On complex semi-symmetric metric F-connection

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Introduction.

Recently K. Yano [5][1] and T. Imai [1, 2] have studied some topics on the curvature tensor of a semi-symmetric metric connection in a Riemannian manifold. Especially, they have given the condition that a semi-symmetric metric connection in a Riemannian manifold has no curvature.

On the other hand K. Yano [6] has introduced the concept of a complex conformal connection in a Kählerian manifold and K. Yano, U. K. Kim [3] and O. Yoon [8] have given the condition that the Bochner curvature tensor of a Kählerian manifold vanishes.

The purpose of the present paper is to introduce the concept of a complex semi-symmetric metric F-connection and study some properties of a complex semi-symmetric metric F-connection in a Kählerian manifold. We study the condition that the Bochner curvature tensor of a Kählerian manifold vanishes. Also we define the holomorphic sectional curvature with respect to a complex semi-symmetric metric F-connection under some assumptions and study another condition that the Bochner curvature tensor of a Kählerian manifold vanishes.

In §1, we give preliminary formulas on a Kählerian manifold. In §2, we define a complex semi-symmetric metric F-connection and give the relation between the components of a complex semi-symmetric metric F-connection and the Christoffel symbols. In §3, we give the curvature tensor of a complex semi-symmetric metric F-connection and obtain the condition that the Bochner curvature tensor of a Kählerian manifold vanishes. In the last section §4, we define the holomorphic sectional curvature with respect to a complex semi-symmetric metric F-connection under some assumptions and obtain some theorems.

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1) Numbers in brackets refer to the references at the end of the paper.

§ 1. Preliminaries.

Let us consider an $n(=2m>2)$ real dimensional Kählerian manifold with local coordinates $\{x^i\}$, where here and in the sequel the indices $h, i, j, k, \cdots$ run over the range $\{1, 2, \cdots, n\}$. Then the positive definite Riemannian metric g_{ji} and the complex structure F_i^h satisfy the following equations

$$(1.1)\qquad \begin{aligned} &F_j^a F_a^i = -\delta_j^i, & &g_{ba}F_j^b F_i^a = g_{ji}, \\ &\nabla_k F_j^h = 0, & &\nabla_k g_{ji} = 0, \end{aligned}$$

where ∇_k denotes the operator of the covariant differentiation with respect to the Christoffel symbols $\left\{ {h \atop ji} \right\}$ constructed from g_{ji}. The Riemannian curvature tensor $K_{kji}{}^h$ of g_{ji} is defined by

$$K_{kji}{}^h = \partial_k \left\{ {h \atop ji} \right\} - \partial_j \left\{ {h \atop ki} \right\} + \left\{ {h \atop ka} \right\}\left\{ {a \atop ji} \right\} - \left\{ {h \atop ja} \right\}\left\{ {a \atop ki} \right\}, \qquad \partial_j = \partial/\partial x^j.$$

We put $K_{kjih} = K_{kji}{}^a g_{ah}$, $K_{ji} = K_{aji}{}^a = K_{bjia} g^{ba}$, $K = K_{ba} g^{ba}$ and $S_{ji} = -K_{ja} F_i^a$, then the following identities are valid

$$(1.2)\qquad K_{aji}{}^h F_k^a = -K_{kai}{}^h F_j^a, \quad K_{kja}{}^h F_i^a = K_{kji}{}^a F_a^h, \quad K_{kjih} = K_{kjba} F_i^b F_h^a,$$

$$(1.3)\qquad K_{ji} = K_{ba} F_j^b F_i^a, \quad F_j^a K_{ai} = -K_{ja} F_i^a, \quad F_j^a K_a^h = K_j^a F_a^h,$$

$$(1.4)\qquad \begin{aligned} &S_{ji} + S_{ij} = 0, \quad S_{ji} = S_{ba} F_j^b F_i^a, \quad F_j^a S_{ai} = -S_{ja} F_i^a = -K_{ji}, \\ &S_{ji} = -\frac{1}{2} K_{jiba} F^{ba} = K_{bjia} F^{ba}. \end{aligned}$$

The Bochner curvature tensor $B_{kji}{}^h$ (S. Tachibana [4]) is given by

$$(1.5)\qquad \begin{aligned} B_{kji}{}^h = {} & K_{kji}{}^h + \delta_k^h L_{ji} - \delta_j^h L_{ki} + L_k^h g_{ji} - L_j^h g_{ki} + F_k^h M_{ji} - F_j^h M_{ki} \\ & + M_k^h F_{ji} - M_j^h F_{ki} - 2(M_{kj} F_i^h + F_{kj} M_i^h), \end{aligned}$$

where

$$(1.6)\qquad \begin{aligned} L_{ji} &= -\frac{1}{n+4} K_{ji} + \frac{1}{2(n+2)(n+4)} K g_{ji}, \\ M_{ji} &= -\frac{1}{n+4} S_{ji} + \frac{1}{2(n+2)(n+4)} K F_{ji} \end{aligned}$$

and $L_k^h = L_{ka} g^{ah}$, $M_k^h = M_{ka} g^{ah}$. Now we consider a tensor B_{kji} given by

$$(1.7)\qquad B_{kji} = -(\nabla_k L_{ji} - \nabla_j L_{ki}) + \frac{1}{2(n+2)(n+4)} (F_{ki} F_j^a - F_{ji} F_k^a + 2F_{kj} F_i^a) \nabla_a K,$$

then we can get the following identity:

$$(1.8)\qquad \nabla_a B_{kji}{}^a = n B_{kji}.$$

§ 2. Complex semi-symmetric metric F-connection.

Let M be an $n(=2m>2)$ real dimensional Kählerian manifold. We consider an affine connection D with components Γ_{ji}^h in M. Let $T_{ji}{}^h$ be the torsion tensor of D which is given by

$$T_{ji}{}^h=\Gamma_{ji}^h-\Gamma_{ij}^h. \tag{2.1}$$

If the connection D satisfies the equation

$$D_k g_{ji}=0, \tag{2.2}$$

then D is called a metric connection. If the connection D satisfies the equation

$$D_k F_i{}^h=0, \tag{2.3}$$

then D is called an F-connection. When the torsion tensor $T_{ji}{}^h$ of D is given by

$$T_{ji}{}^h=\delta_j{}^h p_i-\delta_i{}^h p_j-2F_{ji}q^h, \tag{2.4}$$

where p_i is a covector field and q^h is a vector field, we call the affine connection D a complex semi-symmetric connection.

Let Γ_{ji}^h be the components of a complex semi-symmetric metric F-connection D. If we put

$$\Gamma_{ji}^h=\left\{ {h \atop ji} \right\}+U_{ji}{}^h, \tag{2.5}$$

where $U_{ji}{}^h$ is a tensor field of type (1, 2), then from (2. 1) and (2. 5) we have

$$T_{ji}{}^h=U_{ji}{}^h-U_{ij}{}^h. \tag{2.6}$$

Since the connection D is metric, from (1. 1), (2. 2) and (2. 5) we have

$$U_{kji}+U_{kij}=0, \tag{2.7}$$

where $U_{kji}=U_{kj}{}^a g_{ai}$. From (2. 6) and (2. 7) we have

$$U_{ji}{}^h=\frac{1}{2}\left(T_{ji}{}^h+T^h{}_{ji}+T^h{}_{ij}\right), \tag{2.8}$$

where $T^h{}_{ji}=T_{aj}{}^b g^{ah} g_{bi}$. Since the connection D is an F-connection, from (1. 1), (2. 3) and (2. 5) we have

$$U_{ka}{}^h F_i{}^a-U_{ki}{}^a F_a{}^h=0. \tag{2.9}$$

Substituting (2. 4) into (2. 8), we find

$$U_{ji}{}^h=\delta_j{}^h p_i-g_{ji}p^h+F_j{}^h q_i+F_i{}^h q_j-F_{ji}q^h, \tag{2.10}$$

where $p^h = p_a g^{ah}$, $q_i = q^a g_{ai}$. Substituting (2. 10) into (2. 9) and contracting with respect to k and h, we find

$$q_i = -p_a F_i^{\ a}, \qquad p_i = q_a F_i^{\ a}. \tag{2.11}$$

Therefore the components Γ_{ji}^h of a complex semi-symmetric metric F-connection D are given by

$$\Gamma_{ji}^h = \left\{ {h \atop ji} \right\} + \delta_j^{\ h} p_i - g_{ji} p^h + F_j^{\ h} q_i + F_i^{\ h} q_j - F_{ji} q^h \tag{2.12}$$

with p_i and q_i satisfying the relations (2. 11).

§3. Curvature tensor of a complex semi-symmetric metric F-connection.

We denote by

$$R_{kji}^{\ \ \ h} = \partial_k \Gamma_{ji}^h - \partial_j \Gamma_{ki}^h + \Gamma_{ka}^h \Gamma_{ji}^a - \Gamma_{ja}^h \Gamma_{ki}^a, \qquad \partial_k = \partial/\partial x^k,$$

the curvature tensor of a complex semi-symmetric metric F-connection D with components Γ_{ji}^h. Then by a straightforward computation, we find

$$\begin{aligned} R_{kji}^{\ \ \ h} = K_{kji}^{\ \ \ h} - \delta_k^{\ h} P_{ji} + \delta_j^{\ h} P_{ki} - P_k^{\ h} g_{ji} + P_j^{\ h} g_{ki} - F_k^{\ h} Q_{ji} + F_j^{\ h} Q_{ki} \\ - Q_k^{\ h} F_{ji} + Q_j^{\ h} F_{ki} - \alpha_{kj} F_i^{\ h} - F_{kj} \beta_i^{\ h}, \end{aligned} \tag{3.1}$$

where

$$\begin{gathered} P_{ji} = \nabla_j p_i - p_j p_i + q_j q_i + \frac{1}{2} \lambda g_{ji}, \\ Q_{ji} = \nabla_j q_i - p_j q_i - q_j p_i + \frac{1}{2} \lambda F_{ji}, \\ \alpha_{kj} = -(\nabla_k q_j - \nabla_j q_k), \qquad \beta_i^{\ h} = 2(p_i q^h - q_i p^h), \qquad \lambda = p_a p^a \end{gathered} \tag{3.2}$$

and $P_j^{\ i} = P_{ja} g^{ai}$, $Q_j^{\ i} = Q_{ja} g^{ai}$. Then we get

$$Q_{ji} = -P_{ja} F_i^{\ a}, \qquad P_{ji} = Q_{ja} F_i^{\ a}, \qquad \alpha_{kj} = Q_{jk} - Q_{kj} + \lambda F_{kj}. \tag{3.3}$$

In (3. 1), contracting with respect to k and h, we have

$$R_{ji} = K_{ji} - (n-1) P_{ji} - P g_{ji} + F_j^{\ a} Q_{ai} - Q F_{ji} - \alpha_{aj} F_i^{\ a} - F_{aj} \beta_i^{\ a}, \tag{3.4}$$

where $R_{ji} = R_{aji}^{\ \ \ a}$, $P = P_{ba} g^{ba} = \nabla_a p^a + \frac{n}{2} \lambda$ and $Q = Q_{ba} g^{ba} = \nabla_a q^a$.

By transvecting (3. 4) with g^{ji} we obtain

$$R = K - 2(n+1) P + (n+4) \lambda, \tag{3.5}$$

where $R = R_{ba} g^{ba}$. If we define the tensors V_{kj} and H_{kj} by

$$V_{kj} = \frac{1}{2} R_{kja}^{\ \ \ b} F_b^{\ a} \quad \text{and} \quad H_{kj} = -R_{ka} F_j^{\ a},$$

from (3.1) and (3.4), we have

$$V_{kj}=S_{kj}+\frac{n+4}{2}\alpha_{kj}=S_{kj}+\frac{n+4}{2}(Q_{jk}-Q_{kj}+\lambda F_{kj}), \tag{3.6}$$

$$H_{kj}=S_{kj}-(n-1)Q_{kj}-PF_{kj}-F_k{}^aP_{aj}+Qg_{kj}+\alpha_{kj}+\beta_{kj}. \tag{3.7}$$

By transvecting (3.6) with F^{kj} we have

$$V=K-(n+4)P+\frac{n(n+4)}{2}\lambda, \tag{3.8}$$

where $V=V_{kj}F^{kj}$. From (3.5) and (3.8) we have

$$\begin{aligned}\lambda&=\frac{1}{n^2-4}\left(\frac{2(n+1)}{n+4}V-R\right)-\frac{1}{(n+2)(n+4)}K,\\ P&=\frac{1}{n^2-4}\left(V-\frac{n}{2}R\right)+\frac{1}{2(n+2)}K.\end{aligned} \tag{3.9}$$

If we assume that $R_{kji}{}^h=0$, then we have $R_{ji}=0$, $R=0$, $V_{kj}=0$ and $H_{ji}=0$ and consequently from (3.1), (3.4), (3.6), (3.7) and (3.9) we get

$$\begin{aligned}&K_{kji}{}^h-\delta_k{}^hP_{ji}+\delta_j{}^hP_{ki}-P_k{}^hg_{ji}+P_j{}^hg_{ki}-F_k{}^hQ_{ji}+F_j{}^hQ_{ki}\\ &\qquad -Q_k{}^hF_{ji}+Q_j{}^hF_{ki}-\alpha_{kj}F_i{}^h-F_{kj}\beta_i{}^h=0,\end{aligned} \tag{3.10}$$

$$K_{ji}=(n-1)P_{ji}+Pg_{ji}-F_j{}^aQ_{ai}+QF_{ji}+\alpha_{aj}F_i{}^a+F_{aj}\beta_i{}^a, \tag{3.11}$$

$$S_{kj}=-\frac{n+4}{2}\alpha_{kj}=\frac{n+4}{2}(Q_{kj}-Q_{jk}-\lambda F_{kj}), \tag{3.12}$$

$$S_{kj}=(n-1)Q_{kj}+PF_{kj}+F_k{}^aP_{aj}-Qg_{kj}-\alpha_{kj}-\beta_{kj}, \tag{3.13}$$

$$\lambda=-\frac{1}{(n+2)(n+4)}K,\qquad P=\frac{1}{2(n+2)}K. \tag{3.14}$$

On the other hand we know that K_{ji} is hybrid, i.e.,

$$K_{ji}-K_{ba}F_j{}^bF_i{}^a=0.$$

Taking use of this fact and (3.11), we have

$$(n-1)(P_{ji}+F_j{}^aQ_{ai})-(P_{ij}+F_i{}^aQ_{aj})=0,$$

and the symmetric part of the above equation is written as

$$(n-2)(P_{ji}+F_j{}^aQ_{ai}+P_{ij}+F_i{}^aQ_{aj})=0.$$

Hence from last two equations we obtain

$$P_{ji}=-F_j{}^aQ_{ai},\qquad Q_{ji}=F_j{}^aP_{ai}. \tag{3.15}$$

From (3.13) and (3.15) we have

(3. 16) $$Q_{ji}+Q_{ij}=\frac{2}{n}Qg_{ji},$$

and from (3. 3), (3. 12), (3. 13), (3. 14) and (3. 16) we obtain

(3. 17) $$\begin{aligned} &P_{ji}=-\frac{1}{n}QF_{ji}-L_{ji}, \qquad Q_{ji}=\frac{1}{n}Qg_{ji}-M_{ji},\\ &\alpha_{ji}=2M_{ji}-\frac{1}{(n+2)(n+4)}KF_{ji}, \quad \beta_{ji}=2M_{ji}+\frac{1}{(n+2)(n+4)}KF_{ji}, \end{aligned}$$

where L_{ji} and M_{ji} are defined by (1. 6).

Substituting (3. 17) into (3. 10), we find $B_{kji}{}^{h}=0$, that is, the Bochner curvature tensor of M vanishes. Thus we have

THEOREM 1. *Let M be an $n(=2m>2)$ real dimensional Kählerian manifold. If M admits a complex semi-symmetric metric F-connection which has no curvature, then the Bochner curvature tensor of M vanishes.*

§4. Holomorphic sectional curvature of a complex semi-symmetric metric F-connection.

Let D be a complex semi-symmetric metric F-connection. Then, from (3. 1) we have $R_{kji}{}^{h}=-R_{jki}{}^{h}$. Applying the Ricci's identity on the metric tensor g_{ji} and the complex structure $F_i{}^h$, we have

$$\begin{aligned} &D_kD_jg_{ih}-D_jD_kg_{ih}=-g_{ah}R_{kji}{}^{a}-g_{ia}R_{kjh}{}^{a}-T_{kj}{}^{a}D_ag_{ih},\\ &D_kD_jF_i^h-D_jD_kF_i^h=F_i^aR_{kja}{}^{h}-F_a^hR_{kji}{}^{a}-T_{kj}{}^{a}D_aF_i^h. \end{aligned}$$

Since $D_jg_{ih}=0$ and $D_kF_i^h=0$, we have

$$R_{kjih}=-R_{kjhi}, \qquad R_{kji}{}^{a}F_a^h=R_{kja}{}^{h}F_i^a,$$

where $R_{kjih}=R_{kji}{}^{a}g_{ah}$. By a straightforward computation we find

(4. 1) $$\begin{aligned} R_{kjih}+R_{jikh}+R_{ikjh}=\;&g_{kh}(\nabla_ip_j-\nabla_jp_i)+g_{jh}(\nabla_kp_i-\nabla_ip_k)\\ &+g_{ih}(\nabla_jp_k-\nabla_kp_j)+2F_{ij}(\nabla_kq_h-2q_kp_h+\lambda F_{kh})\\ &+2F_{ki}(\nabla_jq_h-2q_jp_h+\lambda F_{jh})+2F_{jk}(\nabla_iq_h-2q_ip_h+\lambda F_{ih}). \end{aligned}$$

Here and in the sequel we assume that p_i is a gradient vector, i.e., $p_i=\partial_ip$ for a scalar function p and p_i and q_i satisfy the equation

(4. 2) $$\nabla_kq_h-2q_kp_h+\lambda F_{kh}=0.$$

Then, from (4. 1), we have $R_{kjih}+R_{jikh}+R_{ikjh}=0$ and consequently $R_{kjih}=R_{ihkj}$. Therefore we can define the holomorphic sectional curvature for a holomorphic section $\sigma=(u^h, F_a{}^hu^a)$ with respect to a complex semi-symmetric metric F-connection as follows:

$$H(\sigma)=H(u^h)=-\frac{R_{kjih}F_a{}^k u^a u^j F_b{}^i u^b u^h}{g_{kj}u^k u^j g_{ih}u^i u^h}.$$

Then we can easily see that this $H(\sigma)$ is uniquely determined by the holomorphic section σ and is independent of the choice of u^h on σ. If this holomorphic sectional curvature is independent of the holomorphic section at each point of M, then a complex semi-symmetric metric F-connection is said to be of constant holomorphic sectional curvature.

Assume that a complex semi-symmetric metric F-connection D is of constant holomorphic sectional curvature, then we have

$$R_{kji}{}^h=k(\delta_k^h g_{ji}-\delta_j^h g_{ki}+F_k{}^h F_{ji}-F_j{}^h F_{ki}-2F_{kj}F_i{}^h), \tag{4.3}$$

where k is a scalar. From (4.3), we have

$$R_{ji}=k(n+2)g_{ji}, \qquad R=n(n+2)k. \tag{4.4}$$

From our assumption that p_i is a gradient vector and (4.2), we have

$$\begin{aligned} &P_{ji}=P_{ij}, \qquad Q=0, \qquad Q_{ji}+Q_{ij}=0, \\ &V_{ji}=H_{ji}=(n+2)kF_{ji}, \qquad V=n(n+2)k. \end{aligned} \tag{4.5}$$

From (3.9), (4.4) and (4.5), we get

$$\lambda=\frac{R-K}{(n+2)(n+4)}, \qquad P=\frac{K-R}{2(n+2)}. \tag{4.6}$$

Hence, from (3.6), (3.7), (4.4), (4.5) and (4.6), we have

$$\begin{aligned} &P_{kj}=-L_{kj}-\frac{1}{2}kg_{kj}, \qquad Q_{kj}=-M_{kj}-\frac{1}{2}kF_{kj}, \\ &\alpha_{kj}=2M_{kj}+kF_{kj}+\lambda F_{kj}, \qquad \beta_{kj}=2M_{kj}+kF_{kj}-\lambda F_{kj}. \end{aligned} \tag{4.7}$$

Substituting (4.3) and (4.7) into (3.1), we have $B_{kji}{}^h=0$, that is, the Bochner curvature tensor of M vanishes. Thus we have

THEOREM 2. *Let M be an $n(=2m>2)$ real dimensional Kählerian manifold. If M admits a complex semi-symmetric metric F-connection D which satisfies the following:*

(i) *p_i is a gradient vector,*

(ii) $\nabla_k q_h-2q_k p_h+\lambda F_{kh}=0$,

(iii) *D is of constant holomorphic sectional curvature, then the Bochner curvature tensor of M vanishes.*

Next we derive the Bianchi identity with respect to $R_{kji}{}^h$. First, from (4.2), we have

$$D_j q_i=-p_j q_i, \qquad D_j p_i=-p_j p_i. \tag{4.8}$$

Substituting (2. 4) and (4. 8) into

$$D_k D_j q_i - D_j D_k q_i = -R_{kji}{}^a q_a - T_{kj}{}^a D_a q_i,$$

we get $R_{kji}{}^a q_a = 0$. Using the Ricci's identity with respect to D, we have

$$\text{(a)} \qquad D_l D_k D_j w_i - D_k D_l D_j w_i = -R_{lkj}{}^a D_a w_i - R_{lki}{}^a D_j w_a - T_{lk}{}^a D_a D_j w_i,$$

where w_i is an arbitrary covector field. Differentiating covariantly the both sides of the equation

$$-D_k D_j w_i + D_j D_k w_i = R_{kji}{}^a w_a + T_{kj}{}^a D_a w_i,$$

we have

$$\text{(b)} \qquad \begin{aligned} & -D_l D_k D_j w_i + D_l D_j D_k w_i \\ & = (D_l R_{kji}{}^a) w_a + R_{kji}{}^a D_l w_a + (D_l T_{kj}{}^a) D_a w_i + T_{kj}{}^a D_l D_a w_i . \end{aligned}$$

Interchanging indices l, k, j in such a way that $l \to k \to j \to l$ in (a) and (b), we have the other four equations respectively. Summing up these six equations, we obtain

$$\begin{aligned} & (D_l R_{kji}{}^a + D_k R_{jli}{}^a + D_j R_{lki}{}^a) w_a - (R_{lkj}{}^a + R_{kjl}{}^a + R_{jlk}{}^a) D_a w_i \\ & \quad + 2\{p_l (D_k D_j w_i - D_j D_k w_i) + p_k (D_j D_l w_i - D_l D_j w_i) + p_j (D_l D_k w_i \\ & \quad - D_k D_l w_i)\} - 2\{F_{lk} q^a (D_j D_a w_i - D_a D_j w_i - p_j D_a w_i) + F_{kj} q^a (D_l D_a w_i \\ & \quad - D_a D_l w_i - p_l D_a w_i) + F_{jl} q^a (D_k D_a w_i - D_a D_k w_i - p_k D_a w_i)\} = 0 . \end{aligned}$$

Substituting the equations

$$D_k D_j w_i - D_j D_k w_i = -R_{kji}{}^a w_a - p_j D_k w_i + p_k D_j w_i + 2F_{kj} q^a D_a w_i,$$

$$R_{lkj}{}^h + R_{kjl}{}^h + R_{jlk}{}^h = 0, \qquad R_{kji}{}^a q_a = 0$$

into the above equation, we have

$$\text{(4. 9)} \qquad D_l R_{kji}{}^h + D_k R_{jli}{}^h + D_j R_{lki}{}^h = 2(p_l R_{kji}{}^h + p_k R_{jli}{}^h + p_j R_{lki}{}^h).$$

Since $p_i = \partial_i p$, p being a scalar, we have

$$\text{(4. 10)} \qquad D_l (e^{-2p} R_{kji}{}^h) = -2e^{-2p} p_l R_{kji}{}^h + e^{-2p} D_l R_{kji}{}^h .$$

From (4. 9) and (4. 10), we get

$$D_l (e^{-2p} R_{kji}{}^h) + D_k (e^{-2p} R_{jli}{}^h) + D_j (e^{-2p} R_{lki}{}^h) = 0 .$$

Therefore, if we put $\hat{R}_{kji}{}^h = e^{-2p} R_{kji}{}^h$, then we can easily verify the following relations:

$$\text{(4. 11)} \qquad \begin{aligned} & \hat{R}_{kji}{}^h = -\hat{R}_{jki}{}^h, \qquad \hat{R}_{kjih} = -\hat{R}_{kjhi}, \qquad \hat{R}_{kjih} = \hat{R}_{ihkj}, \\ & \hat{R}_{kji}{}^h + \hat{R}_{jik}{}^h + \hat{R}_{ikj}{}^h = 0, \qquad \hat{R}_{kji}{}^a F_a{}^h = \hat{R}_{kja}{}^h F_i{}^a, \\ & D_l \hat{R}_{kji}{}^h + D_k \hat{R}_{jli}{}^h + D_j \hat{R}_{lki}{}^h = 0 . \end{aligned}$$

If a complex semi-symmetric metric F-connection D satisfies the conditions (i), (ii) and (iii) of theorem 2, then we have

(4.12) $$R_{kji}{}^{h}=k(\delta_{k}^{h}g_{ji}-\delta_{j}^{h}g_{ki}+F_{k}{}^{h}F_{ji}-F_{j}{}^{h}F_{ki}-2F_{kj}F_{i}{}^{h}).$$

Multiplying the both sides of (4.12) by e^{-2p}, we have

$$\hat{R}_{kji}{}^{h}=ke^{-2p}(\delta_{k}^{h}g_{ji}-\delta_{j}^{h}g_{ki}+F_{k}{}^{h}F_{ji}-F_{j}{}^{h}F_{ki}-2F_{kj}F_{i}{}^{h}).$$

Then, by using (4.11) we can prove that ke^{-2p} is a constant. Hence, if we put $c=ke^{-2p}$, we find

$$e^{-2p}R_{kji}{}^{h}=c(\delta_{k}^{h}g_{ji}-\delta_{j}^{h}g_{ki}+F_{k}{}^{h}F_{ji}-F_{j}{}^{h}F_{ki}-2F_{kj}F_{i}{}^{h}),$$

c being a constant. Thus we have

THEOREM 3. *Let M be an $n(=2m>2)$ real dimensional Kählerian manifold. If M admits a complex semi-symmetric metric F-connection D which satisfies the following*:

(i) *p_i is a gradient vector, i.e., $p_i=\partial_i p$ for a scalar p,*

(ii) $\nabla_k q_h-2q_k p_h+\lambda F_{kh}=0$,

(iii) *D is of constant holomorphic sectional curvature, then the curvature tensor of D is represented by*

$$R_{kji}{}^{h}=ce^{2p}(\delta_{k}^{h}g_{ji}-\delta_{j}^{h}g_{ki}+F_{k}{}^{h}F_{ji}-F_{j}{}^{h}F_{ki}-2F_{kj}F_{i}{}^{h}),$$

where c is a constant.

References

[1] T. IMAI: Notes on semi-symmetric metric connections, Tensor, N.S., 24 (1972), 293-296.

[2] T. IMAI: Notes on semi-symmetric metric connections II, Tensor, N.S., 27 (1973), 56-58.

[3] U.K. KIM: A note on complex conformal connections, J. Korean Math. Soc., 11 (1974), 91-95.

[4] S. TACHIBANA: On the Bochner curvature tensor, Nat. Sci. Rep. of the Ochanomizu Univ., 18 (1967), 15-19.

[5] K. YANO: On semi-symmetric metric connection, Revue Roumaine de Math. Pures et Appliquées, 15 (1970), 1579-1586.

[6] K. YANO: On complex conformal connections, to appear in Kodai Math. Sem. Rep.

[7] K. YANO: Differential Geometry on Complex and Almost Complex Spaces, Pergamon Press, New York, 1965.

[8] O. YOON: An invariant of complex conformal connections in a Kähler manifold, J. Korean Math. Soc., 11 (1974), 135-139.

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