. *Assume that* Re $a>0$ *and* Re $b<0$, *then the following are equivalent*:

(i) *P is locally solvable at* $t=0$;

(ii) *P is hypoelliptic at* $t=0$;

(iii) *for no integer* n, $c/(a-b)=1-n(k+1)$ *or* $-n(k+1)$.

The case when $A=D_x$ and $H=L^2(\boldsymbol{R}^1)$ or when A positive-definite and $k=1$ under more general set-up were considered by Gilioli and Treves [3] or Treves [6] respectively (see also [2] and [5]). For the case in which k is even, refer to Menikoff [4].

In [3] the sufficiency proof for local solvability was based on the theory of ordinary differential operators of Fuchs' type (see also [1]). However, in the present paper we shall assert that it can be proved in the framework of abstract theory in [6].

1. Preliminary

In order to describe the situation more precisely we here explain our notations and list up some results in [6] and [3] which we must use.

Let A be a densely defind linear operator on H. We shall assume that A is selfadjoint and that $(I-E(0))A$ and $E(0)A$ are unbounded. Here we denote the spectral resolution of A by $E(\lambda)$. (cf. Yosida [7]) For some $\varepsilon>0$, we define an orthogonal decomposition of H;

$$H_+=\big(I-E(\varepsilon)\big)H, \qquad H_\varepsilon=\big(E(\varepsilon)-E(-\varepsilon)\big)H, \qquad H_-=E(-\varepsilon)H.$$

Then we have $A=A_+ + A_\epsilon + A_-$, $H=H_+\oplus H_\epsilon \oplus H_-$. Since A_+ and $-A_-$ are positive-definite operators with bounded inverses respectively in H_+ and H_-, we can introduce the scale of "*Sobolev space*" $(H_+)^s$ and $(H_-)^s$ for $-\infty \leqslant s \leqslant +\infty$, defined by A_+ and $-A_-$. Furthermore we define $H^s = =(H_+)^s \oplus H_\epsilon \oplus (H_-)^s$. Then as in [6], we use the following function spaces. We denote by $C^\infty(J; H^\infty)$ the space of H^∞-valued C^∞-functions in J. Here J is an open interval in $\boldsymbol{R}$. If K is any compact subset of J, we denote by $C_0^\infty(K; H^\infty)$ the subspace of $C^\infty(J; H^\infty)$ consisting of functions vanishing outside K. Being a closed linear subspace of $C^\infty(J; H^\infty)$ with the natural C^∞-topology, it is a Frèchet space. By $C_0^\infty(J; H^\infty)$ we denote the inductive limit of $C_0^\infty(K; H^\infty)$ as K ranges over all compact subsets of J. Its dual space is denoted by $\mathscr{D}'(J; H^{-\infty})$.

DEFINITION 1.1. *P is locally solvable at $t=0$ if there exists an open interval J containing $t=0$ such that for every $f\in C_0^\infty(J; H^\infty)$ there is $u\in\mathscr{D}'(J; H^{-\infty})$ satisfying $Pu=f$ in J.*

We say that P is locally solvable in a subset S of $\boldsymbol{R}$ if P is so at every point of S.

P is hypoelliptic in J if, for any $u\in\mathscr{D}'(J; H^{-\infty})$, $Pu\in C^\infty(J; H^\infty)$ implies $u\in C^\infty(J; H^\infty)$.

We say that P is hypoelliptic at $t=0$ if there exists an open interval J containing $t=0$ such that P is so in J.

Let us write, for convenience, $P=P(c)=XY-cZ$ with a parameter c where

$$X=\partial_t + at^k A\,,$$

$$Y=\partial_t + bt^k A\,,$$

$$Z=t^{k-1}A\,.$$

For our purpose we shall apply the following "*Concatenation method*" which was constructed by [3].

PROPOSITION 1.2. With the above notations, we have

$$(tX+\mu)P(c)=P(c')(tX+\mu-2)\,, \tag{1.1}$$

where $c'=c+(k+1)(a-b)$ and $\mu=k+2+c/(a-b)$.

We shall use an algebraic lemma in order to prove that (iii) implies (ii).

LEMMA 1.3 ([6]). Let E be an abelian group and F a subgroup of E, P, Q, U, V four endomorphisms of E which map into itself having the following properties:

$$UP=QV\,, \tag{1.2}$$

(1.3) $$V^{-1}(F)\cap P^{-1}(F)\subset F.$$

Then $Q^{-1}(F)\subset F$ implies $P^{-1}(F)\subset F$.

2. Proof of the Theorem

This section consists of two parts. In part a) we give a slight extension of subelliptic estimate in [6; Theorem II. 2.1] and its results. In part b) we shall bring the proof to completion.

a) Here we shall consider the special case where A is positive-definite on H and has a bounded inverse A^{-1} and we denote by $\|\cdot\|$, $(,)$ the norm and the inner product respectively. Under these hypotheses we shall prove the following

PROPOSITION 2.1. Suppose that

(2.1) $$-|\delta|^2\leqslant 2/k\,\mathrm{Re}(c\bar{\delta}),\qquad \text{where } \delta=a-b.$$

Then for a suitable constant $C>0$, if $u\in C_0^\infty(J; H^\infty)$,

(2.2) $$\int\left(\|u_t\|^2+\|t^kAu\|^2+\|A^{1/(k+1)}u\|^2\right)dt\leqslant C\left|\int(Pu, u)\,dt\right|.$$

PROOF: First by our hypothesis (2.1), we see

(2.3) $$\int\left(\|u_t\|^2+\|t^kAu\|^2\right)dt\leqslant C\left|\int(Pu, u)\,dt\right|,\quad \text{for all } u\in C_0^\infty(J; H^\infty).$$

In fact if we set $X^+=\partial_t-\bar{a}t^kA=-X^*$, then we have

$$-\mathrm{Re}\left\{\bar{\delta}\int(Pu, u)\,dt\right\}=\mathrm{Re}\left\{\bar{\delta}\int(Yu, X^+u)\,dt\right\}+\mathrm{Re}(c\bar{\delta})\int(t^{k-1}Au, u)\,dt.$$

Here we note that

$$\int(Yu, X^+u)\,dt=\int\left\{\|u_t\|^2-ab\,\|t^kAu\|^2\right\}dt$$
$$+\int\left\{-a(u_t, t^kAu)+b(t^kAu, u_t)\right\}dt.$$

By literal repetition of the arguments of [6; II.2] where (2.1) is used, we can get (2.3). Secondly integration by parts leads to

(2.4) $$\int\|A^{1/(k+1)}u\|^2dt=-2\mathrm{Re}\int(u_t, A^{2/(k+1)}tu)\,dt$$
$$\leqslant\int\|u_t\|^2dt+\int\|A^{2/(k+1)}tu\|^2dt.$$

Repeating the above consideration we have, for every integer $j\geqslant 1$,

$$(2.5)_j \qquad \int \|A^{j/(k+1)} t^{j-1} u\|^2 dt \leqslant \int \|u_t\|^2 dt + \int \|A^{(j+1)/(k+1)} t^j u\|^2 dt .$$

Indeed, by (2.4) we may assume $j \geqslant 2$. Therefore by induction hypothesis, suppose $(2.5)_{j-1}$. Then we have

$$\begin{aligned}
&\int \|A^{j/(k+1)} t^{j-1} u\|^2 dt \\
&= \int (A^{(j-1)/(k+1)} t^{j-2} u, A^{(j+1)/(k+1)} t^j u)\, dt \\
&\leqslant 1/2 \int \|A^{(j-1)/(k+1)} t^{j-2} u\|^2 dt + 1/2 \int \|A^{(j+1)/(k+1)} t^j u\|^2 dt \\
&\leqslant 1/2 \int \|u_t\|^2 dt + 1/2 \int \|A^{j/(k+1)} t^{j-1} u\|^2 dt + 1/2 \int \|A^{(j+1)/(k+1)} t^j u\|^2 dt .
\end{aligned}$$

Hence we get $(2.5)_k$ which means that the left hand side in (2.4) dominated by left in (2.3). Therefore we complete the proof of Proposition 2.1.

In the next part b) the following proposition is useful. If P satisfies (2.2), by the same way as [6; II.3], we see

PROPOSITION 2.2. *P is hypoelliptic in J with loss of $2k/(k+1)$ derivatives.*

b) If we write

$$\begin{aligned}
P &= P\big(I - E(\varepsilon)\big) + P\big(E(\varepsilon) - E(-\varepsilon)\big) + P\big(E(-\varepsilon)\big) \\
&= P_+ + P_\varepsilon + P_- ,
\end{aligned}$$

we can easily find that local solvability of P in H implies one of P_+, P_- in H_+, H_- respectively. Thus if we replace D_x in [3; Section 3] by A_+ or $-A_-$, by the same consideration we can prove that (i) implies (iii). Since the proof that (ii) implies (i) follows from abstract theory [6; II.5], it is sufficient to prove that (iii) implies (ii). In order to do so, we use Lemma 1.3 as follows:

$$U = tX + \mu , \quad P = P(c), \quad Q = P\big(c + (k+1)\,\delta\big), \quad V = tX + \mu - 2 \text{ and}$$

$$E = \mathscr{D}'(J;\, H^{-\infty}), \quad F = C^\infty(J;\, H^\infty).$$

Here (1.2) follows from Proposition 1.2. Furthermore we shall check whether (1.3) is satisfied. By direct computation we observe that

$$\begin{aligned}
Y(tX + \mu - 2) - tP(c) &= X + tXY - t[X, Y] + (\mu - 2)\, Y - tXY + ctZ \\
&= (\mu - 1)\, \partial_t + \big(a + (\mu - 2)\, b + c + k\delta\big)\, tZ .
\end{aligned}$$

Since (1.1) implies $a+(\mu-2)b+c+k\delta=(k+1+c/\delta)a=(\mu-1)a$, it follows from the above equality that

$$Y(tX+\mu-2)-tP(c)=(\mu-1)X.$$

Therefore we get at once

$$tY(tX+\mu-2)-t^2P(c)=(\mu-1)tX$$
$$=(\mu-1)(tX+\mu-2)-(\mu-1)(\mu-2).$$

Thus if $(\mu-1)(\mu-2)\neq 0$, we have

$$\left\{1/(\mu-1)(\mu-2)\right\}\left\{-tY(tX+\mu-2)+t^2P(c)+(\mu-1)(tX+\mu-2)\right\}=1.$$

Hence (1.3) holds. Recalling Proposition 1.2 we have $(\mu-1)(\mu-2)=(c/\delta+k+1)(c/\delta+k)$. Therefore by Lemma 1.3 if $(c/\delta+k+1)(c/\delta+k)\neq 0$, hypoellipticity of $P(c+(k+1)\delta)$ in H implies one of $P(c)$ in H. Here we note that we actually apply the above argument to P_+.

If we iterate the above discussion by replacing $c+(j-1)(k+1)\delta$ with $c+j(k+1)\delta$ for $j=1, 2, \cdots J$ successively, we derive the following: if we assume that

$$\text{(2.6)}\qquad \left(c/\delta-1+j(k+1)\right)\left(c/\delta+j(k+1)\right)\neq 0 \text{ for } j=1, 2, \cdots J,$$

then hypoellipticity of $P_+(c+J(k+1)\delta)$ in H_+ implies that of $P_+(c)$. On the other hand we find that if

$$J\geqslant -k/\left(2(k+1)\right)-\mathrm{Re}\, c/\left(\delta(k+1)\right),$$

$P_+(c+J(k+1)\delta)$ is hypoelliptic at $t=0$ in H_+. In fact under the hypothesis, we can easily check that

$$-|\delta|^2\leqslant(2/k)\,\mathrm{Re}\left(\left(c+J(k+1)\delta\right)\bar{\delta}\right).$$

Therefore we only need to apply Proposition 2.2 putting $P_+(c+J(k+1)\delta)$ and H_+ in place of P and H respectively. By our hypothesis (iii) ensuring (2.6), we see that if for no integer $n\geqslant 1$, $c/(a-b)=1-n(k+1)$ or $-n(k+1)$, then P_+ is hypoelliptic at $t=0$ in H_+.

Finally we may write

$$P_-=\left(\partial_t+(-b)t^k(-A_-)\right)\left(\partial_t+(-a)t^k(-A_-)\right)-\left(-(c+k\delta)\right)t^{k-1}(-A_-) \text{ in } H_-.$$

Here if we note that $\mathrm{Re}(-b)>0$ and $\mathrm{Re}(-a)<0$, we see that P_- in H_- is the same type as P_+ in H_+. Therefore the same consideration as above, we have: suppose that for no integer $n\leqslant 0$, $c/(a-b)=1-n(k+1)$ or

$-n(k+1)$, then P_- is hypoelliptic at $t=0$ in H_-. Note also that hypoellipticity of P_ε in H_ε follows from the boundedness of A_ε. Thus we complete the proof of Theorem.

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Department of Mathematics
Hokkaido University