On balanced projectives and injectives over linearly compact rings

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Dedicated Professor Kiiti Morita on his 60th birthday

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Introduction

Let ${}_RM$ be a left R-module over a ring R[1] and C be the biendomorphism ring of ${}_RM$. Then there exists a canonical ring homomorphism δ of R into C which is defined by $\delta(r)(m)=rm$, $r\in R$, $m\in M$. ${}_RM$ is called balanced if δ is an epimorphism. It is shown that Morita-Suzuki's criterion[2] for δ to be an isomorphism is easily generalized for modules from the view point of reflexivity. Thus we have the following

THEOREM (THEOREM 1). *Let R, S be two rings. Let ${}_RX$ be a left R-module and ${}_RZ_S$ be a two-sided R-S-module. Then the following statements are equivalent:*

(1) *${}_RX$ is Z-reflexive.*

(2) (i) *The Z-dual of ${}_RX$ is Z-reflexive*

(ii) *There exists an exact sequence of left R-modules*

$$0\to X\to \Pi Z\to \Pi Z,$$

where ΠZ's denote the direct products of copies of Z, though the index sets are generally different[3].

Let ${}_RP$ be a finitely generated projective module and ${}_RQ$ be an injective module with essential socle such that each simple homomorphic image of ${}_RP$ is isomorphic to a submodule of ${}_RQ$ and each simple submodule of ${}_RQ$ is a homomorphic image of ${}_RP$. Let S and T be the endomorphism ring of ${}_RP$ and ${}_RQ$, respectively. Then the left S-module ${}_S\mathrm{Hom}_R(P, Q)$ is an injective cogenerator with the endomorphism ring T, and the $\mathrm{Hom}_R(P, Q)$-dual of ${}_SP^*=\mathrm{Hom}_R(P, R)$ is isomorphic to Q_T[4]. It is shown that

1) In what follows we assume that all rings have an identity element and all modules are unital.

2) Cf. [5].

3) That is, Z-dominant dimension of $X\geqq 2$ in the terminology of [5].

4) See Lemma 3 and Theorem 1, [8]. There one can easily replace cofinitely generated injectiveness for ${}_RQ$ by injectiveness with essential socle, as T. Kato pointed out to the author.

${}_SP^*$ is $\mathrm{Hom}_R(P, Q)$-reflexive if and only if ${}_RQ$ satisfies the F_h-condition[5], that is, $\mathrm{Hom}_R(C, Q)$ is canonically isomorphic to Q, C being the biendomorphism ring of ${}_RQ$ (Theorem 2). Thus, in this case, the endomorphism rings of ${}_SP^*$ and Q_T are isomorphic (Corollary to Theorem 2).

Let R be a left linearly compact ring. Then it is shown that every injective left R-module with essential socle satisfies the F_h-condition (Theorem 3).

As an application of our considerations, we obtain the following theorem which generalizes the results in [3].

THEOREM (THEOREM 4). *Let R be a left linearly compact ring. Then the following statements are equivalent*:

(1) *Every (faithful) finitely generated projective right R-module is balanced.*

(2) *Every (faithful) projective right R-module with small radical is balanced.*

(3) *Every (faithful) cofinitely generated injective left R-module is balanced.*

(4) *Every (faithful) injective left R-module with essential socle is balanced.*

§ 1. Regular pairing of modules and endomorphism rings

Let R, S be rings and ${}_RX$, ${}_RZ_S$, Y_S be left R-, two-sided R-S-, right S-modules, respectively. Suppose that there is a bilinear mapping $X \times Y \to (\ ,\ ) \in Z$ which satisfies the following condition:

$(x, y)=0$ for all $x \in X$ implies $y=0$, and $(x, y)=0$ for all $y \in Y$ implies $x=0$.

We call such a pairing a regular pairing. In this case there is a canonical monomorphism $\varphi(\psi)$ of $X(Y)$ into $\mathrm{Hom}_S(Y, Z)(\mathrm{Hom}_R(X, Z))$ which is defined by $\varphi(x)(y)=(x, y)\ (\psi(y)(x)=(x, y))$ for $x \in X$, $y \in Y$.

LEMMA 1. *If both φ and ψ are isomorphisms, then the endomorphism rings of ${}_RX$ and Y_S are isomorphic.*

PROOF. 1. Let t be an endomorphism of ${}_RX$. Then, by assumption, t defines an (unique) endomorphism $\hat{t}$ of Y_S by

$$(x, \hat{t}y)=(xt, y), \quad x \in X, \ y \in Y.$$

Similarly each endomorphism $\hat{t}$ of Y_S defines an (unique) endomorphism t of ${}_RX$ by the above relation. This proves our lemma.

5) Cf. [5].

§ 2. A reflexivity condition for modules

Let ${}_RX$ be a left R-module and ${}_RZ_S$ be a two-sided R-S-module. We denote the Z-dual of ${}_RX$ by $X^*:=\mathrm{Hom}_R(X, Z)$, which is considered as a right S-module. Further by X^{**}, X^{***} we denote the Z-dual of X^*_S, The Z-dual of ${}_RX^{**}$, respectively: $X^{**}=\mathrm{Hom}_S(X^*, Z)$, $X^{***}=\mathrm{Hom}_R(X^{**}, Z)$. Then there exists a natural homomorphism δ_X of X into X^{**} which is defined by

$$\delta_X(x)(f)=f(m) \text{ for } x\in X,\ f\in X^*.$$

When δ_X is an isomorphism (a monomorphism), X is called Z-reflexive (Z-torsionless).

THEOREM 1. *The following statements are equivalent*:

(1) ${}_RX$ *is* Z-*reflexive*

(2) (i) X^*_S *is* Z-*reflexive*

(ii) *There exists an exact sequence of left* R-*modules*

$$0\to X\to \Pi Z\to \Pi Z,$$

where ΠZ*'s denote the direct products of copies of* Z, *though the index sets are generally different.*

PROOF. (1)$\Rightarrow$(2). As is easily verified we have $\delta^*_X\delta_{X^*}=1_{X^*}$, where $\delta^*_X=\mathrm{Hom}(\delta_X, 1_Z): X^{***}\to X^*$. Since δ_X is an isomorphism, δ^*_X, whence δ_{X^*} is an isomorphism. Thus X^* is Z-reflexive. Let $F_1\to F_2\to X^*\to 0$ be an exact sequence of right S-modules, where F_1, F_2 are free right S-modules. Then we have the exact sequence of left R-modules $0\to X^{**}\to \mathrm{Hom}_S(F_2, Z)\to \mathrm{Hom}_S(F_1, Z)$. Since X is Z-reflexive and $\mathrm{Hom}_S(F_i, Z)$, $i=1, 2$, are isomorphic to direct products of copies of Z, this proves the assertion (ii).

(2)$\Rightarrow$(1). From the exact sequece $0\to X\to \Pi Z\to \Pi Z$, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccc} 0\to \mathrm{Hom}_R(X^{**}, X) & \to & \Pi\mathrm{Hom}_R(X^{**}, Z) & \to & \Pi\mathrm{Hom}_R(X^{**}, Z) \\ \downarrow{\scriptstyle \mathrm{Hom}(\delta_X, 1_X)} & & \downarrow{\scriptstyle \Pi\delta^*_X} & & \downarrow{\scriptstyle \Pi\delta^*_X} \\ 0\to \mathrm{Hom}_R(X, X) & \to & \Pi\mathrm{Hom}_R(X, Z) & \to & \Pi\mathrm{Hom}_R(X, Z). \end{array}$$

Since, by assumption, δ_{X^*}, whence δ^*_X is an isomorphism, $\Pi\delta^*_X$'s are isomorphisms. It follows that $\mathrm{Hom}(\delta_X, 1_X)$ is an isomorphism. Let φ be an element of $\mathrm{Hom}_R(X^{**}, X)$ such that $\varphi\cdot\delta_X=1_X$. Then φ is an epimorphism, and $\delta^*_X\varphi^*=1_{X^*}$, where $\varphi^*=\mathrm{Hom}(\varphi, 1_Z)$. Since δ^*_X is an isomorphism, together with the relation $\delta^*_X\delta_{X^*}=1_{X^*}$, we have $\varphi^*=\delta_{X^*}$. Let $g\in X^{**}$ such that $\varphi(g)=0$. Then we have $f(\varphi(g))=0$ for all $f\in X^*$. But $\{f\varphi=\varphi^*(f) \mid f\in X^*\}=\{\delta_{X^*}(f)\mid$

$f\in X^*\}=X^{***}$, because δ_{X^*} is an isomorphism. Since X^{**} is, as the Z-dual of X^*, Z-torsionless, it follows that $g=0$. Thus φ is a monomorphism, whence an isomorphism. It follows that δ_X is an isomorphism, that is, ${}_RX$ is Z-reflexive.

Let ${}_RM$ be a left R-module, S the endomorphism ring of ${}_RM$, and, C be the endomorphism ring of the right S-module M_S. Then by setting ${}_RX={}_RR$, ${}_RZ_S={}_RM_S$ in Theorem 1 we have the following

COROLLARY (Morita-Suzuki). *The following statements are equivalent*:

(1) ${}_RM$ *is faithfull and balanced*

(2) (i) $Hom_R(C, M)$ *is isomorphic to* M *under the mapping* $Hom_R(C, M)\ni f\to f(1)\in M$.

(ii) *There exists an exact sequence of left R-modules*:

$$0\to R\to \Pi M\to \Pi M, \quad \textit{that is},$$

M-dominant dimension of ${}_RR\geqq 2$.

The condition (i) in (2) is called the F_h-condition for ${}_RM$.

§ 3. Generalized RZ-pairs

Let ${}_RP$ be a finitely generated projective module and ${}_RQ$ be an injective module with an essential socle. We call the pair $\{P, Q\}$ forms a generalized RZ-pair if every simple homomorphic image of ${}_RP$ is isomorphic to a submodule of ${}_RQ$, and, every simple submodule of ${}_RQ$ is a homomorphic image of ${}_RP$.

LEMMA 2. *Let* $\{P, Q\}$ *forms a generalized RZ-pair and* S, T *be the endomorphism rings of* ${}_RP$, ${}_RQ$, *respectively. Then the left S-module* ${}_SHom_R(P, Q)$ *is an injective cogenerator and the endomorphism ring of* ${}_SHom_R(P, Q)$ *is naturally isomorphic to* T. *Further, the* $Hom_R(P, Q)$*-dual of* ${}_SP^*$ *is isomorphic to* Q_T, *where* P^* *is the R-dual of* $P_R: P^*=Hom_R(P, R)$.

PROOF. The first assertion follows from Theorem 1, [9], while the latter assertions follows from Lemma 2, [9].

THEOREM 2. *Under the same assumptions as in Lemma* 2, *the following statements are equivalent*:

(1) ${}_SP^*$ *is* $Hom_R(P, Q)$*-reflexive*

(2) ${}_RQ$ *satisfies the* F_h*-condition*

(3) Q_T *is* $Hom_R(P, Q)$*-reflexive.*

PROOF. Let C be the endomorphism ring of Q_T. The $\mathrm{Hom}_R(P, Q)$-

dual of Q_T, $\mathrm{Hom}_T(Q, \mathrm{Hom}_R(P, Q))$ is isomorphic to $\mathrm{Hom}_R(P, Q)$[6], and, the $\mathrm{Hom}_R(P, Q)$-dual of ${}_S\mathrm{Hom}_R(P, C)$ is $\mathrm{Hom}_S(\mathrm{Hom}_R(P, C), \mathrm{Hom}_R(P, Q))$, which is isomorphic to $\mathrm{Hom}_R(C, Q)$ by Lemma 2, [9]. Thus we see that Q_T is $\mathrm{Hom}_R(P, Q)$-reflexive if and only if ${}_RQ$ satisfies the F_h-condition. This proves the equivalence (2)$\Longleftrightarrow$(3).

On the other hand, since ${}_S\mathrm{Hom}_R(P, Q)$ is an injective cogenerator and Q_T is the $\mathrm{Hom}_R(P, Q)$-dual of ${}_SP^*$, we see that ${}_SP^*$ is $\mathrm{Hom}_R(P, Q)$-reflexive if and only if Q_T is $\mathrm{Hom}_R(P, Q)$-reflexive by Theroem 1. This implies the equivalence (1)$\Longleftrightarrow$(3).

COROLLARY. *If one of the equivalence conditions in Theorem* 2 *is satisfied, then P_R^* is balanced if and only if ${}_RQ$ is balanced.*

PROOF. Since ${}_RP$ is finitely generated projective, the endomorphism ring of P_R^* is isomorphic to S[7]. Consider the regular pairing of ${}_SP^*$ and Q_T in ${}_S\mathrm{Hom}_R(P, Q)_T$ which is defined by

$$(f, q)(p) = f(p)\,q\,, \quad f\in P^*\,, \quad q\in Q\,, \quad p\in P\,.$$

This is a regular pairing by Lemma 3, [9]. Further, by assumption, ${}_SP^*$, Q_T are the $\mathrm{Hom}_R(P, Q)$-dual of each others. The corollary follows then direct from Lemma 1.

§4. Injective modules with essential socles over linearly compact rings.

Let ${}_RM$ be a left R-module. ${}_RM$ is called linearly compact if every finitely solvable system of congruences

$$x \equiv m_\alpha (\mathrm{mod}\ \mathcal{U}_\alpha)\,, \quad \alpha\in I\,,$$

is solvable, where m_α's are elements of ${}_RM$, $\mathcal{U}_\alpha$'s are submodules of ${}_RM$, and I is an index set. A ring R is called left (right) linearly compact if ${}_RR\,(R_R)$ is linearly compact. It is known that a one-sided linearly compact ring is a semi-perfect ring[8].

LEMMA 3. *Let R be a left linearly compact ring and ${}_RQ$ be a quasi-injective left R-module with an essential socle. Let S be the endomorphism ring of ${}_RQ$. For every natural number n, we define the bilinear mapping $[\ ,\ ]$ of $R^{(n)}\times Q^{(n)}$ into ${}_RQ_S$ by $\left[(r_1, \cdots, r_n), (q_1, \cdots, q_n)\right] = \sum_{i=1}^{n} r_i\, q_i \in Q$, where $R^{(n)}$,*

6) Cf. [1], p. 32, Exercise 4.

7) Cf. [6], Folgerung 2, Beweis.

8) Cf. [7], Corolloary to Theorem 5.

$Q^{(n)}$ are the direct sums of n-copies of R, Q, respectively. Then for every S-submodule $\mathscr{U}$ of $Q^{(n)}$, we have Ann $_{Q^{(n)}}$(Ann $_{R^{(n)}}(\mathscr{U}))=\mathscr{U}$, where, as usual, $Ann_Y(X)$ denotes the annihilator of X in Y. with respect to the given bilinear mapping[9].

PROOF. Let $q=(q_1, \cdots, q_n)$ be an element of Ann $_{Q^{(n)}}$(Ann $_{R^{(n)}}(\mathscr{U})$). Then we have Ann $_{R^{(n)}}(q)\supseteq$ Ann $_{R^{(n)}}(\mathscr{U})=\bigcap_{u\in\mathscr{U}}$ Ann $_{R^{(n)}}(u)$. Since $R^{(n)}/$Ann $_{R^{(n)}}(q)$ is R-isomorphic to the submodule $[R^{(n)}, q]$ of Q, which is, as a homomorphic image of $R^{(n)}$, linearly compact[10], whence cofinitely generated[11], there exists a finite number of elements $u_1, \cdots, u_t$ of $\mathscr{U}$ such that Ann $_{R^{(n)}}(q)\supseteq\bigcap_{i=1}^{t}$ Ann $_{R^{(n)}}(u_i)$. Let $X=\{([r, u_1], \cdots, [r, u_t])|r\in R^{(n)}\}$ and define the well defined R-homomorphism φ of X into $Q^{(n)}$:

$$X\ni([r, u_1], \cdots, [r, u_t])\xrightarrow{\varphi}[r, q]\in Q.$$

Since $_RQ$ is quasi-injective, φ is extended to an R-homomorphism of $Q^{(n)}$ into Q. Thus there exist elements $s_1, \cdots, s_n$ of S such that $[r, q]=[r, \sum_{i=1}^{t} u_i s_i]$ for all $r\in R^{(n)}$, and, from which we have $q=\sum_{i=1}^{t} u_i s_i\in\mathscr{U}$.

THEOREM 3. *Let $_RQ$ be an injective left R-module with an essential socle over a left linearly compact ring R. Let S be the endomorphism ring of $_RQ$ and C the endomorphism ring of Q_S. Then Q is isomorphic to $Hom_R(C, Q)$ under the mapping $Hom_R(C, Q)\ni f\rightarrow f(1)$, that is, $_RQ$ satisfies the F_h-condition. Further, the left C-module $_CQ$ is injective.*

PROOF. Let $f\in$ Hom$_R(C, Q)$. Then for each $c\in C$, we have $f(c)\in cQ$. Because if $f(c)\notin cQ$ then by Lemma 3 there exists an element $r\in R$ such that $rcQ=0$, $rf(c)\neq 0$. But this is a contradiction. Let $f(c)=cq_c$, $c\in C$, $q_c\in Q$. Then again by Lemma 3 we see that the system of congruences,

$$x\equiv q_c(\text{mod Ann}_Q(c)), \quad c\in C$$

is finitely solvable. Since Q_S is linearly compact[12], there exists an element $q_0\in Q$ such that $q_0\equiv q_c$(mod Ann$_Q(c)$) for all $c\in C$. This implies that $cq_0=f(c)$, $c\in C$, and, from which it is easy to see that $_RQ$ satisfies the F_h-condition. The last assertion is also proved in a similar way.

9) Cf. [8], Proposition 4.
10) Cf. [7], Proposition 8.
11) Cf. [7], Proposition 3.
12) Cf. [8], Proposition 4.

§ 5. An application

As an application of our considerations we have the following

THEOREM 4. *Let R be aleft linearly compact ring. Then the following statements are equivalent*:

(1) *Every (faithful) finitely generated projective right R-module is balanced.*

(2) *Every (faithful) projective right R-module with small radical is balanced.*

(3) *Every (faithful) cofinitely generated injective left R-module is balanced.*

(4) *Every (faithful) injective left R-module with essential socle is balanced.*

PROOF. It is obvious that (2) implies (1) and (4) implies (3). (1)$\Rightarrow$(2). Let P_R be a (faithful) projective right R-module with small radical. P_R is isomorphic to $\bigoplus_{\alpha\in\Lambda} e_\alpha R$, where e_α's are primitive idempotents of R. Let $e_{\alpha_1}R, \cdots, e_{\alpha_t}R$ be a complete set of representatives of non-isomorphic modules of $\{e_\alpha R\,;\ \alpha\in\Lambda\}$. Then $P_0=\bigoplus_{i=1}^{t} e_{\alpha_i}R$, is a (faithful) finitely generated projective module and $P=P_0\oplus P_1$, where P_0 generates and cogenerates P_1. Our assertion follows then from Lemma, [3]. (3)$\Rightarrow$(1). Let P_R be a (faithful) finitely generated projective right R-module, and, S be the endomorphism ring of P_R. Let ${}_RQ$ be a cofinitely generated injective left R-module such that $\{{}_RP^*, {}_RQ\}$ forms an RZ-pair[13]. In this case, if P_R is faithful, then ${}_RQ$ is also faithful[14]. Our assertion follows then from Corollary to Theorem 2, because P_R is R-reflexive and ${}_RQ$ is balanced. (1)$\Rightarrow$(3). Let ${}_RQ$ be a (faithful) injective left R-module with essential socle. Let ${}_RP$ be a finitely generated projective left R-module such that $\{{}_RP, {}_RQ\}$ forms a generalized RZ-pair. In this case, if ${}_RQ$ is faithful, then P^*_R is also faithful. Further, the endomorphism ring of P^*_R is isomorphic to that of ${}_RP$ because ${}_RP$ is finitely generated projective. Since, by Theorem 3, ${}_RQ$ satisfies the F_h-condition, our assertion follows also from Corollary to Theorem 2.

COROLLARY[15]. *Let R be a left artinian ring. Then the following statements are equivalent*:

13) Cf. [8], § 2.

14) Cf. [8], Lemma 3.

15) Cf. [3].

(1) *Every (faithful) finitely generated projective right R-module is balanced.*
(2) *Every (faithful) projective right R-module is balanced.*
(3) *Every (faithful) cofinitely generated injective left R-module is balanced.*
(4) *Every (faithful) injective left R-module is balanced.*

ADDENDUM:

Recently K. Morita has sent the author his unpublished manuscript titled "Localization in category of modules IV", where one can see that our Theorem 1 is also obtained independently.

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