Invariant hypersurfaces of $S^n \times S^n$ with constant mean curvature

Yoshio MATSUYAMA

(Received May 14, 1975)

0. Introduction

Ludden and Okumura [1] studied minimal hypersurfaces of the product $S^n \times S^n$ of two n-spheres. Some of their results are as follows:

a. If a compact orientable minimal hypersurface M of $S^n \times S^n$ $(n>1)$ satisfies

$$\int_M (S^2-(n-1)S)\,dM \geqq \int_M \|\nabla H\|^2 dM$$

(in particular, $\nabla H=0$ and $S \geqq n-1$), then the tangent space of M is invariant under an almost product structure on $S^n \times S^n$ (for simplicity, we say that M is an invariant hypersurface), where $S=\text{trace } H^2$.

b. Let M be a compact orientable invariant minimal hypersurface of $S^n \times S^n$. Then either M is the totally geodesic hypersurface or $S \equiv n-1$, or $S(x)>n-1$ at some $x \in M$.

c. $S^{n-1}(1) \times S^n(1)$ and

$$S^m(\sqrt{m/(n-1)}) \times S^{n-m-1}(\sqrt{(n-m-1)/(n-1)}) \times S^n(1)$$

are the only compact orientable invariant minimal hypersurfaces of $S^n \times S^n$ satisfying $S \leqq n-1$.

In the present paper, we further investigate hypersurfaces of $S^n \times S^n$ under the assumption of non-negative sectional curvature.

That is, we obtain the following results:

A. A compact orientable minimal hypersurface with non-negative sectional curvature of $S^n \times S^n$ $(n>1)$ which satisfies

$$\int_M (S^2-(n-1)S)\,dM \geqq 0$$

(in particular, $S \geqq n-1$) is an invariant hypersurface (Theorem 1.2 and Corollary 1.3).

B. Let M be a compact orientable invariant minimal hypersurface with non-negative sectional curvature of $S^n \times S^n$. Then either M is the totally geodesic hypersurface or $S \equiv n-1$ (Theorem 2.1).

C. $S^{n-1}(r) \times S^n(1)$ and $S^m(s) \times S^{n-m-1}(\sqrt{1-s^2}) \times S^n(1)$ are the only comapct orientable invariant hypersurfaces of $S^n \times S^n$ with constant mean curvature and non-negative sectional curvature (Theorem 2.5).

Moreover, as a special case of C, we have:
$S^{n-1}(1) \times S^n(1)$ and

$$S^m(\sqrt{m/(n-1)}) \times S^{n-m-1}(\sqrt{(n-m-1)/(n-1)}) \times S^n(1)$$

are the only compact orientable invariant minimal hypersurfaces of $S^n \times S^n$ with non-negative sectional curvature (Corollary 2.6).

The author would like to express sincere thanks Professor K. Ogiue for his many valuable suggestions.

1. Preliminaries

Let S^n be an n-sphere of radius 1, and consider $S^n \times S^n$. We denote by $\bar{P}$ and $\bar{Q}$ the projections of the tangent space of $S^n \times S^n$ to each component respectively. Then the product Riemannian metric $\bar{g}$ on $S^n \times S^n$ is given by

$$\bar{g}(\bar{X}, \bar{Y}) = g'(\bar{P}\bar{X}, \bar{P}\bar{Y}) + g'(\bar{Q}\bar{X}, \bar{Q}\bar{Y}),$$

where g' is the Riemannian metric of S^n. We put

$$\bar{J} = \bar{P} - \bar{Q}.$$

Then we have ([1])

$$\bar{P} + \bar{Q} = I, \qquad \bar{P}\bar{Q} = \bar{Q}\bar{P} = 0,$$

$$\bar{P}^2 = \bar{P}, \qquad \bar{Q}^2 = \bar{Q},$$

$$\bar{J}^2 = I, \qquad \text{trace } \bar{J} = 0,$$

$$\bar{g}(\bar{J}\bar{X}, \bar{Y}) = \bar{g}(\bar{X}, \bar{J}\bar{Y}), \qquad \bar{\nabla}_{\bar{X}}\bar{J} = 0,$$

$$\bar{R}(\bar{X}, \bar{Y})\bar{Z} = \frac{1}{2}\left\{\bar{g}(\bar{Y}, \bar{Z})\bar{X} - \bar{g}(\bar{X}, \bar{Z})\bar{Y} + \bar{g}(\bar{J}\bar{Y}, \bar{Z})\bar{J}\bar{X} - \bar{g}(\bar{J}\bar{X}, \bar{Z})\bar{J}\bar{Y}\right\},$$

where $\bar{\nabla}$ and $\bar{R}$ denote the operator of covariant differentiation with respect to the Riemannian connection of $\bar{g}$ and the curvature tensor of $S^n \times S^n$, respectively. We call $\bar{J}$ an *almost product structure* on $S^n \times S^n$.

Now let M be a hypersurface of $S^n \times S^n$, and B the differential of the immersion i of M into $S^n \times S^n$. Let g be the induced Riemannian metric, and ∇ denote the operator of covariant differentiation with respect to the Riemannian connection of g. Let X, Y and Z be tangent to M and N a unit normal vector. Then we have the following relations ([1]):

(1. 1) $\qquad \bar{J}BX = BfX + u(X)N,$

$$(1.2)\qquad \bar{J}N = BU + \lambda N,$$

$$g(U, X) = u(X),$$

$$(1.3)\qquad \bar{\nabla}_{BX} BY = B\nabla_X Y + h(X, Y)N,$$

$$(1.4)\qquad \bar{\nabla}_{BX} N = -BHX,$$

$$h(X, Y) = g(HX, Y),$$

$$(1.5)\qquad \begin{aligned} R(X, Y)Z = \frac{1}{2}\Big\{ & g(Y, Z)X - g(X, Z)Y + g(fY, Z)fX \\ & - g(fX, Z)fY\Big\} + h(Y, Z)HX - h(X, Z)HY, \end{aligned}$$

$$(1.6)\qquad (\nabla_X H)Y - (\nabla_Y H)X = \frac{1}{2}(u(X)fY - u(Y)fX),$$

$$(1.7)\qquad f^2 X = X - u(X)U,$$

$$(1.8)\qquad u(U) = g(U, U) = 1 - \lambda^2,$$

$$(1.9)\qquad \text{trace } f = -\lambda,$$

$$(1.10)\qquad \nabla_X U = -fHX + \lambda HX,$$

$$(1.11)\qquad X\cdot\lambda = -2h(U, X) = -2u(HX),$$

where f, u, U, λ, h and R define a symmetric linear transformation of the tangent bundle of M, a 1-form, a vector field, a function on M, the second fundamental tensor of the hypersurface and the curvature tensor of M, respectively.

If u is identically zero, then M is said to be an *invariant* hypersurface, that is, the tangent space $T_x(M)$ is invariant under $\bar{J}$. We can easily see by (1.8) that this is equivalent to $\lambda^2 = 1$.

We consider the function $S = \text{trace } H^2$ which is globally defined on M and wish to compute its Laplacian ΔS. We now assume that the hypersurface M has constant mean curvature, that is, trace H is constant. Then it is know that

$$(1.12)\qquad \begin{aligned} \frac{1}{2}\Delta S = & -\lambda \,\text{trace } fH^2 + \text{trace } (fH)^2 + \frac{1}{2}(\text{trace } H)\, g(HU, U) \\ & -(\text{trace } Hf)^2 + \frac{1}{2}\lambda (\text{trace } H) \text{ trace } fH + g(HU, HU) \\ & -\frac{1}{2}(\text{trace } H)^2 - S(S-(n-1)) + (\text{trace } H) \text{ trace } H^3 \\ & + g(\nabla H, \nabla H), \end{aligned}$$

$$\begin{aligned}(1.13)\quad &\operatorname{div}((\text{trace } fH)\, U - fHU) \\ &\quad = g(HU, HU) - (\text{trace } fH)^2 + \lambda(\text{trace } H) \text{ trace } fH \\ &\qquad - (\text{trace } H)\, g(HU, U) + \text{trace } (fH)^2 - \lambda \text{ trace } fH^2 \\ &\qquad + (n-1)(1-\lambda^2),\end{aligned}$$

$$(1.14)\quad \operatorname{div}((\text{trace } H)\, U) = -(\text{trace } H) \text{ trace } fH + \lambda(\text{trace } H)^2$$

(See [1, (2.5), (2.10) and (2.12)]).

Let $\{e_1, \cdots, e_{2n-1}\}$ be an orthonormal basis for $T_x(M)$. Then because of (1.5), (1.7) and (1.9) we have

$$\begin{aligned}(1.15)\quad &\sum_{i,j=1}^{2n-1} \{g(R(e_i, e_j)\, e_j, H^2 e_i) - 2g(R(e_i, He_j)\, e_j, He_i) \\ &\qquad\qquad + g(R(e_i, He_j)\, He_j, e_i)\} \\ &\quad = 2S((n-1) - S) - (\text{trace } H)^2 - \lambda \text{ trace } fH^2 \\ &\qquad + \text{trace } (fH)^2 - (\text{trace } fH)^2 + g(HU, HU) \\ &\qquad + 2(\text{rtace } H) \text{ trace } H^3.\end{aligned}$$

If we choose $\{e_1, \cdots, e_{2n-1}\}$ such that $He_i = \lambda_i e_i$, $1 \leqq i \leqq 2n-1$, then from (1.12) and (1.15) we obtain

$$\begin{aligned}(1.16)\quad \frac{1}{2}\Delta S &= \sum_{i,j=1}^{2n-1} (\lambda_i - \lambda_j)^2\, g(R(e_i, e_j)\, e_j, e_i) \\ &\quad + S(S-(n-1)) + \frac{1}{2}(\text{trace } H)^2 + \frac{1}{2}\lambda(\text{trace } H) \text{ trace } fH \\ &\quad + \frac{1}{2}(\text{trace } H)\, g(HU, U) - (\text{trace } H) \text{ trace } H^3 \\ &\quad + g(\nabla H, \nabla H),\end{aligned}$$

Adding up both sides of (1.12) and (1.16), we get

$$\begin{aligned}(1.17)\quad \Delta S &= \sum_{i,j=1}^{2n-1} (\lambda_i - \lambda_j)^2\, g(R(e_i, e_j)\, e_j, e_i) - \lambda \text{ trace } fH^2 \\ &\quad + \text{trace } (fH)^2 + (\text{trace } H)\, g(HU, U) - (\text{trace } Hf)^2 \\ &\quad + \lambda(\text{trace } H) \text{ trace } fH + g(HU, HU) + 2g(\nabla H, \nabla H)\end{aligned}$$

In particular, if the hypersurface is minimal, then

$$\begin{aligned}(1.16)'\quad \frac{1}{2}\Delta S &= \sum_{i,j=1}^{2n-1} (\lambda_i - \lambda_j)^2\, g(R(e_i, e_j)\, e_j, e_i) \\ &\quad + S(S-(n-1)) + g(\nabla H, \nabla H).\end{aligned}$$

From (1.16)′ we easily get

PROPOSITION 1.1. *If M is a compact minimal hypersurface with non-negative sectional curvature of $S^n\times S^n$ satisfying $S(S-(n-1))\geqq 0$, then either $S\equiv 0$ (i. e., M is totally geodesic) or $\nabla H=0$ and $S\equiv n-1$.*

Next we compute div ((trace H) λU). Since M has constant mean curvature, we have

$$\begin{aligned}\nabla_X((\mathrm{trace}\ H)\lambda U)&=(\mathrm{trace}\ H)(X\cdot\lambda)U+\lambda(\mathrm{trace}\ H)\nabla_X U\\ &=-2(\mathrm{trace}\ H)\,u(HX)\,U\\ &\quad+\lambda(\mathrm{trace}\ H)(-fHX+\lambda HX),\end{aligned}$$

which implies that

$$\begin{aligned}\mathrm{div}\ ((\mathrm{trace}\ H)\lambda U)&=-2(\mathrm{trace}\ H)\,g(HU,U)\\ &\quad-\lambda(\mathrm{trace}\ H)\,\mathrm{trace}\ fH+\lambda^2(\mathrm{trace}\ H)^2.\end{aligned}\tag{1.18}$$

Subtracting (1.12) from (1.16) we get

$$\begin{aligned}&\sum_{i,j=1}^{2n-1}(\lambda_i-\lambda_j)^2\,g(R(e_i,e_j)e_j,e_i)+2S(S-(n-1))\\ &\quad+(\mathrm{trace}\ H)^2-2(\mathrm{trace}\ H)\,\mathrm{trace}\ H^3+\lambda\ \mathrm{trace}\ fH^2\\ &\quad-\mathrm{trace}(fH)^2+(\mathrm{trace}\ Hf)^2-g(HU,HU)=0.\end{aligned}\tag{1.19}$$

From (1.13), (1.14), (1.18) and (1.19), we obtain

$$\begin{aligned}&\mathrm{div}\ ((\mathrm{trace}\ fH)U-fHU)-\frac{1}{2}\mathrm{div}\ ((\mathrm{trace}\ H)\lambda U\\ &\qquad\qquad+\frac{3}{2}\lambda\,\mathrm{div}\ ((\mathrm{trace}\ H)U)\\ &=\sum_{i,j=1}^{2n-1}(\lambda_i-\lambda_j)^2\,g(R(e_i,e_j)e_j,e_i)+(n-1)(1-\lambda^2)\\ &\quad+2S(S-(n-1))+(1+\lambda^2)(\mathrm{trace}\ H)^2\\ &\quad-2(\mathrm{trace}\ H)\,\mathrm{trace}\ H^3.\end{aligned}$$

On the other hand, from (1.13), (1.14), (1.17) and (1.18) we have

$$\begin{aligned}&\Delta S-\mathrm{div}\ ((\mathrm{trace}\ fH)U-fHU)+\mathrm{div}\ ((\mathrm{trace}\ H)\lambda U)\\ &\qquad\qquad-\lambda\,\mathrm{div}\ ((\mathrm{trace}\ H)U)\\ &=\sum_{i,j=1}^{2n-1}(\lambda_i-\lambda_j)^2\,g(R(e_i,e_j)e_j,e_i)\\ &\quad-(n-1)(1-\lambda^2)+2g(\nabla H,\nabla H).\end{aligned}$$

Assume that the hypersurface M is compact and orientable and that $\lambda=$ constant. Integrating the above equations over M and using Green-Stokes' theorem, we get

$$\int_M \Big\{ \sum_{i,j=1}^{2n-1} (\lambda_i - \lambda_j)^2 g(R(e_i, e_j) e_j, e_i) + (n-1)(1-\lambda^2) + 2S(S-(n-1)) + (1+\lambda^2)(\text{trace } H)^2 - 2(\text{trace } H) \text{ trace } H^3 \Big\} dM = 0, \tag{1.20}$$

$$\int_M \Big\{ \sum_{i,j=1}^{2n-1} (\lambda_i - \lambda_j)^2 g(R(e_i, e_j) e_j, e_i) - (n-1)(1-\lambda^2) + 2g(\Delta H, \nabla H) \Big\} dM = 0. \tag{1.21}$$

In particular, if the hypersurface M is minimal, then without the assumption of λ=constant

$$\int_M \Big\{ \sum_{i,j=1}^{2n-1} (\lambda_i - \lambda_j)^2 g(R(e_i, e_j) e_j, e_i) + (n-1)(1-\lambda^2) + 2S(S-(n-1)) \Big\} dM = 0. \tag{1.20$'$}$$

From (1.20)$'$ we easily get

THEOREM 1.2. *A compact orientable minimal hypersurface with non-negative sectional curvature of $S^n \times S^n$ $(n>1)$ satisfying*

$$\int_M (S^2 - (n-1)S)\, dM \geqq 0$$

is an invariant hypersurface.

COROLLARY 1.3. *A compact orientable minimal hypersurface with non-negative sectional curvature of $S^n \times S^n$ satisfying $S \geqq n-1$ is an invariant hypersurface.*

2. Main results

In this section we assume that the hypersurface M is invariant. Then formulas (1.20), (1.21) and (1.20)$'$ become

$$\int_M \Big\{ \sum_{i,j=1}^{2n-1} (\lambda_i - \lambda_j)^2 g(R(e_i, e_j) e_j, e_i) + 2S(S-(n-1)) + 2(\text{trace } H)^2 - 2(\text{trace } H) \text{ trace } H^3 \Big\} dM = 0, \tag{2.1}$$

$$\int_M \Big\{ \sum_{i,j=1}^{2n-1} (\lambda_i - \lambda_j)^2 g(R(e_i, e_j) e_j, e_i) + 2g(\nabla H, \nabla H) \Big\} dM = 0, \tag{2.2}$$

$$\int_M \Big\{ \sum_{i,j=1}^{2n-1} (\lambda_i - \lambda_j)^2 \, g(R(e_i, e_j) e_j, e_i) + 2S(S-(n-1)) \Big\} dM = 0 , \tag{2.1'}$$

respectively. Thus combining (2. 2) and (2. 1)′ we get

THEOREM 2. 1. *Let M be a compact orientable invariant minimal hypersurface with non-negative sectional curvature of $S^n \times S^n$. Then either M is the totally geodesic hypersurface or $S \equiv n-1$.*

Moreover, from (2. 2) we obtain

PROPOSITION 2. 2. *If M is a compact orientable invariant hypersurface of $S^n \times S^n$ with constant mean curvature and non-negative sectional curvature, then the second fundamental form of M is parallel.*

The following results are basic:

LEMMA 2. 3 (Ludden and Okumura [1]). *A complete invariant hypersurface of $S^n \times S^n$ is a product manifold $M' \times S^n$, where M' is a hypersurface of S^n.*

LEMMA 2. 4 (Ludden and Okumura ([1]). *In Lemma* 2. 3, *denoting the second fundamental tensor of M' in S^n by H', we have*

$$\text{trace } H^p = \text{trace } H'^p$$

for any positive integer p.

By virtue of Lemmas 2. 3 and 2. 4 we can easily see that if a hypersurface M of $S^n \times S^n$ is of constant mean curvature and non-negative sectional curvature, then so is M', a hypersurface immersed in S^n. Applying Nomizu and Smyth's result (See [2], Theorem 2 or Corollary 2), we have $M' = S^{n-1}(r)$ or $M' = S^m(s) \times S^{n-m-1}(\sqrt{1-s^2})$, where we denote the radius of spheres in the parentheses and $H' = \frac{\sqrt{1-r^2}}{r} I$, $m\left(\frac{\sqrt{1-s^2}}{s}\right) + (n-m-1)\left(-\frac{s}{\sqrt{1-s^2}}\right) = \text{trace } H'$. Hence we have $M = S^{n-1}(r) \times S^n(1)$ or $M = S^m(s) \times S^{n-m-1}(\sqrt{1-s^2}) \times S^n(1)$.

THEOREM 2. 5. *$S^{n-1}(r) \times S^n(1)$ and $S^m(s) \times S^{n-m-1}(\sqrt{1-s^2}) \times S^2(1)$ are the only compact orientable invariant hypersurfaces of $S^n \times S^n$ with constant mean curvature and non-negative sectional curvature.*

COROLLARY 2. 6. *$S^{n-1}(1) \times S^n(1)$ and $S^m(\sqrt{m/(n-1)}) \times S^{n-m-1}(\sqrt{n-m-1)/(n-1)}) \times S^n(1)$ are the only compact orientable invariant minimal hypersurfaces of $S^n \times S^n$ with non-negative sectional curvature.*

References

[1] G. D. LUDDEN and M. OKUMURA: Some integral formulas and their applications to hypersurfaces of $S^n \times S^n$, J. Differential Geometry 9 (1974), 617–631.

[2] K. NOMIZU and B. SMYTH: A formula of Simons type and hypersurfaces with constant mean curvature, J. Differential Geometry 3 (1969), 367–377.

Department of Mathematics
Tokyo Metropolitan University