Positive approximants of normal operators

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1. Introduction. We consider the problem of approximation for a given bounded linear operator on a fixed Hilbert space by positive operators where positivety means non-negative semi-definite. Study of this problem was initiated by P. R. Halmos [4], who proved that the distance of an operator to the set of all positive operators is completely determined. The results proved by him can be formulated as follows.

Let A be a bounded linear operator on a Hilbert space $\mathscr{H}$. Put $A=B+iC$ where B and C denote the real part Re A and the imaginary part Im A of A respectively.

(1) *Put*

$$\delta=\inf\left\{\|A-P\|:P\geqq 0\right\}.$$

Then

$$\delta=\inf\left\{r\geqq 0:r^2\geqq C^2,\ B+(r^2-C^2)^{\frac{1}{2}}\geqq 0\right\}.$$

(2) *Define another norm* $|||\ \ |||$ *by*

$$|||A|||=\left\|(\mathrm{Re}\ A)^2+(\mathrm{Im}\ A)^2\right\|^{\frac{1}{2}}.$$

Then

$$\frac{1}{2}\|A\|\leqq|||A|||\leqq\|A\|$$

and

$$\delta=\inf\left\{|||A-P|||:P\geqq 0\right\}.$$

(3) *Put*

$$\mathscr{P}(A)=\left\{P\geqq 0:\|A-P\|=\delta\right\}$$

and

$$\mathscr{P}_n(A)=\left\{P\geqq 0:|||A-P|||=\delta\right\}.$$

Then both $\mathscr{P}(A)$ *and* $\mathscr{P}_n(A)$ *are convex sets and* $\mathscr{P}(A)\subseteqq\mathscr{P}_n(A)$. *The operators in* $\mathscr{P}(A)$ *and* $\mathscr{P}_n(A)$ *are called positive approximants and positive near-approximants respectively.*

(4) *The operator* P_0 *defined by*

$$P_0 = B + (\delta^2 - C^2)^{\frac{1}{2}}$$

is maximum in both $\mathscr{P}(A)$ *and* $\mathscr{P}_n(A)$, *that is,* $P_0 \in \mathscr{P}(A)$ *and* $P \leqq P_0$ *for any operator* P *in* $\mathscr{P}_n(A)$.

In the present paper we consider the problem raised by R. Bouldin [2], that is, a necessary and sufficient condition for that $\mathscr{P}(A)$ coincides with $\mathscr{P}_n(A)$ in the case A is a normal operator. Since both $\mathscr{P}(A)$ and $\mathscr{P}_n(A)$ are weakly compact convex sets, these sets are the convex closures of respective extremal points. By this result we show that the set of all extremal points of $\mathscr{P}(A)$ is either finite or uncountable in the case A is a normal operator.

In this paper operators are bounded linear operators on a complex Hilbert space $\mathscr{H}$. Put $B = \mathrm{Re}\ A$ and $C = \mathrm{Im}\ A$ for a given operator A. B_+ and B_- denote the positive and the negative parts of a Hermitian operator B respectively. $\mathrm{Ran}(A)$ denotes the range of an operator A. $A|_{\mathscr{M}}$ denotes the restriction of A on an A-reducing subspace $\mathscr{M}$. $\{A\}'$ and $\{A\}''$ denote the commutant and the double commutant of A respectively. The dimension of a subspace $\mathscr{M}$ is denoted by dim $\mathscr{M}$. $\mathscr{M}^{\perp}$ denotes the orthogonal complement of $\mathscr{M}$. N^- denotes the closure of a set $\mathscr{N}$.

2. Positive approximants and positive near approximants. Put

$$\mathscr{H}_0 = \mathrm{Ran}(P_0)^- \cap \mathrm{Ran}(\delta^2 - C^2)^-,$$

then $\mathrm{Ran}(P_0 - P)^-$ is included in $\mathscr{H}_0$ for any operaotr P in $\mathscr{P}_n(A)$ since $(B-P)^2 + C^2 \leqq \delta^2$ and $0 \leqq P \leqq P_0$. In the case A is a normal operator, $\mathscr{H}_0$ is an A-reducing subspace, hence $\mathscr{H}_0$ is a reducing subspace for each operator P in $\mathscr{P}_n(A)$.

THEOREM 2.1. *Let A be a normal operator. If the operator* $(\mathrm{Im}\ A)|_{\mathscr{H}_0}$ *is non-scalar, then there exists a positive operator P such that*

(a) $P \notin \mathscr{P}(A)$ *and* $P \in \mathscr{P}_n(A)$,

(b) $P|_{\mathscr{H}_0}$ *does not commute with* $(\mathrm{Im}\ A)|_{\mathscr{H}_0}$.

PROOF. Obviously $C|_{\mathscr{H}_0}$ is scalar if dim $\mathscr{H}_0$ is zero or one. Hence it can be assumed that dim $\mathscr{H}_0 \geqq 2$. Let $E(\sigma)$ denote the spectral measure of A. $\sigma(A)$ and $\sigma_p(A)$ denote the spectrum and the point spectrum of A respectively. Put

$$\Gamma_\delta = \{z : |z| = \delta,\ \mathrm{Re}\ z \leqq 0\} \cup \{z : |\mathrm{Im}\ z| = \delta\}$$

and

$$\sigma'(A)=\sigma(A)-\Gamma_\delta.$$

Obviously $\mathscr{K}_0=\mathrm{Ran}(E(\sigma'(A)^-))$ holds. Suppose $C|_{\mathscr{K}_0}$ is non-scalar. Im σ denotes $\{\mathrm{Im}\ z:z\in\sigma\}$ for a set σ included in $\sigma(A)$. The set Im $\sigma'(A)$ contains more than two points. There exist non-empty and sufficiently small closed sets σ_1 and σ_2 included in $\sigma'(A)$ such that

(i) *both* σ_1 *and* σ_2 *are connected sets,*

(ii) *both* σ_1 *and* σ_2 *have positive distances from the set* Γ_δ,

(iii) Im $\sigma_1\cap$ Im $\sigma_2=\phi$.

By condition (i), Im σ_i is either a one point set or a closed interval for $i=1, 2$.

(1) Im σ_i *is a closed interval,*

(2) σ_i *is a segment paralell to the real axis,*

(3) σ_i *is a one point set* $\{\lambda_i\}$ *(then* $\lambda_i\in\sigma_p(A)$).

Put $\mathscr{M}_i=\mathrm{Ran}(E(\sigma_i))$ for $i=1, 2$. In the case condition (1) it can be assumed that $\sigma_p(C|_{\mathscr{M}_i})=\phi$ and moreover dim $\mathscr{M}_i$ is countably infinite. In fact if dim $\mathscr{M}_i$ is uncountable, then choose a subspace $\mathscr{M}'_i$ instead of $\mathscr{M}_i$ where $\mathscr{M}'_i$ is the minimal C-reducing subspace generated by a non-zero vector in $\mathscr{M}_i$. dim $\mathscr{M}'_i$ is countably infinite and the set Im $\sigma(C|_{\mathscr{M}'_i})$ contains more than two points and connected since $\sigma_p(C|_{\mathscr{M}'_i})=\phi$. Similarly in the case condition (2) it can be assumed that $\sigma_p(B|_{\mathscr{M}_i})=\phi$ and dim $\mathscr{M}_i$ is countably infinite. In the case condition (3) it can be assumed that dim $\mathscr{M}_i=1$. The proof is reduced to the following cases.

Case. I. Both σ_1 *and* σ_2 *satisfy condition* (1). Put Im $\sigma_i=[\alpha_i, \beta_i]$ for $i=1, 2$. Without loss of generality, it can be moreover assumed that

(1–1) $0<\alpha_i<\beta_i$ *or* $\alpha_i<\beta_i<0$ *for* $i=1, 2$,

(1–2) $\beta_i-\alpha_i=\varepsilon_1>0$ *for* $i=1, 2$,

(1–3) *all numbers* $|\alpha_1|$, $|\alpha_2|$, $|\beta_1|$ *and* $|\beta_2|$ *are distinct.*

Put $a_i=\beta_i$ and $b_i=\alpha_i$ if $0<\alpha_i<\beta_i$, and put $a_i=|\alpha_i|$ and $b_i=|\beta_i|$ if $\alpha_i<\beta_i<0$ for $i=1, 2$, then $a_i=\|C|_{\mathscr{M}_i}\|$ and $b_i=\inf\ \{\|C|_{\mathscr{M}_i}x\|:x\in\mathscr{M}_i, \|x\|=1\}$ for $i=1, 2$. Put $\mathscr{M}=\mathscr{M}_1\oplus\mathscr{M}_2$ where the symbol $\oplus$ means orthogonal direct sum. The operators $B|_{\mathscr{M}}$, $C|_{\mathscr{M}}$ and $P_0|_{\mathscr{M}}$ can be represented as matrices of operators on $\mathscr{M}=\mathscr{M}_1\oplus\mathscr{M}_2$:

$$B|_{\mathscr{M}}=\begin{bmatrix}B_1 & 0\\ 0 & B_2\end{bmatrix},\qquad C|_{\mathscr{M}}=\begin{bmatrix}C_1 & 0\\ 0 & C_2\end{bmatrix}$$

and

$$P_0|_{\mathscr{M}}=\begin{bmatrix}B_1+(\delta^2-C_1^2)^{\frac{1}{2}} & 0\\ 0 & B_2+(\delta^2-C_2^2)^{\frac{1}{2}}\end{bmatrix}.$$

By condition (ii) there exists a positive number ε_0 such that for $i=1, 2$,

$$B_i+(\delta^2-C_i^2)^{\frac{1}{2}}\geqq\varepsilon_0 \text{ and } (\delta^2-C_i^2)^{\frac{1}{2}}\geqq\varepsilon_0 .$$

Put

$$\sigma_i(s)=\left\{z: z\in\sigma_i,\ a_i-s\leqq|\mathrm{Im}\ z|\leqq a_i\right\}$$

for each positive number s such that $0<s\leqq\varepsilon_1$. Choose a unitary operator U mapping $\mathcal{M}_2$ onto $\mathcal{M}_1$ such that

$$U\left(\mathrm{Ran}\left(E(\sigma_2(s))\right)\right)=\mathrm{Ran}\left(E(\sigma_1(s))\right)$$

for each s. Define a positive operator Q_t on $\mathcal{M}$ for each real number t such that $0<t\leqq\varepsilon_0$ by

$$Q_t=\begin{bmatrix}(\delta^2-C_1^2)^{\frac{1}{2}}-\varepsilon_0+t & tU \\ tU^* & (\delta^2-C_2^2)^{\frac{1}{2}}-\varepsilon_0+t\end{bmatrix}.$$

Moreover define a positive operator P_t on $\mathcal{H}$ for each t such that $\mathcal{M}$ is a P_t-reducing subspace for each t,

$$P_t\big|_{\mathcal{M}}=Q_t+B\big|_{\mathcal{M}} \text{ and } P_t\big|_{\mathcal{M}^\perp}=P_0\big|_{\mathcal{M}^\perp} .$$

Then

$$(A-P_t)\big|_{\mathcal{M}^\perp}=\left\{-(\delta^2-C^2)^{\frac{1}{2}}+iC\right\}\Big|_{\mathcal{M}^\perp}$$

is a saclar multiple of a unitary operator on $\mathcal{M}^\perp$ with norm δ while

$$(A-P_t)\big|_{\mathcal{M}}=-Q_t+iC\big|_{\mathcal{M}} .$$

Define the operators D_i and F_i for $i=1, 2$ by

$$D_i=(\delta^2-C_i^2)^{\frac{1}{2}}-\varepsilon_0+t ,$$

and

$$\begin{aligned}F_i&=D_i^2+t^2+C_i^2\\&=\delta^2+t^2+(\varepsilon_0-t)^2-2(\varepsilon_0-t)(\delta^2-C_i^2)^{\frac{1}{2}} .\end{aligned}$$

Then

$$Q_t^2+\left(C\big|_{\mathcal{M}}\right)^2=\begin{bmatrix}F_1 & t(D_1U+UD_2) \\ t(U^*D_1+D_2U^*) & F_2\end{bmatrix}$$

and

$$\begin{aligned}&\left(-Q_t+iC\big|_{\mathcal{M}}\right)^*\left(-Q_t+iC\big|_{\mathcal{M}}\right)\\&=\begin{bmatrix}F_1 & t(D_1U+UD_2+iC_1U-iUD_2) \\ t(U^*D_1+D_2U^*-iU^*C_1+iC_2U^*) & F_2\end{bmatrix}.\end{aligned}$$

Obviously both $\|(Q_t)^2+(C|_{\mathcal{M}})^2\|$ and $\|-Q_t+iC|_{\mathcal{M}}\|^2$ are continuous functions with respect to t. Since

$$\begin{aligned}\|-Q_t+iC|_{\mathcal{M}}\|^2 &\geqq \|(Q_t)^2+(C|_{\mathcal{M}})^2\| \\ &\geqq \max\left\{\|F_1\|^2, \|F_2\|^2\right\}\end{aligned}$$

hence

$$\begin{aligned}\|-Q_{\varepsilon_0}+iC|_{\mathcal{M}}\|^2 &\geqq \|(Q_{\varepsilon_0})^2+(C|_{\mathcal{M}})^2\| \\ &\geqq \delta^2+\varepsilon_0^2 .\end{aligned}$$

It can be shown that for each t

$$\|(Q_t)^2+(C|_{\mathcal{M}})^2\|^{\frac{1}{2}}<\|-Q_t+iC|_{\mathcal{M}}\| .$$

In fact any unit vector x in $\mathcal{M}$ can be represented as $x=\cos\theta\, x_1\oplus\sin\theta\, x_2$ where $x_i\in\mathcal{M}_i$ and $\|x_i\|=1$ for $i=1, 2$, and $0\leqq\theta\leqq\frac{\pi}{2}$. Then

$$\begin{aligned}\left(\left\{(Q_t)^2+(C|_{\mathcal{M}})^2\right\} x, x\right) &= \cos^2\theta\,(E_1x_1, x_1)+\sin^2\theta\,(E_2x, x_2) \\ &+2t\sin\theta\cos\theta\,\mathrm{Re}\left\{(D_1Ux_2, x_1)+(UD_2x_2, x_1)\right\} .\end{aligned}$$

Since $a_i=\|C|_{\mathcal{M}_i}\|$ and $b_i=\inf\,\{\|C|_{\mathcal{M}_i}x\| : x\in\mathcal{M}_i,\ \|x\|=1\}$, it holds that for each t such that $0<t\leqq\varepsilon_0$ and for $i=1, 2$

$$\|D_i\| = (\delta^2-b_i^2)^{\frac{1}{2}}-\varepsilon_0+t$$

and

$$\|F_i\| = \delta^2+t^2-2(\varepsilon_0-t)(\delta^2-a_i^2)^{\frac{1}{2}} .$$

Put

$$X_i = \|F_i\| \text{ for } i=1, 2$$

and

$$Y=2t(\|D_1\|+\|D_2\|) .$$

Then

$$\begin{aligned}&\left(\left\{(Q_t)^2+(C|_{\mathcal{M}})^2\right\} x, x\right) \\ \leqq\ & \sup\left\{X_1\cos^2\theta+X_2\sin^2\theta+Y\sin\theta\cos\theta : 0\leqq\theta\leqq\frac{\pi}{2}\right\} \\ =\ & \sup\left\{\frac{1}{2}(X_1+X_2)+\frac{1}{2}(X_1-X_2)\cos 2\theta+\frac{1}{2}Y\sin 2\theta : 0\leqq\theta\leqq\frac{\pi}{2}\right\} \\ =\ & \frac{1}{2}(X_1+X_2)^2+\frac{1}{2}\left\{(X_1-X_2)^2+Y^2\right\}^{\frac{1}{2}} .\end{aligned}$$

Hence

$$\|(Q_t)^2+(C|_{\mathcal{M}})^2\|\leqq\frac{1}{2}(X_1+X_2)^2+\frac{1}{2}\left\{(X_1-X_2)^2+Y^2\right\}^{\frac{1}{2}} .$$

Put

$$Z=2t\left[\left\{(\delta^2-a_1^2)^{\frac{1}{2}}+(\delta^2-a_2^2)^{\frac{1}{2}}-2\varepsilon_0+2t\right\}^2+(a_1-a_2)^2\right]^{\frac{1}{2}}.$$

Choose a sequence $\{x_n\}_{n=1}^{\infty}$ of unit vectors in $\mathcal{M}$ as follows:

$$x_n=\cos\theta\, x_{1(n)}\oplus\sin\theta\, x_{2(n)}$$

where $x_{i(n)}\in E(\mathcal{M}_i)$, $\|x_{i(n)}\|=1$ $(n=1, 2, \cdots)$ for $i=1, 2$, θ is a constant such that $0\leqq\theta\leqq\frac{\pi}{2}$,

$$\lim_{u\to\infty}\left\{C_2\, x_{2(n)}-a_2\, x_{2(n)}\right\}=0,$$

and

$x_{1(n)}=zUx_{2(n)}$ $(n=1, 2, \cdots)$ where z is a complex number such that

$$2t\left\{(\delta^2-a_1^2)^{\frac{1}{2}}+(\delta^2-a_2^2)^{\frac{1}{2}}-2\varepsilon_0+2t+i(a_1-a_2)\right\}z=Z.$$

It is easy that

$$\lim_{n\to\infty}\left\{C_1\, x_{1(n)}-a_1\, x_{1(n)}\right\}=0.$$

Then

$$\begin{aligned}&\lim_{n\to\infty}\|(-Q_t+iC|_{\mathcal{M}})\, x_n\|^2\\&=\lim_{n\to 8}\Big[\cos^2\theta(F_1\, x_{1(n)},\, x_{1(n)})+\sin^2\theta(F_2\, x_{2(n)},\, x_{2(n)})\\&+2t\sin\theta\cos\theta\ \mathrm{Re}\left\{((D_1U+UD_2+iC_1U-iUC_2)\, x_{2(n)},\, x_{1(n)})\right\}\Big]\\&=X_1\cos^2\theta+X_2\sin^2\theta+Z\ \sin\theta\cos\theta.\end{aligned}$$

Hence

$$\begin{aligned}&\|-Q_t+iC|_{\mathcal{M}}\|^2\\&\geqq\sup\left\{X_1\cos^2\theta+X_2\sin^2\theta+Z\ \sin\theta\cos\theta: 0\leqq\theta\leqq\frac{\pi}{2}\right\}\\&=\frac{1}{2}(X_1+X_2)+\frac{1}{2}\left\{(X_1-X_2)^2+Z^2\right\}^{\frac{1}{2}}.\end{aligned}$$

Obviously for sufficiently small ε_1, Z is larger than Y. Hence for each t such that $0<t\leqq\varepsilon_0$,

$$\|(Q_t)^2+(C|_{\mathcal{M}})^2\|^{\frac{1}{2}}<\|-Q_t+iC|_{\mathcal{M}}\|.$$

Since $\|-Q_t+iC|_{\mathcal{M}}\|<\delta$ for sufficiently small t and $\|(Q_t)^2+(C|_{\mathcal{M}})^2\|^{\frac{1}{2}}>\delta$ for t sufficiently near ε_0, there exists a positive number t_0 such that $t_0<\varepsilon_0$,

$$\|(Q_{t_0})^2+(C|_{\mathcal{M}})^2\|^{\frac{1}{2}}=\delta \text{ and } \|-Q_{t_0}+iC|_{\mathcal{M}}\|>\delta.$$

Then

$$|||A-P_{t_0}|||=\delta \text{ and } \|A-P_{t_0}\|>\delta\,.$$

Hence P_{t_0} is contained in $\mathscr{P}_n(A)$ but not in $\mathscr{P}(A)$. C_1U is not equal to UC_2 since $\sigma(C|_{\mathscr{M}_1})\neq\sigma(C|_{\mathscr{M}_2})$ hence P_{t_0} does not commute wit $C|_{\mathscr{H}_0}$.

Other cases can be similarly proved.

Case II. σ_1 *satisfies condition* (1) *or* (2) *and* σ_2 *satisfies condition* (2). Since $C|_{\mathscr{M}_2}$ is scalar, by choosing an arbitrary unitary operator U in the proof of Case *I*, the proof can be shown similarly as Case *I*.

Case III. σ_1 *and* σ_2 *satisfy condition* (1) *and condition* (3) *respectively.* Choose an isometric operator V such that there exists a positive number s_0 less than ε but sufficiently near ε_1 and

$$V(\mathscr{M}_2)\subseteqq \operatorname{Ran}(\sigma_1(s_0))$$

instead of a unitary operator U in the proof of Case *I*, and define a positive operator F_1 in the proof of Case *I* by

$$F_1=\sigma^2+t^2VV^*+(\varepsilon_0-t)^2-2(\varepsilon_0-t)(\delta^2-C_1^2)^{\frac{1}{2}}\,.$$

Case IV. σ_1 *and* σ_2 *satisfy condition* (2) *and condition* (3) *respectively.* An isometric operator V in the proof of Case *III* can be chosen arbitrarily.

Case V. Both σ_1 *and* σ_2 *satisfy condition* (3). Since $\dim \mathscr{M}_1=\dim \mathscr{M}_2=1$, the proof is obvious. The proof is completed.

We show a sufficient and necessary condition for that $\mathscr{P}(A)$ coincides with $\mathscr{P}_n(A)$ as corollary of Theorem 2. 1.

COROLLARY 2. 2. *Let A be a normal operator. The following conditions are equivalent*:

(a) $\mathscr{P}(A)\subseteqq\{\operatorname{Im} A\}'$,

(b) $(\operatorname{Im} A)_{\mathscr{H}_0}=\lambda I_{\mathscr{H}_0}$ *where* $I_{\mathscr{H}_0}$ *is the identity operator on* $\mathscr{H}_0$ *and* λ *is a real number.*

(c) $\sigma(A)\subseteqq\Gamma_\delta\cup\{z:\operatorname{Im} z=\lambda\}$,

(d) $\mathscr{P}(A)=\mathscr{P}_n(A)$.

PROOF. The implications (b)$\Leftrightarrow$(c), (b)$\Rightarrow$(a) and (b)$\Rightarrow$(d) are obvious since $A-P$ is a normal operator for any P in $\mathscr{P}_n(A)$. By the proof of Theorem 2. 1 the implication (d)$\Rightarrow$(b) holds, and moreover for sufficiently small positive number t there exists a positive operator P_t in $\mathscr{P}(A)$ such that $P_t|_{\mathscr{H}_0}$ does not commute with $C|_{\mathscr{H}_0}$. Hence the implication (a)$\Rightarrow$(b) holds.

COROLLARY 2. 3. *Let A be a normal operator. The following con-*

ditions are equivalent:

(a) $\mathscr{P}(A)\subseteqq\{A\}'$,

(b) $A|_{\mathscr{H}_0}=\lambda I_{\mathscr{H}_0}$ *where* λ *is a complex number.*

(d) $\sigma(A)\subseteqq\Gamma_\delta\cup\{\lambda\}$.

PROOF. The implications (b)$\Longleftrightarrow$(c) and (b)$\Longleftrightarrow$(a) are obvious.

(a)$\Rightarrow$(b): $\mathscr{P}(A)\subseteqq\{C\}'$ holds since $\mathscr{P}(A)\subseteqq\{A\}'$, hence $C|_{\mathscr{H}_0}$ is scalar. Moreover $\mathscr{P}(A)\subseteqq\{B\}'$ holds. Suppose $B|_{\mathscr{H}_0}$ is non-scalar. Choose two non-trivial orthogonal subspace $\mathscr{M}_1$ and $\mathscr{M}_2$ included in $\mathscr{H}_0$ such that $\mathscr{M}_i$ is the range of a spectral projection of B for $i=1,2$, and there exists a positive number ε_2 such that for $i=1,2$

$$\left\{(B_-)^2+C^2\right\}\Big|_{\mathscr{M}_i}\leqq\delta^2-\varepsilon_2\,.$$

Define a positive operator P_t on $\mathscr{H}$ for sufficiently small positive number t such that the subspace $\mathscr{M}=\mathscr{M}_1\oplus\mathscr{M}_2$ is a P_t-reducing subspace, $P_t|_{\mathscr{M}^\perp}=P_0|_{\mathscr{M}^\perp}$ and $P_t|_{\mathscr{M}}$ is represented as matrix of operators on $\mathscr{M}=\mathscr{M}_1\oplus\mathscr{M}_2$:

$$P_t|_{\mathscr{M}}=\begin{bmatrix} t & tU\\ tU^* & t\end{bmatrix}+B_+|_{\mathscr{M}}$$

where U is a partially isometric operator mapping $\mathscr{M}_2$ into $\mathscr{M}_1$. For sufficiently small t,

$$\|(A-P_t)|_{\mathscr{M}}\|=|||(A-P_t)|_{\mathscr{M}}|||\leqq\delta$$

since $C|_{\mathscr{H}_0}$ is scalar. Hence P_t is contained in $\mathscr{P}(A)$ and does not commute with B. This contradicts to condition (a). The proof is completed.

3. The extremal points of $\mathscr{P}(A)$. In this section we consider a condition for that $\exp(\mathscr{P}(A))$ is a finite set where $\mathrm{ext}(\mathscr{P}(A))$ denotes the set of all extremal points of $\mathscr{P}(A)$.

THEOREM 3.1. *Let* A *be a normal operator. The following conditions are equivalent*:

(a) $\mathscr{P}(A)\subseteqq\{A\}''$,

(b) $\mathrm{ext}(\mathscr{P}(A))$ *consists of at most countable operators,*

(c) $\mathrm{ext}(\mathscr{P}(A))$ *consists of at most two operators,*

(d) $\dim\mathscr{H}_0\leqq1$,

(e) P *is a linear combination of* $(\mathrm{Re}\,A)_+$ *and* P_0 *for any operator* P *in* $\mathscr{P}(A)$.

PROOF. The implications (c)$\Rightarrow$(d)$\Rightarrow$(e) hold by the result in [1]. Moreover, the implications (e)$\Rightarrow$(a) and (c)$\Longleftrightarrow$(b) hold obviously.

(a)$\Rightarrow$(d): Since $\mathscr{P}(A)\subsetneqq\{A\}''$, $\mathscr{P}(A)\subsetneqq\{A\}'$ holds. By Corollary 2.3 $A|_{\mathscr{H}_0}$ is scalar. If dim $\mathscr{H}_0\geqq 2$ holds, then by the proof of Corollary 2.3 there exist a subspace $\mathscr{M}$ included in $\mathscr{H}_0$ such that dim $\mathscr{M}\geqq 2$ and a positive operator P in $\mathscr{P}(A)$ such that $\mathscr{M}$ is a P-reducing subspace and $P|_{\mathscr{M}}$ is non-scalar. This is a contradiction.

(b)$\Rightarrow$(d): Suppose ext$(\mathscr{P}(A))$ is at most countable and dim $\mathscr{H}\geqq 2$. For any closed subspace $\mathscr{M}$ included in $\mathscr{H}_0$ such that $\mathscr{M}$ is the range of a spectral projection of A, there exists a positive operator P_1 in $\mathscr{P}(A)$ such that P_1 differs from P_0, $P_1|_{\mathscr{M}^\perp}=P_0|_{\mathscr{M}^\perp}$ and $\mathrm{Ran}(P_0-P_1)^-\subseteqq\mathscr{M}$. If P_1 is not contained in ext$(\mathscr{P}(A))$, there exist two operators P_2 and P_3 in ext$(\mathscr{P}(A))$ and a positive number λ such that $0<\lambda<1$,

$$P_1=\lambda P_2+(1-\lambda)P_3 \text{ and } P_2\neq P_0.$$

Since $P_0-P_1=\lambda(P_0-P_1)+(1-\lambda)(P_0-P_3)$ and all operators P_0-P_1, P_0-P_2 and P_0-P_3 are positive, by Douglas' theorem [3]

$$\mathrm{Ran}(P_0-P_1)^{\frac{1}{2}}\supseteqq\mathrm{Ran}(P_0-P_2)^{\frac{1}{2}}$$

holds. Hence

$$\mathscr{M}\supseteqq\mathrm{Ran}(P_0-P_1)^-\supseteqq\mathrm{Ran}(P_0-P_2)^-.$$

By choosing P_2 instead of P_1, it can be assumed that $P_1\in$ext$(\mathscr{P}(A))$. If any operator P in ext$(\mathscr{P}(A))$ is commuting with all spectral projections of A, then $\mathscr{P}(A)\subseteqq\{A\}'$. This contradicts to dim $\mathscr{H}_0\geqq 2$ by the proof of the implication (a)$\Rightarrow$(d). Hence there exist two non-trivial orthogonal subspace $\mathscr{M}_1$ and $\mathscr{M}_2$ included in $\mathscr{H}_0$ such that $\mathscr{M}_i$ is the range of a spectral projection G_i of A for $i=1, 2$, and a positive operator P in ext$(\mathscr{P}(A))$ such that P does not commute with both G_1 and G_2, and

$$\mathrm{Ran}(P_0-P)^-\subseteqq\mathscr{M}_1\oplus\mathscr{M}_2.$$

For any unitary operator U commuting with A, $U^*PU\in$ext$(\mathscr{P}(A))$ holds. Choose a unitary operator U_θ commiting with A such that $U_\theta|_{\mathscr{M}}$ is defined as matrix of operators on $\mathscr{M}=\mathscr{M}_1\oplus\mathscr{M}_2$:

$$U_\theta|_{\mathscr{M}}=\begin{bmatrix}1 & 0\\ 0 & e^{i\theta}\end{bmatrix}$$

and

$$U_\theta|_{\mathscr{M}^\perp}=I_{\mathscr{M}^\perp}.$$

Put

$$P|_{\mathscr{M}}=\begin{bmatrix}P_{11} & P_{12}\\ P_{12}^* & P_{22}\end{bmatrix}$$

as matrix of operators on $\mathscr{M}=\mathscr{M}_1 \oplus \mathscr{M}_2$.
Then

$$(U_\theta^* PU_\theta)|_{\mathscr{M}} = \begin{bmatrix} P_{11} & e^{i\theta}P_{12} \\ e^{-i\theta} P_{12}^* & P_{22} \end{bmatrix} \text{ and } (U_\theta^* PU_\theta)|_{\mathscr{M}^\perp} = P_0|_{\mathscr{M}^\perp}.$$

Obviously $\{U_\theta^* PU_\theta : 0 \leqq \theta < 2\pi\}$ is uncountable, this contradicts to condition (b). Hence $\dim \mathscr{K}_0 \leqq 1$. The proof is completed.

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