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Remarks on L^2-well posedness of mixed problems for hyperbolic systems

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§1. Introduction and main theorem

In the present paper we are concerned with the boundary value problem for a $2m\times 2m$ strictly x_0-hyperbolic system P of order one:

$$(P, B)\begin{cases} P(x, D)u=f & \text{in } \Omega, \\ B(x')u=g & \text{on } \Gamma, \end{cases}$$

where Ω is the open half space $\{x=(x', x_n)=(x_0, x'', x_n);\ x_0\in R^1, x''\in R^{n-1}, x_n>0\}$ $(n\geqq 2)$, its boundary Γ is noncharacteristic for P and $B(x')$ is an $m\times 2m$ matrix of rank m for every $x'\in\Gamma$. All coefficients of differential operators considered here are assumed to be smooth in $\bar{\Omega}$ and constant outside of a compact subset of $\bar{\Omega}$. Then the problem (P, B) is said to be *L^2-well posed* if and only if there exist positive constants γ_0 and C such that for every $\gamma\geqq\gamma_0$, $f\in H_{1,\gamma}(\Omega)$ and $g\in H_{3/2,\gamma}(\Gamma)$ (P, B) has a unique solution u in $H_{1,\gamma}(\Omega)$ satisfying

$$\gamma^2\|u\|_{0,\gamma}^2\leqq C\left(\|f\|_{0,\gamma}^2+|g|_{1/2,\gamma}^2\right). \tag{1.1}$$

This definition of L^2-well posedness is weaker than that in Kreiss [7], but it implies a certain well posedness of the corresponding mixed problem with initial data on $x_0=0$. (See §5 in Kubota [8]).

In a recent paper Sato and Shirota [15] have refined the results at §7 in Ohkubo and Shirota [11] who investigated the above problem under the condition that all of constant coefficients problems frozen the coefficients at Γ are L^2-well posed. In this paper we try to complete some of the results at §§5, 6 and 8 in [11]. Here we shall use the same terminologies as in [11] unless otherwise indicated, but we denote by $\sqrt{\zeta}$ the branch of square roots of ζ such that $\sqrt{1}=1$ (see [15]).

Throughout the present article we assume the following conditions (i) and (ii):

CONDITION (i) (with respect to the principal part P^0 of P). *Let* (η, σ, λ) *be the covector of* (x^0, x'', x_n). *Then for every* $(x, \eta, \sigma)\in\Gamma\times(R^n\backslash 0)$ *the* *real zeros of the polynomial* det $P^0(x, \eta, \sigma, \lambda)$ *in* λ *are at most of double multiplicity and it does not have more than one double real zero. Furthermore* det $P^0(x, 0, \sigma, \lambda)$ *is elliptic for* $x \in \Gamma$.

CONDITION (*ii*) (with respect to Lopatinskii determinant). *Let* $L(x', \tau, \sigma)$ *be the Lopatinskii determinant of* (P^0, B), *where* $\tau=\eta-i\gamma$ *and* γ *is considered as a real parameter.* (*See Definition* 4.1 *in* [11] *or* (2.9)). *Then if* $L(x', \tau, \sigma)$ *vanishes at a point* $(x^0, \eta^0, \sigma^0) \in \Gamma \times (R^n \backslash 0)$,

$$\left|L(x^0, \eta^0-i\gamma, \sigma^0)\right| \geqq C\gamma^{1/2} \quad \text{or} \quad C\gamma$$

for small $\gamma>0$ *according as* det $P^0(x^0, \eta^0, \sigma^0, \lambda)$ *has a double real zero or not, where* C *is a positive constant independent of* γ.

Condition (i) is the same one as the condition (I) in [11] and Condition (ii) means that for fixed (x^0, σ^0) $L(x^0, \eta, \sigma^0)$ is simply characteristic in a certain sense. (See the condition (II) in [11], (2.11) and (2.13)). In particular, if $L(x^0, \eta^0, \sigma^0)=0$ and det $P^0(x^0, \eta^0, \sigma^0, \lambda)$ has a double real zero, then L is decomposed as

$$L(x', \tau, \sigma)=\left(\sqrt{\tau-\theta(x', \sigma)}-D(x', \sigma)\right)\cdot(\text{nonzero factor})$$

in a conic neighborhood of (x^0, η^0, σ^0) with $\operatorname{Im}\tau \leqq 0$.

Using the above root $D(x', \sigma)$, we finally impose the following additional

CONDITION (*iii*).

$$\left|\operatorname{Im} D(x', \sigma)\right| \leqq -C\operatorname{Re} D(x', \sigma) \quad \text{or} \quad \left|\operatorname{Re} D(x', \sigma)\right| \leqq C\operatorname{Im} D(x', \sigma)$$

in a neighborhood of (x^0, σ^0) *according to the case* (*a*) *or* (*b*) *of* (2.4) *below, where* $C=C(x^0, \eta^0, \sigma^0)$ *is a positive constant.*

This holds under the conditions (II) β), γ) and (III) in [11].

Now we shall state our main

THEOREM. *Under the assumptions* (i), (ii) *and* (iii) *described above the following three conditions are equivalent*:

(A) *The problem* (P, B) *is* L^2-*well posed.*

(B) *Every constant coefficients problem* $(P^0, B)_{x'}$ *obtained by freezing the coefficients of* P^0 *and* B *at* $x' \in \Gamma$ *is* L^2-*well posed and the constants* C *in* (1.1) *with respect to these problems are independent of the parameter* x'.

(C) *Hersh's condition holds, that is,*

$$L(x', \tau, \sigma) \neq 0, \quad \text{if} \quad \operatorname{Im}\tau<0.$$

And for every $(\tilde{x}', \tilde{\eta}, \tilde{\sigma}) \in \Gamma \times (R^n \backslash 0)$ *there exist a constant* $C=C(\tilde{x}', \tilde{\eta}, \tilde{\sigma})$ *and a neighborhood* $U(\tilde{x}', \tilde{\eta}, \tilde{\sigma})$ *in* $\Gamma \times C \times R^{n-1}$ *such that*

$$(1.2)\qquad |b_{ij}(x',\tau,\sigma)|^2 \leqq C|\mathrm{Im}\,\tau|^{-2}|\mathrm{Im}\,\lambda_i^+(x',\tau,\sigma)\,\mathrm{Im}\,\lambda_j^-(x',\tau,\sigma)|$$

for all $(x',\tau,\sigma)\in U(\tilde{x}',\tilde{\eta},\tilde{\sigma})\cap\{\mathrm{Im}\,\tau<0\}$ *and* $i,j=1,\cdots,m$. Here b_{ij} is the coupling coefficient of (P^0,B) (see Definition 4.2 in [11] or (2.19)) and $\lambda_i^+(x,\tau,\sigma)$, $\lambda_j^-(x,\tau,\sigma)$ are zeros of $\det P^0(x,\tau,\sigma,\lambda)$ with positive imaginary part and negative one when $\mathrm{Im}\,\tau<0$ respectively.

The implication $(A)\rightarrow(B)$ in our theorem follows from Theorem 1 and Lemma 2.2 in Agemi [1], and $(B)\rightarrow(C)$ is proved in Lemm 3.1 below. To prove the implication $(C)\rightarrow(A)$ we shall derive in section 3 certain relations among the coupling coefficients, the Lopatinskii determinant and its zeros. In fact, we obtain the main inequalities (3.2) and (3.13), by virtue of which we may simplify the proofs in [11] and, in particular, from the latter we can derive the inequality (6.5) in [11] which is assumed there. Using those inequalities we construct in sections 4 and 5 such modified symmetrizers as in [15] which give us a priori estimates microlocally. In section 6 we complete the proof of the imlication $(C)\rightarrow(A)$ by showing that an adjoint problem of (P,B) also satisfies Conditions (i), (ii), (iii) and (C). Thus we may assert that under Conditions (i) and (ii) the microlocal structures of L^2-well posed problems are completely described by our theorem in the case where there is no point $(x^0,\eta^0,\sigma^0)\in\Gamma\times(R^n\backslash 0)$ such that $L(x^0,\eta^0,\sigma^0)=0$ and $\det P^0(x^0,\eta^0,\sigma^0,\lambda)$ has a double real zero, but it remains to investigate the microlocal one in the other case. (See [15]).

As an application of our theorem we shall give in section 7 full proofs of Theorem and Corollary in the preceding paper [9] which characterize the L^2-well posedness of such an iterated mixed problem as in Mizohata [10] or Agemi [2, 3]. The method used in proving those can be also applied to another iterated mixed problem treated in Sakamoto [13]. (See Remark 7.3). Furthermore it should be pointed out that in general the L^2-well posedness of a variable coefficients problem need not follow from that of the frozen constant coefficients problems, as we have already remarked in [9].

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§ 2. Notations and preliminaries

2.1. In order to express our assertions in simple and precise form it is convenient to introduce the following notations. By x' we often denote a boundary point $(x', 0)$. Let Σ be the closed hemishere $\{(\tau, \sigma)=(\eta-i\gamma, \sigma)$; $\eta\in R^1$, $\sigma\in R^{n-1}$, $\gamma\geqq 0$, $|\tau|^2+|\sigma|^2=1\}$. Then by $\partial\Sigma$ and Σ_0 we denote the boundary of Σ and $\{(\eta, \sigma); (\eta-i\gamma, \sigma)\in\Sigma\}$ respectively. Moreover Σ is often identified with $\{(\eta, \sigma, \gamma)$; $(\eta, \sigma)\in R^n, \gamma\geqq 0, \eta^2+|\sigma|^2+\gamma^2=1\}$. Let

$$\Lambda(\tau, \sigma)=\left(|\tau|^2+|\sigma|^2\right)^{\frac{1}{2}}.$$

Then we use the notations:

$$\tau'=\tau\Lambda(\tau, \sigma)^{-1} \quad \text{and} \quad \gamma'=\gamma\Lambda(\tau, \sigma)^{-1} \quad \text{etc.}$$

By a (conic) neighborhood of a point $(x^0, \eta^0, \sigma^0)\in\Gamma\times\partial\Sigma$ we mean one in $\Gamma\times C\times R^{n-1}$ or $\bar{\Omega}\times C\times R^{n-1}$ according as Γ or $\bar{\Omega}$ is considered as the base space respectively and by $U(x^0, \eta^0, \sigma^0)$ its appropriate neighborhoods, unless otherwise indicated. E_k stands for the identity matrix of size k which is omitted if there is no ambiguity. Furthermore we use the function spaces $H_{k,\gamma}(\Omega)$ $(k=0, 1, 2, \cdots)$ and $H_{s,\gamma}(\Gamma)$ (s: real) depending a real parameter $\gamma(\gamma\neq 0)$ with the norms $\|\cdot\|_{k,\gamma}$, $|\cdot|_{s,\gamma}$ and the inner products $(\cdot, \cdot)_{k,\gamma}$, $\langle\cdot,\cdot\rangle_{s,\gamma}$ respectively. (See for instance §2 in [8]). We also use the same class S^q_+ (q: real) of symbols of (tangential) pseudo-differential operators with non-negative parameters x_n and γ as in §2 of [11]. With a symbol $a(x', \eta, \sigma, \gamma)\in S^q_+$ we associate a pseudo-differential operator by

$$a(x', D', \gamma)\, u(x')=(2\pi)^{-n}\int_{R^n} e^{i(\tau x_0+\sigma x'')}\, a(x', \eta, \sigma, \gamma)\, \hat{u}(\tau, \sigma)\, d\eta\, d\sigma$$

for $u\in H_{q,\gamma}(\Gamma)$, where

$$\hat{u}(\tau, \sigma)=\int_{\Gamma} e^{-i(\tau x_0+\sigma x'')}\, u(x')\, dx',$$

and for simplicity we denote $a(x', D', \gamma)$ by $a(x', D')$. If a C^∞-function $a(x, \eta, \sigma, \gamma)$ defined in a neighborhood of $(x^0, \eta^0, \sigma^0, \gamma^0)\in\Gamma\times\Sigma$ is homogeneous of degree q in (η, σ, γ), then we extend it to $\bar{\Omega}\times((R^n\times[0, \infty))\backslash 0)$ so that the extended one belongs to S^q_+ and is denoted also by $a(x, \eta, \sigma, \gamma)$.

2.2. In this and the following two subsections we shall give a summary of fundamental concepts for our problems. (See §§3, 4 and 6 in [11] and §2 in [15]).

α) For every point $(x^0, \eta^0, \sigma^0)\in\Gamma\times\partial\Sigma$ such that det $P^0(x^0, \eta^0, \sigma^0, \lambda)$ has a double real zero, there exists a smooth and real valued function $\theta(x, \sigma)$ defined in a conic neighborhood of (x^0, σ^0) which is homogeneous of degree one in σ and such that $\theta(x^0, \sigma^0)=\eta^0$ and det $P^0(x, \tau, \sigma, \lambda)$ has a double real zero precisely on the surface $\tau=\theta(x, \sigma)$ in a conic neighborhood of (x^0, η^0, γ^0). (See Lemma 3.1 in [11]). Furthermore, as in [15], we shall often use the function:

$$\zeta=\tau-\theta(x, \sigma).$$

β) By virtue of the assumptions with respect to P and Γ we may assume without loss of generality that near Γ the coefficient of D_n in $P^0(x, D)$ is E. Let $(x^0, \tau^0, \sigma^0)\in\Gamma\times\Sigma$. Then there exists a smooth and nonsingular matrix $S(x, \tau, \sigma)$ defined in a conic neighborhood of (x^0, τ^0, σ^0) which is homogeneous of degree zero in (τ, σ), analytic in τ and satisfies

$$S^{-1}P^0(x, \tau, \sigma, \lambda)\, S=\lambda E-M(x, \tau, \sigma), \tag{2.1}$$

where in general

$$M(x, \tau, \sigma)=\begin{bmatrix} \lambda_{\mathrm{I}}^{+} & & & & \\ & \lambda_{\mathrm{I}}^{-} & & & \\ & & M_{\mathrm{II}} & & \\ & & & M_{\mathrm{III}}^{+} & \\ & & & & M_{\mathrm{III}}^{-} \end{bmatrix}(x, \tau, \sigma), \tag{2.2}$$

$$S(x, \tau, \sigma)=(h_{\mathrm{I}}^{+}, h_{\mathrm{I}}^{-}, h_{\mathrm{II}}', h_{\mathrm{II}}'', h_{\mathrm{III}}^{+}, h_{\mathrm{III}}^{-})(x, \tau, \sigma). \tag{2.3}$$

Here denoting by $2l$ the number of the real zeros of det $P^0(x^0, \tau^0, \sigma^0, \lambda)$, we set $\mathrm{III}=\{l+1, \cdots, m\}$, $\mathrm{I}=\{1, \cdots, l-1\}$ or $\{1, \cdots, l\}$ and $|\mathrm{I}|=l-1$ or l according as the polynomial has a double real zero or not respectively. More precisely

1) $\lambda_{\mathrm{I}}^{\pm}(x, \tau, \sigma)$ are $|\mathrm{I}|\times|\mathrm{I}|$ diagonal matrices whose eigenvalues are simple zeros of det $P^0(x, \tau, \sigma, \lambda)$ with positive imaginary parts or negative ones when Im $\tau<0$ respectively and are real when Im $\tau=0$.

2) $M_{\mathrm{II}}(x, \tau, \sigma)$ is a 2×2 matrix which has eigenvalues $\lambda_{\mathrm{II}}^{\pm}(x, \tau, \sigma)$ with positive imaginary part or negative one when Im $\tau<0$ respectively. Moreover if $\tau=\theta(x, \sigma)$, then they are equal to the double real zero of det $P^0(x, \tau, \sigma, \lambda)$, and $h_{\mathrm{II}}'(x, \tau, \sigma)$ or $h_{\mathrm{II}}''(x, \tau, \sigma)$ is an eigenvector or a generalized eigenvector of $M_{\mathrm{II}}(x, \tau, \sigma)$ associated with the eigenvalue $\lambda_{\mathrm{II}}^{+}(x, \theta(x, \sigma), \sigma)$ respectively.

3) $\pm M_{\mathrm{III}}^{\pm}(x, \tau, \sigma)$ are $(m-l)\times(m-l)$ matrices whose eigenvalues have positive imaginary parts at (x^0, τ^0, σ^0) and, in fact, whose imaginary parts are positive definite. (See [7] and §3 in [11]).

γ) Next let $(x^0, \eta^0, \sigma^0)\in\Gamma\times\partial\Sigma$ be a point such that det $P^0(x^0, \eta^0, \sigma^0, \lambda)$ has the double real zero λ^0. Then the eigenvalues $\lambda_{\mathrm{II}}^{\pm}(x, \tau, \sigma)$ of $M_{\mathrm{II}}(x, \tau, \sigma)$ are functions defined in a conic neighborhood $U(x^0, \eta^0, \sigma^0)$ with Im $\tau\leqq 0$ and are represented as

$$\text{(2.4)}\qquad \begin{aligned}&(a)\quad \lambda_{\mathrm{II}}^{\pm}(x,\tau,\sigma)=\lambda_1(x,\tau,\sigma)\mp\sqrt{\zeta}\,\lambda_2(x,\tau,\sigma)\quad\text{or}\\ &(b)\quad \lambda_{\mathrm{II}}^{\pm}(x,\tau,\sigma)=\lambda_1(x,\tau,\sigma)\pm i\sqrt{\zeta}\,\lambda_2'(x,\tau,\sigma)\end{aligned}$$

according as the normal cone cut by $x=x^0$ and $\sigma=\sigma^0$ is convex or concave with respect to η at (η^0, λ^0) respectively. Here λ_1, λ_2 and λ_2' are real valued for real τ, analytic in ζ, $\lambda_1\in S_+^1$, $\lambda_2, \lambda_2'\in S_+^{\frac{1}{2}}$ and λ_2, λ_2' are positive at (x^0, η^0, σ^0). (See Lemma 3.1 in [11]). Moreover M_{II} has the following expansion:

$$\text{(2.5)}\qquad \begin{aligned}M_{\mathrm{II}}(x,\tau,\sigma)&=\begin{bmatrix}\lambda_1(x,\theta(x,\sigma),\sigma) & \Lambda(\tau,\sigma)\\ 0 & \lambda_1(x,\theta(x,\sigma),\sigma)\end{bmatrix}\\ &+\mathrm{Re}\,\zeta\begin{bmatrix}e_{11} & e_{12}\\ e_{21} & e_{22}\end{bmatrix}(x,\eta,\sigma)-i\gamma\begin{bmatrix}h_{11} & h_{12}\\ h_{21} & h_{22}\end{bmatrix}(x,\eta,\sigma)+O(\gamma^2\Lambda^{-1}),\end{aligned}$$

where $e_{ij}(x,\eta,\sigma)$ and $h_{ij}(x,\eta,\sigma)$ belong to S_+^0, e_{ij} are real valued,

$$\text{(2.6)}\qquad e_{11}=e_{22}$$

(see P. 132 in [15]) and at (x^0, η^0, σ^0)

$$\text{(2.7)}\qquad e_{21}=h_{21}>0\quad\text{or}\quad<0$$

according to the case (a) or (b) of (2.4) respectively.

In what follows for simplicity of description we shall be concerned with only the case (a) above unless otherwise indicated.

2.3. In order to define Lopatinskii determinant let ${}^t(1, s(x,\tau,\sigma))$ be an eigenvector of $M_{\mathrm{II}}(x,\tau,\sigma)$ associated with the eigenvalue $\lambda_{\mathrm{II}}^{+}(x,\tau,\sigma)$ and set

$$\text{(2.8)}\qquad h_{\mathrm{II}}^{+}=h_{\mathrm{II}}'+sh_{\mathrm{II}}''.$$

Then $h_{\mathrm{II}}^{+}(x,\tau,\sigma)$ is an eigenvector of $M(x,\tau,\sigma)$ associated with $\lambda_{\mathrm{II}}^{+}(x,\tau,\sigma)$ and the function

$$\text{(2.9)}\qquad L(x',\tau,\sigma)=\det B(x')(h_{\mathrm{I}}^{+}, h_{\mathrm{II}}^{+}, h_{\mathrm{III}}^{+})(x',\tau,\sigma)$$

is said to be *Lopatinskii determinant* of the problem (P^0, B). (See Definition 4.1 in [11]). By (2.8) and the fact that $s(x,\theta(x,\sigma),\sigma)=0$ (see (2.4) and (2.5)), (2.9) becomes

$$\text{(2.10)}\qquad L(x',\tau,\sigma)=\det B(x')(h_{\mathrm{I}}^{+}, h_{\mathrm{II}}', h_{\mathrm{III}}^{+})(x',\tau,\sigma),\quad\text{if}\quad \tau=\theta(x',\sigma).$$

Let $(x^0, \eta^0, \sigma^0)\in\Gamma'\times\partial\Sigma$ be a point such that $L(x^0, \eta^0, \sigma^0)=0$ and $\det P^0(x^0, \eta^0, \sigma^0, \lambda)$ has no double real zero. Then $L(x',\tau,\sigma)$ is defined in a conic neighborhood of (x^0, η^0, σ^0), analytic in τ and smooth in $(x', \eta, \sigma, \gamma)$. Moreover it is represented as

$$\text{(2.11)}\qquad L(x',\tau,\sigma)=\big(\tau-\nu(x',\sigma)\big)L^{(1)}(x',\tau,\sigma)$$

in a conic neighborhood of (x^0, η^0, σ^0), where $\nu(x', \sigma)$ is smooth, homogeneous of degree one in σ, $\nu(x^0, \sigma^0)=\eta^0$, and $L^{(1)}(x', \tau, \sigma)$ is smooth, analytic in τ, homogeneous of degree -1 in (τ, σ) and $L^{(1)}(x^0, \eta^0, \sigma^0)=(\partial L/\partial\tau)(x^0, \eta^0, \sigma^0) \neq 0$. (See P. 111 in [11]).

Next let $(x^0, \eta^0, \sigma^0) \in \Gamma \times \partial\Sigma$ be a point such that $L(x^0, \eta^0, \sigma^0)=0$ and det $P^0(x^0, \eta^0, \sigma^0, \lambda)$ has a double real zero. Then we find by virtue of (2. 4), (2. 8) and (2. 9) that $L(x', \tau, \sigma)$ is defined in a conic neighborhood of (x^0, η^0, σ^0) with Im $\tau \leqq 0$, homogeneous of degree zero in (τ, σ), analytic in $\sqrt{\zeta}$ and smooth in $(x', \sqrt{\zeta}, \sigma)$, so it is continous, since $s(x', \tau, \sigma)$ is so. Moreover we obtain

$$\det B(x^0)(h_{\mathrm{I}}^+, h_{\mathrm{II}}'', h_{\mathrm{III}}^+)(x^0, \eta^0, \sigma^0) \neq 0 \tag{2.12}$$

(see P. 120 in [11]). Therefore L is represented as

$$L(x', \tau, \sigma) = \left(\sqrt{\zeta} - D(x', \sigma)\right) \cdot (\text{nonzero factor}) \tag{2.13}$$

in a conic neighborhood of (x^0, η^0, σ^0) with Im $\tau \leqq 0$, where $D(x', \sigma)$ is smooth, homogeneous of degree 1/2 in σ and $D(x^0, \sigma^0)=0$. (See P. 84 in [11]).

Now let us define a function Q by

$$Q(x', \tau, \sigma) = \left(\det B(h_{\mathrm{I}}^+, h_{\mathrm{II}}'', h_{\mathrm{III}}^+)\right)^{-1} \left(\det B(h_{\mathrm{I}}^+, h_{\mathrm{II}}', h_{\mathrm{III}}^+)\right)(x', \tau, \sigma). \tag{2.14}$$

(See Lemma 6. 1 in [11]). Then by the implicit function theorem we find a smooth and real valued function $\rho(x', \sigma)$ defined in a conic neighborhood of (x^0, σ^0) which is homogeneous of degree one in σ and such that $\rho(x^0, \sigma^0) = \eta^0$ and $\eta' = \rho(x', \sigma')$ satisfies

$$\left(|Q|^2 + e_{21} \operatorname{Re} \zeta (1 + e_{12} \operatorname{Re} \zeta)^{-1}\right)(x', \eta', \sigma') = 0. \tag{2.15}$$

(See Remark 7. 1 in [11] or Lemma 2. 3 in [15]). Moreover let $L_{\mathrm{II}}(x', \tau, \sigma)$ be the Lopatinskii determinant of the problem:

$$\begin{cases} (D_n - M_{\mathrm{II}}(x, D'))\, u_{\mathrm{II}} = f & \text{in } \Omega, \\ u_{\mathrm{II}}'' + Q(x', D')\, u_{\mathrm{II}}' = g & \text{on } \Gamma, \end{cases}$$

where $u_{\mathrm{II}} = {}^t(u_{\mathrm{II}}', u_{\mathrm{II}}'')$. (See Lemma 6. 3 in [11]). Then from (6. 4. 3) in [11] we have

$$\begin{aligned} L_{\mathrm{II}}(x', \tau, \sigma) &= Q(x', \tau, \sigma) + s(x', \tau, \sigma) \\ &= L(x', \tau, \sigma) \cdot (\text{nonzero factor}). \end{aligned} \tag{2.16}$$

Hence we find from (2. 8), (2. 20) and (2. 21) in [15] that in the case (*a*) of (2. 4)

$$\left(|Q|^2+e_{21}\operatorname{Re}\zeta(1+e_{12}\operatorname{Re}\zeta)^{-1}\right)(x',\eta',\sigma') = \left((\operatorname{Re}\zeta+|D|^2)A-4\operatorname{Re}\zeta\, B\operatorname{Re}D\right)(x',\eta',\sigma'), \tag{2.17}$$

$$(s_2\operatorname{Re}Q)(x',\eta',\sigma')=\left(A\operatorname{Re}D-(\operatorname{Re}\zeta+|D|^2)B\right)(x',\eta',\sigma'), \tag{2.18}$$

where $s_2(x',\eta,\sigma)>0$, $A(x',\eta,\sigma)>0$ and $B(x',\eta,\sigma)$ is real.

2.4. Finally let $L(x',\tau,\sigma)\neq 0$. Then the elements b_{ij}, $i,j=1,\cdots,m$ of the following matrix are said to be *coupling coefficients* of (P^0,B):

$$\begin{aligned}&\left(b_{ij}(x',\tau,\sigma)\,;\ i\downarrow 1,\cdots,m, j\rightarrow 1,\cdots,m\right)\\ &=\begin{bmatrix} b_{\mathrm{I\,I}} & b_{\mathrm{I\,II}} & b_{\mathrm{I\,III}}\\ b_{\mathrm{II\,I}} & b_{\mathrm{II\,II}} & b_{\mathrm{II\,III}}\\ b_{\mathrm{III\,I}} & b_{\mathrm{III\,II}} & b_{\mathrm{III\,III}}\end{bmatrix}(x',\tau,\sigma)\\ &=\left(B(h_{\mathrm{I}}^{+},h_{\mathrm{II}}^{+},h_{\mathrm{III}}^{+})\right)^{-1}B(h_{\mathrm{I}}^{-},h_{\mathrm{II}}'',h_{\mathrm{III}}^{-}).\end{aligned} \tag{2.19}$$

(See Definition 4.2 in [11]).

§ 3. Necessary conditions for (P,B) to be L^2-well posed

First of all we shall show the following

LEMMA 3.1. *The implication (B)→(C) in Theorem is valid.*

PROOF. It is known that (B) and (C) are equivalent in the case of constant coefficients. (See Theorem 4.1 in [11]). Therefore we only have to show that the constant C in (1.2) is independent of x'. Let $(P^0,B)_{y'}$ be L^2-well posed for a point $y'\in\Gamma$. Then, if $f(x)$ belongs to $H_{1,\gamma}(\Omega)$ and vanishes when $x_0<0$, the solution $u(x)$ of the problem $(P^0,B)_{y'}$ with $g=0$ is so. (See §5, Remark 3) in [8]). Hence, applying the method used in the proof of Theorem 4.1 in [5], we find that under the condition (B) the norm of the Green's operator $\in B(L^2(0,\infty),L^2(0,\infty))$ of the boundary value problem for an ordinary differential system depending parameters (x',τ,σ) $(\operatorname{Im}\tau<0)$:

$$\begin{cases} P^0(x',\tau,\sigma,D_n)\,u(x_n)=f(x_n) & \text{in} \quad x_n>0,\\ B(x')\,u(x_n)=0 & \text{on} \quad x_n=0\end{cases}$$

is estimated by $\operatorname{Im}|\tau|^{-1}$ uniformly with respect to (x',τ,σ) near a fixed point $(\tilde{x}',\tilde{\eta},\tilde{\sigma})\in\Gamma\times\partial\Sigma$. Therefore we can obtain (1.2) by the same procedure as in the proof of Theorem 4.1 in [11]. This completes the proof.

In what follows $(x',\eta-i\gamma,\sigma)$ varies only near a given point (x^0,η^0,σ^0) $\in\Gamma\times\partial\Sigma$ and throughout sections 3, 4 and 5 we assume that the condition

(C) holds, so every constant coefficients problem $(P^0, B)_{x'}$ is L^2-well posed.

Let $(x^0, \eta^0, \sigma^0)\in\Gamma\times\partial\Sigma$ be a point such that $L(x^0, \eta^0, \sigma^0)=0$ and the polynomial det $P^0(x^0, \eta^0, \sigma^0, \lambda)$ has no double real zero. Then we have

LEMMA 3.2. *There exist indexes i and j with* Im $\lambda_i^+(x^0, \eta^0, \sigma^0)>0$ *and* Im $\lambda_j^-(x^0, \eta^0, \sigma^0)<0$ *such that*

1) *The matrix resulting from replacing the i-th column of the Lopatinskii determinant by Bh_j^- does not vanish at (x^0, η^0, σ^0), that is,*

$$\det B(x^0)(h_1^+, \cdots, h_{i-1}^+, h_j^-, h_{i+1}^+, \cdots, h_m^+)(x^0, \eta^0, \sigma^0)\neq 0, \tag{3.1}$$

where $h_p^+, p=1, \cdots, m$ and h_j^- are generalized eigenvectors of M associated with λ_p^+ and λ_j^- at (x^0, η^0, σ^0) respectively.

2) *There is a positive constant $C=C(x^0, \eta^0, \sigma^0)$ satisfying*

$$\left|(b_{i\mathrm{I}}L)(x', \eta, \sigma)\right|^2+\left|(b_{\mathrm{I}j}L)(x', \eta, \sigma)\right|^2\leqq C\,\mathrm{Im}\,\nu(x', \sigma) \tag{3.2}$$

for $\eta=\mathrm{Re}\,\nu(x', \sigma)$. Here $\nu(x', \sigma)$ is the function in (2.11).

PROOF. We first observe from (2.9) and (2.19) that the left hand side of (3.1) is equal to $(b_{ij}L)(x^0, \eta^0, \sigma^0)$, for every $i, j=1, \cdots, m$. On the other hand, since $L(x^0, \eta^0-i\gamma, \sigma^0)=O(\gamma)$, it follows from (1.2) with $(x', \tau, \sigma)=(x^0, \eta^0-i\gamma, \sigma^0)$ that $(b_{ij}L)(x^0, \eta^0, \sigma^0)=0$ if $i\in \mathrm{I}$ or $j\in\mathrm{I}$. Hence we find indexes $i, j\in\mathrm{III}$ satisfying (3.1), because of (2.11) and rank $(BS)(x^0, \eta^0, \sigma^0)=m$. (See also the proof of Lemma 5.2, (i) in [11]). Furthermore (2.11) and the Hersh's condition imply that Im $\nu(x', \sigma)\geqq 0$.

To derive (3.2) let det $P^0(x^0, \eta^0, \sigma^0, \lambda)$ have simple real zeros. Then it follows from (1.2) that for $\gamma>0$ and some $C>0$

$$\left|(b_{\mathrm{I}j}L)(x', \eta-i\gamma, \sigma)\right|^2\leqq C\gamma^{-1}\left|L(x', \eta-i\gamma, \sigma)\right|^2, \tag{3.3}$$

because of β), 1) and 3) in §2.2. On the other hand from (2.11) we have for some C

$$\left|L(x', \eta-i\gamma, \sigma)\right|^2\leqq C\left(\gamma^2+(\mathrm{Im}\,\nu(x', \sigma))^2\right), \quad \text{if} \quad \eta=\mathrm{Re}\,\nu(x', \sigma). \tag{3.4}$$

Furthermore the left hand side of (3.3) is estimated from below by

$$2^{-1}\left|(b_{\mathrm{I}j}L)(x', \eta, \sigma)\right|^2-0(\gamma^2),$$

since $(b_{\mathrm{I}j}L)(x', \eta-i\gamma, \sigma)$ is smooth. Thus from (3.3) and (3.4) we obtain

$$\left|(b_{\mathrm{I}j}L)(x', \mathrm{Re}\,\nu(x', \sigma), \sigma)\right|^2\leqq C\left\{\gamma+\gamma^{-1}(\mathrm{Im}\,\nu(x', \sigma))^2\right\} \tag{3.5}$$

for small $\gamma>0$ and some $C>0$. Here we may assume that $0\leqq\mathrm{Im}\,\nu(x', \sigma)<\delta$ for some $\delta>0$ and that (3.5) is valid if $0<\gamma<\delta$ with the same δ.

Now let us fix (x', σ). Then if $\mathrm{Im}\, \nu(x', \sigma)=0$, the left hand side of (3.5) must vanish, since γ is independent of (x', σ). Otherwise let $\gamma=\mathrm{Im}\, \nu(x', \sigma)$. Then from (3.5) we also obtain for some $C>0$

$$\left|(b_{\mathrm{I}j}L)(x', \mathrm{Re}\, \nu(x', \sigma), \sigma)\right|^2 \leqq C\, \mathrm{Im}\, \nu(x', \sigma).$$

Similarly we can conclude that (3.2) is valid. The proof is complete.

Next we consider the case where the double real zero exists. Hereafter in this section let $(x^0, \eta^0, \sigma^0) \in \Gamma \times \partial\Sigma$ be a point such that $L(x^0, \eta^0, \sigma^0)=0$ and $\det P^0(x^0, \eta^0, \sigma^0, \lambda)$ has a double real zero. Then we obtain

LEMMA 3.3.

$$\left|(b_{\mathrm{I\,II}}L)(x', \eta, \sigma)\right|^2+\left|(b_{\mathrm{II\,I}}L)(x', \eta, \sigma)\right|^2 \leqq C\left|D(x', \sigma)\right| \tag{3.6}$$

for $\eta=\rho(x', \sigma)$, *where* $\rho(x', \sigma)$ *is the function defined by* (2.15) *and* C *is a positive constant.*

PROOF. It follows from (1.2) and (2.13) that for $\gamma>0$

$$\left|(b_{\mathrm{II\,I}}L)(x', \eta-i\gamma, \sigma)\right|^2 \leqq C\gamma^{-1}\left|\mathrm{Im}\, \lambda_{\mathrm{II}}^{+}(x', \eta-i\gamma, \sigma)\right|\left|\sqrt{\zeta}-D(x', \sigma)\right|^2, \tag{3.7}$$

because of β), 1) in §2.2. Hereafter in the proof let C denote various positive constants independent of $(x', \eta, \sigma, \gamma)$. Then by (2.4) we have

$$\left|\mathrm{Im}\, \lambda_{\mathrm{II}}^{+}(x', \eta-i\gamma, \sigma)\right| \leqq C\left|\mathrm{Im}\sqrt{\zeta}\right|+O(\gamma). \tag{3.8}$$

Furthermore the choise of the branch $\sqrt{\zeta}$ implies

$$C|\zeta|^{\frac{1}{2}} \leqq -\mathrm{Im}\sqrt{\zeta} \leqq C^{-1}|\zeta|^{\frac{1}{2}}, \quad \text{if} \quad \mathrm{Re}\, \zeta \leqq 0. \tag{3.9}$$

On the other hand from (2.8), (2.9) and (2.19) we have for $j \in \mathrm{I}$

$$\begin{aligned} b_{\mathrm{II}j}L &= \det B(h_{\mathrm{I}}^{+}, h_j^{-}, h_{\mathrm{III}}^{+}), \\ b_{j\mathrm{II}}L &= \det B(h_1^{+}, \cdots, h_{j-1}^{+}, h_{\mathrm{II}}'', h_{j+1}^{+}, \cdots, h_{|\mathrm{I}|}^{+}, h_{\mathrm{II}}', h_{\mathrm{III}}^{+}). \end{aligned} \tag{3.10}$$

Hence $(b_{\mathrm{II}\,j}L)(x', \eta-i\gamma, \sigma)$ is smooth in $(x', \eta, \sigma, \gamma)$. Therefore from (3.7), (3.8) and (3.9) we obtain

$$\left|(b_{\mathrm{II\,I}}L)(x', \eta, \sigma)\right|^2 \leqq C\gamma^{-1}|\zeta|^{\frac{1}{2}}\left(|\zeta|+|D(x', \sigma)|^2\right) \tag{3.11}$$

for small $\gamma>0$ and $\mathrm{Re}\, \zeta \leqq 0$, as in deriving (3.5). Furthermore (2.15) and (2.17) imply

$$C^{-1}|\mathrm{Re}\, \zeta| \leqq \left|D(x', \sigma)\right|^2 \leqq C|\mathrm{Re}\, \zeta|, \quad \text{if} \quad \eta=\rho(x', \sigma). \tag{3.12}$$

Hence from (3.11) we have

$$\left|(b_{\mathrm{II\,I}}L)(x', \eta, \sigma)\right|^2 \leqq C\gamma^{-1}\left(\gamma+|D(x', \sigma)|^2\right)^{\frac{3}{2}}$$

for small $\gamma>0$ and $\eta=\rho(x',\sigma)$, since $\operatorname{Im}\zeta=-\gamma$. Therefore we obtain

$$\left|(b_{\mathrm{III\,I}}L)(x',\rho(x',\sigma),\sigma)\right|^2\leqq C\left|D(x',\sigma)\right|,$$

as in the proof of (3.2). Similarly we conclude that (3.6) is valid, since it follows from (3.10) that $(b_{j\,\mathrm{II}}L)(x',\eta-i\gamma,\sigma)$ is smooth in (x',η,σ,γ).

COROLLARY 3.4. *Under the hypotheses of Lemma* 3.3 *Condition* (*iii*) *implies that*

$$\begin{aligned}\left|(b_{\mathrm{I\,II}}L)(x',\eta,\sigma)\right|^2+\left|(b_{\mathrm{II\,I}}L)(x',\eta,\sigma)\right|^2\\ \leqq -C\operatorname{Re}D(x',\sigma)\quad or\quad C\operatorname{Im}D(x',\sigma)\end{aligned}\tag{3.13}$$

for $\eta=\rho(x',\sigma)$ *and some constant* $C>0$ *according to the case* (*a*) *or* (*b*) *of* (2.4) *respectively, where the left hand side of* (3.13) *is regarded as zero if* $\det P^0(x^0,\eta^0,\sigma^0,\lambda)$ *has no simple real zero.*

§ 4. Construction of modified symmetrizers I

In this section we shall show that (3.2) is sufficient for a modified symmetrizer to be constructed in the case where the double real zero is absent and then we derive microlocally a priori estimate.

Let $(x^0,\eta^0,\sigma^0)\in\Gamma\times\partial\Sigma$ be such a point as in Lemma 3.2 and suppose that the condition (*C*) holds. Then we have

LEMMA 4.1. *There exists a* $2m\times 2m$ *hermitian matrix* $R(x,\eta,\sigma,\gamma)\in S^0_+$, *homogeneous in* (η,σ,γ) *such that*

$$\operatorname{Re}(iRM)(x,\eta,\sigma,\gamma)\,u\cdot u\geqq a_0\gamma|u|^2\tag{4.1}$$

for $(x,\eta,\sigma,\gamma)\in\bar{\Omega}\times\Sigma$ *and vector*:

$$u={}^t({}^tu^+_{\mathrm{I}},{}^tu^-_{\mathrm{I}},{}^tu^+_{\mathrm{III}},{}^tu^-_{\mathrm{III}})\in C^{2m}\tag{4.2}$$

satisfying $u^-_{\mathrm{III}}=0$, *where*

$${}^tu^{\pm}_{\mathrm{I}}=(u^{\pm}_1,\cdots,u^{\pm}_{|\mathrm{I}|}),\ {}^tu^{\pm}_{\mathrm{III}}=(u^{\pm}_{l+1},\cdots,u^{\pm}_m),$$

$a_0=a_0(x^0,\eta^0,\sigma^0)$ *is a positive constant and* $v\cdot u$ *stands for the inner product in* C^{2m}. *Furthermore*

$$R(x',\eta,\sigma,\gamma)\,u\cdot u\geqq 0\tag{4.3}$$

for $(x',\eta,\sigma,\gamma)\in\Gamma\times\Sigma$ *and* $u\in C^{2m}$ *of the form* (4.2) *such that*

$$B(x')\,S(x',\eta-i\gamma,\sigma)\,u=0\tag{4.4}$$

and

$${}^tu^-_{\mathrm{III}}=(0,\cdots,0,u^-_j,0,\cdots,0),\tag{4.5}$$

where j is the index in Lemma 3.2 *and* $S(x, \tau, \sigma)$ *is the matrix in* (2.1).

REMARK 4.2. The restrictions $u_{\mathrm{III}}^{-}=0$ in (4.1) and (4.5) on u are due to the fact that the Dirichlet problem for $D_n-M_{\mathrm{III}}^{-}(x, D')$ is coercive. (See Lemma 4.3 below).

PROOF of LEMMA 4.1. We may assumed without loss of generality that $i=j=m$ in Lemma 3.2. Put

$$
\begin{aligned}
B'(x', \tau, \sigma) &= B(x')(h_{\mathrm{I}}^{-}, h_{\mathrm{III}'}^{-}, h_m^{+})(x', \tau, \sigma), \\
B''(x', \tau, \sigma) &= B(x')(h_{\mathrm{I}}^{+}, h_{\mathrm{III}'}^{+}, h_m^{-})(x', \tau, \sigma),
\end{aligned} \tag{4.6}
$$

where $\mathrm{III}'=\{l+1, \cdots, m-1\}$. Then (3.1) becomes

$$\det B''(x^0, \eta^0, \sigma^0) \neq 0. \tag{4.7}$$

Now we shall define a hermitian matrix $R(x, \eta, \sigma, \gamma)$ in the following way:

$$
R(x, \eta, \sigma, \gamma)=\left[\begin{array}{cc|cc}
-E_l & & & \\
& c_{\mathrm{I}}^{-} E_l & & \\
\hline
& & -\gamma' E_{m-l} & R_0 \\
& & R_0^* & R_{\mathrm{III}}^{-}
\end{array}\right] \tag{4.8}
$$

where R_{III}^{-} and R_0 are matrices whose elements equal zero except the lowest right ones r_{III}^{-} and r_{mm} respectively, and where c_{I}^{-} and r_{III}^{-} are positive constants to be chosen later. Furthermore

$$r_{mm}(x, \eta, \sigma, \gamma)=(2i)^{-1} a (L^{(1)})^{-1} \det B''(x', \tau', \sigma'), \tag{4.9}$$

where a is a positive constant to be chosen later and $L^{(1)}(x', \tau, \sigma)$ is the function in (2.11). Then we see easily that the matrix $R(x, \eta, \sigma, \gamma)$ thus defined belongs to S_{+}^{0}, is homogeneous in (η, σ, γ) and satisfies (4.1) when c_{I}^{-}, r_{III}^{-} and a are arbitrary positive constants. Therefore it is enough to show that (4.3) is valid. Hereafter in the proof we consider only such $u \in C^{2m}$ as described in the statement of the lemma. Then (4.2) and (4.8) imply

$$
\begin{aligned}
R(x', \eta, \sigma, \gamma) u \cdot u = & -u_{\mathrm{I}}^{+} \cdot u_{\mathrm{I}}^{+} + c_{\mathrm{I}}^{-} u_{\mathrm{I}}^{-} \cdot u_{\mathrm{I}}^{-} \\
& -\gamma' u_{\mathrm{III}}^{+} \cdot u_{\mathrm{III}}^{+} + r_{\mathrm{III}}^{-} u_m^{-} \cdot u_m^{-} + 2\mathrm{Re}\, r_{mm} u_m^{-} \cdot u_m^{+}.
\end{aligned} \tag{4.10}
$$

Let

$$
(B''^{-1} B')(x', \tau, \sigma)=\begin{bmatrix}
k_{\mathrm{I}\,\mathrm{I}} & k_{\mathrm{I}\,\mathrm{III}'} & k_{\mathrm{I}m} \\
k_{\mathrm{III}'\,\mathrm{I}} & k_{\mathrm{III}'\,\mathrm{III}'} & k_{\mathrm{III}'\,m} \\
k_{m\mathrm{I}} & k_{m\,\mathrm{III}'} & k_{mm}
\end{bmatrix}(x', \tau, \sigma). \tag{4.11}
$$

Then it follows from (2. 9), (2. 19) and (4. 6) that

(4. 12) $$k_{mm}=(\det B'')^{-1}L,$$

(4. 13) $$b_{m\mathrm{I}}L=k_{m\mathrm{I}}\det B'',\ b_{\mathrm{I}m}L=-k_{\mathrm{I}m}\det B''.$$

Furthermore (4. 4) may be written as

(4. 14) $$\begin{bmatrix} u_{\mathrm{I}}^{+} \\ u_{\mathrm{III}'}^{+} \\ u_m^{-} \end{bmatrix} = -\begin{bmatrix} k_{\mathrm{I}\,\mathrm{I}} & k_{\mathrm{I}m} \\ k_{\mathrm{III}'\mathrm{I}} & k_{\mathrm{III}'m} \\ k_{m\mathrm{I}} & k_{mm} \end{bmatrix}(x', \tau, \sigma)\begin{bmatrix} u_{\mathrm{I}}^{-} \\ u_m^{+} \end{bmatrix},$$

if (4. 5) is satisfied, i. e., $u_{\mathrm{III}'}^{-}=0$.

Now we insert (4. 14) into (4. 10). Then by a simple calculation we have

(4. 15) $$R(x', \eta, \sigma, \gamma)\,u\cdot u=\begin{bmatrix} q_{\mathrm{I}\,\mathrm{I}} & q_{\mathrm{I}m}^{*} \\ q_{\mathrm{I}m} & q_{mm} \end{bmatrix}(x', \eta, \sigma, \gamma)\begin{bmatrix} u_{\mathrm{I}}^{-} \\ u_m^{+} \end{bmatrix}\cdot\begin{bmatrix} u_{\mathrm{I}}^{-} \\ u_m^{+} \end{bmatrix},$$

where

(4. 16) $$q_{\mathrm{I}\,\mathrm{I}}(x', \eta, \sigma, \gamma)=\left(c_{\mathrm{I}}^{-}+r_{\mathrm{III}}^{-}|k_{m\mathrm{I}}|^2\right)E_l-k_{\mathrm{I}\,\mathrm{I}}^{*}k_{\mathrm{I}\,\mathrm{I}}+O(\gamma),$$

(4. 17) $$q_{\mathrm{I}m}(x', \eta, \sigma, \gamma)=-k_{\mathrm{I}m}^{*}k_{\mathrm{I}\,\mathrm{I}}+r_{\mathrm{III}}^{-}\bar{k}_{mm}k_{m\mathrm{I}}-r_{mm}k_{m\mathrm{I}}+O(\gamma)$$

and

(4. 18) $$q_{mm}(x', \eta, \sigma, \gamma)=-2\mathrm{Re}(r_{mm}k_{mm})+r_{\mathrm{III}}^{-}|k_{mm}|^2-|k_{\mathrm{I}m}|^2-\gamma\left(1+|k_{\mathrm{III}'m}|^2\right).$$

Here and in what follows we suppose that $(\eta, \sigma, \gamma)\in\Sigma$. On the other hand it follows from (4. 9), (4. 12) and (2. 11) that

(4. 19) $$-2\mathrm{Re}(r_{mm}k_{mm})(x', \eta, \sigma, \gamma)=a\left(\gamma+\mathrm{Im}\,\nu(x', \sigma)\right)$$

and

(4. 20) $$\left|k_{mm}(x', \eta-i\gamma, \sigma)\right|^2=\left\{(\eta-\mathrm{Re}\,\nu(x', \sigma))^2+(\gamma+\mathrm{Im}\,\nu(x', \sigma))^2\right\}|k_{mm}^{(1)}|^2,$$

where

$$k_{mm}^{(1)}(x^0, \eta^0, \sigma^0)=(\det B'')^{-1}L^{(1)}(x^0, \eta^0, \sigma^0)\neq 0.$$

Here we shall use (3. 2). Then from (4. 7) and (4. 13) we have for some constant $C>0$

(4. 21) $$\begin{aligned} &\left|k_{\mathrm{I}m}(x', \eta-i\gamma, \sigma)\right|^2+\left|k_{m\mathrm{I}}(x', \eta-i\gamma, \sigma)\right|^2 \\ &\qquad\leqq C\left\{\mathrm{Im}\,\nu(x', \sigma)+(\eta-\mathrm{Re}\,\nu(x', \sigma))^2+\gamma^2\right\}. \end{aligned}$$

Therefore taking a, r_{III}^{-} sufficiently large and using (4. 18)–(4. 21) we obtain

(4. 22) $$q_{mm}(x', \eta, \sigma, \gamma)\geqq\gamma+\mathrm{Im}\,\nu(x', \sigma)+\left(\eta-\mathrm{Re}\,\nu(x', \sigma)\right)^2.$$

Hence from (4.16), (4.17) and (4.21) we have

$$\begin{bmatrix} q_{\mathrm{I}\,\mathrm{I}} & q^*_{\mathrm{I}m} \\ q_{\mathrm{I}m} & q_{mm} \end{bmatrix}(x', \eta, \sigma, \gamma) \geqq \frac{1}{2}\begin{bmatrix} E_l & 0 \\ 0 & \gamma \end{bmatrix} \tag{4.23}$$

for $(x', \eta, \sigma, \gamma) \in (\Gamma \times \Sigma) \cap U(x^0, \eta^0, \sigma^0)$, if we take c_{I}^- sufficiently large. Thus (4.15) and (4.23) give (4.3). This completes the proof.

To obtain a priori estimate we also use the following

LEMMA 4.3. *For every $(x^0, \tau^0, \sigma^0) \in \Gamma \times \Sigma$ there are positive constants C, γ_0 and a neighborhood $U(x^0, \tau^0, \sigma^0)$ such that*

$$\|\phi v\|^2_{1,\gamma} + |\phi v|^2_{\frac{1}{2},\gamma} \leqq C\left\|(D_n - M_{\bar{\mathrm{I}}\mathrm{I}\mathrm{I}})\phi v\right\|^2_{0,\gamma} + C_\phi\left(\|v\|^2_{0,\gamma} + |v|^2_{-\frac{1}{2},\gamma}\right)$$

for all $v \in H_{1,\gamma}(\Omega)$ and $\gamma \geqq \gamma_0$, where $\phi(x, \eta, \sigma, \gamma) \in S^0_+$, supp $\phi \cap (\bar{\Omega} \times \Sigma) \subset U(x^0, \tau^0, \sigma^0)$ *and C_ϕ is a constant independent of v and γ.*

This is well-known. Concerning its proof see for instance Lemma 5.1 in [11].

LEMMA 4.4. *Let $(x^0, \eta^0, \sigma^0) \in \Gamma \times \partial\Sigma$ be such a point as in Lemma 4.1. Then there are positive constants C, γ_0 and a neighborhood $U(x^0, \eta^0, \sigma^0)$ such that*

$$\gamma^2 \|\phi u\|^2_{0,\gamma} \leqq C\left(\left\|(D_n - M)\phi u\right\|^2_{0,\gamma} + |BS\phi u|^2_{\frac{1}{2},\gamma}\right) + C_\phi \gamma\left(\|u\|^2_{0,\gamma} + |u|^2_{-\frac{1}{2},\gamma}\right) \tag{4.24}$$

for all $u \in H_{1,\gamma}(\Omega)$ and $\gamma \geqq \gamma_0$, where ϕ and C_ϕ are analogous to those in the previous lemma.

PROOF. We keep using the notations in the proof of Lemma 4.1. Let $U(x^0, \eta^0, \sigma^0)$ be an appropriate neighborhood and $\phi, \psi \in S^0_+$ be symbols such that supp $\psi \cap (\bar{\Omega} \times \Sigma) \subset U(x^0, \eta^0, \sigma^0)$, $\psi(x, \eta, \sigma, \gamma) = 1$ on supp ϕ and $|\psi| \leqq 1$. Furthermore let $u \in H_{1,\gamma}(\Omega)$ have the form (4.2) and put

$$\begin{cases} \left(D_n - M(x, D')\right)\psi\phi u = f & \text{in } \Omega, \\ \phi u'' + B''^{-1}B'\phi u' = g & \text{on } \Gamma, \end{cases} \tag{4.25}$$

where

$$u' = {}^t({}^t u_{\mathrm{I}}^-, {}^t u_{\mathrm{I}\mathrm{I}\mathrm{I}'}^-, u_m^+), \quad u'' = {}^t({}^t u_{\mathrm{I}}^+, {}^t u_{\mathrm{I}\mathrm{I}\mathrm{I}'}^+, u_m^-).$$

(See (4.6) and (4.11)). Then we have

$$-\mathrm{Re}(iRf, \psi\phi u)_{0,\gamma} = \mathrm{Re}(iRM\psi\phi u, \psi\phi u)_{0,\gamma} + \frac{1}{2}\mathrm{Re}\langle R\psi\phi u, \psi\phi u\rangle_{0,\gamma}, \tag{4.26}$$

where $R(x, \eta, \sigma, \gamma)$ is the symbol defined by (4.8) and (4.9).

We first show that for sufficiently large γ

$$\mathrm{Re}\langle R\psi\phi u, \psi\phi u\rangle_{0,\gamma} \geqq -C\gamma^{-1}\left(|\phi u_{\bar{\mathrm{I}}\mathrm{I}\mathrm{I}'}|^2_{\frac{1}{2},\gamma} + |g|^2_{\frac{1}{2},\gamma}\right) - C_{\phi,\psi}|u|^2_{-\frac{1}{2},\gamma}. \tag{4.27}$$

Hereafter in the proof let C and $C_{\phi,\psi}$ denote various positive constants independent of u and γ. Let

$$u=T\begin{bmatrix}u'\\u''\end{bmatrix}\quad\text{and}\quad \tilde{R}=T^*RT,$$

where T is a constant orthogonal matrix. Then we have

$$\begin{aligned}\langle R\psi\phi u,\psi\phi u\rangle_{0,\gamma}&=\left\langle\tilde{R}\psi\begin{bmatrix}\phi u'\\\phi u''\end{bmatrix},\psi\begin{bmatrix}\phi u'\\\phi u''\end{bmatrix}\right\rangle_{0,\gamma}\\&=\left\langle\tilde{R}\psi\begin{bmatrix}\phi u'\\\phi u''-g\end{bmatrix},\psi\begin{bmatrix}\phi u'\\\phi u''-g\end{bmatrix}\right\rangle_{0,\gamma}+\left\langle\tilde{R}\psi\begin{bmatrix}\phi u'\\\phi u''-g\end{bmatrix},\psi\begin{bmatrix}0\\g\end{bmatrix}\right\rangle_{0,\gamma}\\&+\left\langle\tilde{R}\psi\begin{bmatrix}0\\g\end{bmatrix},\psi\begin{bmatrix}\phi u'\\\phi u''-g\end{bmatrix}\right\rangle_{0,\gamma}+\left\langle R\psi\begin{bmatrix}0\\g\end{bmatrix},\psi\begin{bmatrix}0\\g\end{bmatrix}\right\rangle_{0,\gamma}.\end{aligned}$$

Since $|\psi|\leqq 1$ and $\phi u''-g=-B''^{-1}B'\phi u'$ by (4. 25), the last three terms in the right hand side are estimated from below by

$$-8^{-1}\gamma|\phi u'|^2_{-\frac12,\gamma}-C\gamma^{-1}|g|^2_{\frac12,\gamma}-C_{\phi,\psi}|u|^2_{-\frac12,\gamma}.$$

Hence we see that for $\gamma>0$

$$\begin{aligned}\text{(4. 28)}\quad&\operatorname{Re}\langle R\psi\phi u,\psi\phi u\rangle_{0,\gamma}\geqq\operatorname{Re}\left\langle\tilde{R}\psi\begin{bmatrix}\phi u'\\-B''^{-1}B'\phi u'\end{bmatrix},\psi\begin{bmatrix}\phi u'\\-B''^{-1}B'\phi u'\end{bmatrix}\right\rangle_{0,\gamma}\\&\qquad-8^{-1}\gamma|\phi u'|^2_{-\frac12,\gamma}-C\gamma^{-1}|g|^2_{\frac12,\gamma}-C_{\phi,\psi}|u|^2_{-\frac12,\gamma}.\end{aligned}$$

Now we shall consider the first term in the right hand side of (4. 28). Let us decompose u' as

$$u'={}^t({}^tu_{\mathrm{I}}^-,0,u_m^+)+{}^t(0,{}^tu_{\mathrm{III}'}^-,0).$$

Then, by the same way as above, we have

$$\begin{aligned}&\operatorname{Re}\left\langle\tilde{R}\psi\begin{bmatrix}\phi u'\\-B''^{-1}B'\phi u'\end{bmatrix},\psi\begin{bmatrix}\phi u'\\-B''^{-1}B'\phi u'\end{bmatrix}\right\rangle_{0,\gamma}\\\text{(4. 29)}\quad&\geqq\operatorname{Re}\left\langle q_\psi\phi\begin{bmatrix}u_{\mathrm{I}}^-\\u_m^+\end{bmatrix},\phi\begin{bmatrix}u_{\mathrm{I}}^-\\u_m^+\end{bmatrix}\right\rangle_{0,\gamma}-8^{-1}\gamma\left(|\phi u_{\mathrm{I}}^-|^2_{-\frac12,\gamma}+|\phi u_m^+|^2_{-\frac12,\gamma}\right)\\&-C\gamma^{-1}|\phi u_{\mathrm{III}'}^-|^2_{\frac12,\gamma}-C_{\phi,\psi}|u|^2_{-\frac12,\gamma},\end{aligned}$$

where on the support of ϕ the principal symbol of q_ψ is equal to the matrix in the right hand side of (4. 15). Therefore we see by virtue of (4. 23) and the sharp form of Gårding's inequality that the first term in the right hand side of (4. 29) is estimated from below by

$$4^{-1}\left(|\phi u_{\mathrm{I}}^-|^2_{0,\gamma}+\gamma\,|\phi u_m^+|^2_{-\frac12,\gamma}\right)-C_{\phi,\psi}|u|^2_{-\frac12,\gamma}.$$

From this, (4. 28) and (4. 29) we obtain (4. 27).

On the other hand it follows from (4. 1) that

(4. 30) $\quad \mathrm{Re}\,(iRM\psi\phi u, \psi\phi u)_{0,\gamma} \geqq C\gamma\|\phi u\|^2_{0,\gamma} - C^{-1}\gamma^{-1}\|\phi u^-_{\mathrm{III}}\|^2_{1,\gamma} - C_{\phi,\psi}\|u\|^2_{0,\gamma}$

for $\gamma \geqq \gamma_1 > 0$. Thus we obtain (4. 24) from (4. 25), (4. 26), (4. 27), (4. 30) and Lemma 4. 3.

§ 5. Construction of modified symmetrizers II

In this section we shall establish the analogues of Lemmas 4. 1 and 4. 4 for the case where the double real zero exists.

Let $(x^0, \eta^0, \sigma^0) \in \Gamma \times \partial\Sigma$ be such a point as in Lemma 3. 3. Then from Lemma 3. 2 in [15] we have the following

LEMMA 5. 1. *Let $R_{\mathrm{II}}(x, \eta, \sigma, \gamma) \in S^0_+$ be a 2×2 hermitian matrix, homogeneous in (η, σ, γ) such that*

$$R_{\mathrm{II}}(x, \eta, \sigma, \gamma) = \begin{bmatrix} b & d_1 \\ d_1 & d_2 \end{bmatrix}(x, \eta', \sigma') + \gamma' \begin{bmatrix} 0 & if \\ -if & 0 \end{bmatrix}, \tag{5. 1}$$

where d_1 and f are positive constants, $d_2(x, \eta, \sigma) \in S^0_+$ is an arbitrary real valued function which is homogeneous in (η, σ) and

$$b(x, \eta, \sigma) = \left(d_2 e_{21} \mathrm{Re}\,\zeta (1 + e_{12} \mathrm{Re}\,\zeta)^{-1}\right)(x, \eta', \sigma'). \tag{5. 2}$$

Then for every positive number δ there is a neighborhood $U_\delta(x^0, \eta^0, \sigma^0)$ such that

$$\mathrm{Re}(iR_{\mathrm{II}} M_{\mathrm{II}})(x, \eta, \sigma, \gamma) \geqq \gamma \begin{bmatrix} a_0 & 0 \\ 0 & \delta^{-1} \end{bmatrix} \tag{5. 3}$$

for all $(x, \eta, \sigma, \gamma) \in (\bar{\Omega} \times \Sigma) \cap U_\delta(x^0, \eta^0, \sigma^0)$ if we take f sufficiently large according to d_1, d_2 and δ, where a_0 is a positive constant dependent only on d_1 and M_{II} is the matrix in (2. 2).

Now suppose that the conditions (*iii*) and (C) hold. Then we have

LEMMA 5. 2. *There exists a $2m \times 2m$ hermitian matrix $R(x, \eta, \sigma, \gamma)$, homogeneous in (η, σ, γ) such that the inequality* (4. 1) *is valid for $(x, \eta, \sigma, \gamma) \in \bar{\Omega} \times \Sigma$ and vector*:

$$u = {}^t({}^t u^+_{\mathrm{I}}, {}^t u^-_{\mathrm{I}}, u'_{\mathrm{II}}, u''_{\mathrm{II}}, {}^t u^+_{\mathrm{III}}, {}^t u^-_{\mathrm{III}}) \in C^{2m}, \tag{5. 4}$$

where $u^\pm_{\mathrm{I}}$ and $u^\pm_{\mathrm{III}}$ are those described below (4. 2) *and u'_{II}, u''_{II} are complex numbers. Furthermore*

$$R(x', \eta, \sigma, \gamma)\, u \cdot u \geqq -a_1 \gamma |u'_{\mathrm{II}}|^2 \tag{5. 5}$$

is valid for $(x', \eta, \sigma, \gamma) \in \Gamma \times \Sigma$ and such u satisfying (4. 4), *where a_1 is a positive constant independent of $(x', \eta, \sigma, \gamma)$ and u.*

PROOF. We define a $2m \times 2m$ hermitian matrix R by

$$R(x, \eta, \sigma, \gamma)=\begin{bmatrix} R_{\mathrm{I}}^{+} & & & & \\ & R_{\mathrm{I}}^{-} & & & \\ & & R_{\mathrm{II}} & & \\ & & & R_{\mathrm{III}}^{+} & \\ & & & & R_{\mathrm{III}}^{-} \end{bmatrix}(x, \eta, \sigma, \gamma), \tag{5.6}$$

where

$$R_{\mathrm{I}}^{+}=-E_{l-1},\ R_{\mathrm{I}}^{-}=c_{\mathrm{I}}^{-}E_{l-1},\ R_{\mathrm{III}}^{+}=-\gamma' E_{m-l},\ R_{\mathrm{III}}^{-}=0,$$

c_{I}^{-} is a positive constant to be chosen later and $R_{\mathrm{II}}(x, \eta, \sigma, \gamma)$ is the mataix in Lemma 5.1. Then it follows from Lemma 5.1 and β), 1) and 3) in §2.2 that for every $\delta>0$ there is a neighborhood $U_{\delta}(x^0, \eta^0, \sigma^0)$ such that

$$\mathrm{Re}\,(iRM)(x, \eta, \sigma, \gamma)\,u\cdot u \geqq a_0\gamma|u|^2+\delta^{-1}\gamma|u_{\mathrm{II}}''|^2 \tag{5.7}$$

for all $(x, \eta, \sigma, \gamma)\in(\bar{\Omega}\times\Sigma)\cap U_{\delta}(x^0, \eta^0, \sigma^0)$ and $u\in C^{2m}$ of the form (5.4) satisfying $u_{\mathrm{III}}^{-}=0$, where a_0 is a positive constant independent of δ.

Next to derive (5.5) we set

$$\begin{aligned} B'(x', \tau, \sigma)&=B(x')(h_{\mathrm{I}}^{-}, h_{\mathrm{II}}', h_{\mathrm{III}}^{-})(x', \tau, \sigma), \\ B''(x', \tau, \sigma)&=B(x')(h_{\mathrm{I}}^{+}, h_{\mathrm{II}}'', h_{\mathrm{III}}^{+})(x', \tau, \sigma). \end{aligned} \tag{5.8}$$

Notice that the matrix B' is different from one used in [11]. Then we see from (2.12) that $B''(x', \tau, \sigma)$ is nonsingular. Furthermore put

$$(B''^{-1}B')(x', \tau, \sigma)=\begin{bmatrix} k_{\mathrm{I\,I}} & k_{\mathrm{I\,II}} & k_{\mathrm{I\,III}} \\ k_{\mathrm{II\,I}} & k_{\mathrm{II\,II}} & k_{\mathrm{II\,III}} \\ k_{\mathrm{III\,I}} & k_{\mathrm{III\,II}} & k_{\mathrm{III\,III}} \end{bmatrix}(x', \tau, \sigma) \tag{5.9}$$

(see (6.3) in [11]) and

$$u'={}^t({}^tu_{\mathrm{I}}^{-}, u_{\mathrm{II}}', {}^tu_{\mathrm{III}}^{-}), \qquad u''={}^t({}^tu_{\mathrm{I}}^{+}, u_{\mathrm{II}}'', {}^tu_{\mathrm{III}}^{+}). \tag{5.10}$$

Hereafter in the proof we shall consider only such $u\in C^{2m}$ as described in the statement of the lemma. Then it follows from (2.3) and (5.8)-(5.10) that the boundary condition $BSu=0$ may be written as

$$u''=-\begin{bmatrix} k_{\mathrm{I\,I}} & k_{\mathrm{I\,II}} \\ k_{\mathrm{II\,I}} & k_{\mathrm{II\,II}} \\ k_{\mathrm{III\,I}} & k_{\mathrm{III\,II}} \end{bmatrix}\begin{bmatrix} u_{\mathrm{I}}^{-} \\ u_{\mathrm{II}}' \end{bmatrix}. \tag{5.11}$$

Here we remark that (5.8), (5.9) and (2.14) give

$$k_{\mathrm{II\,II}}(x', \tau, \sigma)=Q(x', \tau, \sigma). \tag{5.12}$$

Moreover (3.10), (5.8) and (5.9) imply

(5. 13) $$b_{\mathrm{II\,I}}L=k_{\mathrm{II\,I}}\det B'',\quad b_{\mathrm{I\,II}}L=-k_{\mathrm{I\,II}}\det B''.$$

Now we shall consider the left hand side of (5. 5). After a simple calculation with (5. 6), (5. 1) and (5. 8)-(5. 12) we have

(5. 14) $$R(x',\eta,\sigma,\gamma)u\cdot u=\begin{bmatrix} q_{\mathrm{I\,I}} & q^*_{\mathrm{I\,II}} \\ q_{\mathrm{I\,II}} & q_{\mathrm{II\,II}}\end{bmatrix}(x',\eta,\sigma,\gamma)\begin{bmatrix}u^-_{\mathrm{I}}\\ u'_{\mathrm{II}}\end{bmatrix}\cdot\begin{bmatrix}u^-_{\mathrm{I}}\\ u'_{\mathrm{II}}\end{bmatrix}.$$

Here, when $\gamma=0$,

(5. 15) $$q_{\mathrm{I\,I}}=c^-_{\mathrm{I}}E_{l-1}-k^*_{\mathrm{I\,I}}k_{\mathrm{I\,I}}+d_2k^*_{\mathrm{II\,I}}k_{\mathrm{II\,I}},$$

(5. 16) $$q_{\mathrm{I\,II}}=-(d_1-d_2\bar{Q})\,k_{\mathrm{II\,I}}-k^*_{\mathrm{I\,II}}k_{\mathrm{I\,I}},$$

(5. 17) $$q_{\mathrm{II\,II}}=-2d_1\operatorname{Re}Q+b+d_2|Q|^2-|k_{\mathrm{I\,II}}|^2.$$

Furthermore it follows from (5. 17) and (5. 2) that

(5. 18) $$\begin{aligned}&q_{\mathrm{II\,II}}(x',\eta,\sigma,0)\\ &=-2d_1\operatorname{Re}Q+d_2\left(|Q|^2+e_{21}\operatorname{Re}\zeta(1+e_{12}\operatorname{Re}\zeta)^{-1}\right)-|k_{\mathrm{I\,II}}|^2,\ \text{if }(\eta,\sigma)\in\Sigma_0.\end{aligned}$$

We now show that there exist a positive constant C_0 and a real valued function $d(x',\eta,\sigma)\in S^0_+$, homogeneous in (η,σ) such that

(5. 19) $$\begin{aligned}&\left|k_{\mathrm{I\,II}}(x',\eta,\sigma)\right|^2+\left|k_{\mathrm{II\,I}}(x',\eta,\sigma)\right|^2\\ &\leqq -C_0\operatorname{Re}Q(x',\eta,\sigma)+d\left(|Q|^2+e_{21}\operatorname{Re}\zeta(1+e_{12}\operatorname{Re}\zeta)^{-1}\right)(x',\eta,\sigma)\end{aligned}$$

for $(\eta,\sigma)\in\Sigma_0$, in the case (a) of (2. 4). To do it we shall use the inequality (3. 13). Let $\eta=\rho(x',\sigma)$. Then it follows from (3. 13) and (5. 13) that for some $C>0$

(5. 20) $$\left(|k_{\mathrm{I\,II}}|^2+|k_{\mathrm{II\,I}}|^2\right)\left(x',\rho(x',\sigma),\sigma\right)\leqq -C\operatorname{Re}D(x',\sigma).$$

Furthermore from (2. 17) and (2. 18) we have

(5. 21) $$\operatorname{Re}D(x'\,\sigma)=\operatorname{Re}Q\left(x',\rho(x',\sigma),\sigma\right)\cdot(\text{positive factor}).$$

Hence (5. 20) implies that for $\eta=\rho(x',\sigma)$ and some $C_0>0$

(5. 22) $$\left(|k_{\mathrm{I\,II}}|^2+|k_{\mathrm{II\,I}}|^2\right)(x',\eta,\sigma)\leqq -C_0\operatorname{Re}Q(x',\eta,\sigma).$$

Here we shall apply the method used in the proof of Lemma 3. 3 in [15].

Let $w(x',\eta',\sigma')$ denote the left hand side of (2. 15). Then we see that (x',w,σ') may be taken as new variables in stead of (x',η',σ'), since (2. 10) and (2. 14) imply

(5. 23) $$Q(x^0,\eta^0,\sigma^0)=0,$$

so $(\partial w/\partial\eta')(x^0,\eta^0,\sigma^0)\neq 0$ by (2. 7). Put

(5.24) $$G(x', w, \sigma') = \left(|k_{\mathrm{I\,II}}|^2 + |k_{\mathrm{II\,I}}|^2 + C_0 \operatorname{Re} Q\right)(x', \eta', \sigma').$$

Then (5.22) implies

(5.25) $$G(x', 0, \sigma') \leqq 0.$$

On the other hand Taylor's formula gives

$$G(x', w, \sigma) = G(x', 0, \sigma) + wG^{(1)}(x', w, \sigma).$$

Now we define d by

$$d(x', \eta, \sigma) = G^{(1)}\left(x', w(x', \eta', \sigma'), \sigma'\right).$$

Then (5.19) follows from (5.24) and (5.25) and $d(x', \eta, \sigma)$ has the required property.

Now let

$$d_1 = C_0,\ \ d_2(x, \eta, \sigma) = 2d(x', \eta, \sigma).$$

Then from (5.18) and (5.19) we have

(5.26) $$q_{\mathrm{II\,II}}(x', \eta, \sigma, 0) \geqq \left(|k_{\mathrm{I\,II}}|^2 + |k_{\mathrm{II\,I}}|^2\right)(x', \eta, \sigma).$$

Thus it follows from (5.15), (5.16) and (5.26) that

(5.27) $$\begin{bmatrix} q_{\mathrm{I\,I}} & q^*_{\mathrm{I\,II}} \\ q_{\mathrm{I\,II}} & q_{\mathrm{II\,II}} \end{bmatrix}(x', \eta, \sigma, 0) \geqq 2\begin{bmatrix} E_{l-1} & 0 \\ 0 & 0 \end{bmatrix}$$

if we take c_{I}^- sufficiently large. Consequently from (5.27) and (5.14) we obtain

(5.28) $$R(x', \eta, \sigma, \gamma)\, u\cdot u \geqq |u_{\mathrm{I}}^-|^2 - a_1\gamma|u'_{\mathrm{II}}|^2$$

for all $(x', \eta, \gamma, \sigma) \in (\Gamma\times\Sigma)\cap U(x^0, \eta^0, \sigma^0)$ and $u \in C^{2m}$. The proof is complete.

Using (5.7), (5.28), Lemma 4.3 and [15], Lemma 4.1 we can prove the following lemma, as in the proof of Lemma 4.4. (See also the one of Lemma 4.2 in [15]).

LEMMA 5.3. *There are positive constants C and* γ_0 *such that*

$$\gamma^2\|\phi u\|^2_{0,\gamma} \leqq C\left(\|(D_n - M)\,\phi u\|^2_{0,\gamma} + |BS\phi u|^2_{\frac12,\gamma}\right) + C_\phi\gamma\left(\|u\|^2_{0,\gamma} + |u|^2_{-\frac12,\gamma}\right)$$

for all $u \in H_{1,\gamma}(\Omega)$ *and* $\gamma \geqq \gamma_0$, *where* $\phi(x, \eta, \sigma, \gamma) \in S^0_+$ *and* supp $\phi \cap (\bar\Omega\times\Sigma) \subset U(x^0, \eta^0, \sigma^0)$.

§6. Proof of the implication (C)→(A)

Since Γ is noncharacteristic for P, from the main estimate in [7], Lemmas 4.4 and 5.3 we obtain

LEMMA 6.1. (*Global a priori estimate*). *Suppose that the conditions* (*iii*) *and* (*C*) *hold. Then there exist positive constants* C *and* γ_0 *such that*

$$\gamma^2\|u\|_{0,\gamma}^2 \leqq C\left(\|Pu\|_{0,\gamma}^2 + |Bu|_{\frac{1}{2},\gamma}^2\right) \tag{6.1}$$

for all $u \in H_{1,\gamma}(\Omega)$ *and* $\gamma \geqq \gamma_0$.

PROOF OF (C)→(A). Since the L^2-well posedness of (P, B) follows from (6.1) and the analogous estimate for its adjoint problem according to §5, Remark 1) in [8], in view of Lemma 6.1 it is enough to show that the hypotheses of the lemma are also fulfilled by an adjoint problem of (P, B):

$$(P^{(*)}, B^{(*)}) \begin{cases} P^{(*)}(x, D)\, v = f & \text{in } \Omega, \\ B^{(*)}(x')\, v = g & \text{on } \Gamma. \end{cases}$$

Here for real (τ, σ, λ)

$$P^{(*)}(x, \tau, \sigma, \lambda) = P'(-x_0, x'', x_n, -\tau, \sigma, \lambda), \tag{6.2}$$

$$B^{(*)}(x') = B'(-x_0, x''), \tag{6.3}$$

$P'(x, D)$ is the principal part of the formal adjoint of $P(x, D)$ and

$$B'(x') = \left(b_1'(x'), \cdots, b_m'(x')\right)^*, \tag{6.4}$$

where $\{b_j'(x')\,;\, j=1, \cdots, m\}$ is a base of *ker* $B(x')$ which is smoothly varying on Γ. Notice the relation

$$B(x')\left(B'(x')\right)^* = 0, \quad x' \in \Gamma. \tag{6.5}$$

Now (6.2) implies

$$P^{(*)}(x, \tau, \sigma, \lambda) = \left(P^0(-x_0, x'', x_n, -\bar{\tau}, \sigma, \bar{\lambda})\right)^* \tag{6.6}$$

from which Condition (i) follows. By $L^{(*)}$ etc. denote the Lopatinskii determinant etc. of $(P^{(*)}, B^{(*)})$ respectively. Then (6.6) gives

$$\theta^{(*)}(x, \sigma) = -\theta(-x_0, x'', x_n, \sigma). \tag{6.7}$$

Furthermore it follows from Lemmas 9.1 and 9.2 in [11] respectively that

$$b_{ij}^{(*)}(x', \tau, \sigma) = \overline{-b_{ji}(-x_0, x'', -\bar{\tau}, \sigma)}, \quad i, j = 1, \cdots, m \tag{6.8}$$

and that the Hersh's condition holds. Hence (6.6) and (6.8) imply (C). Next for a moment let Condition (ii) be fulfilled. Then from (9.10) and (9.11) in [11] we have

$$L^{(*)}(x', \tau, \sigma) = \overline{-L(-x_0, x'', -\bar{\tau}, \sigma)} \cdot (\text{nonzero factor}). \tag{6.9}$$

From this, (2.13) and (6.7) we obtain

$$-\mathrm{Re}\, D^{(*)}(x', \sigma) = \mathrm{Im}\, D(-x_0, x'', \sigma), \quad \mathrm{Im}\, D^{(*)}(x', \sigma) = -\mathrm{Re}\, D(-x_0, x'', \sigma).$$

Therefore Condition (*iii*) is satisfied.

To prove the validity of Condition (*ii*), the boundary matrices B and B' may be assumed to have the following forms:

$$B(x')=(E_m, 0) \quad \text{and} \quad B'(x')=(0, E_m) \tag{6.10}$$

after appropriate transformations of the dependent variables, because of (6.5). Let $(x^0, \eta^0, \sigma^0)\in\Gamma\times\partial\Sigma$ and $L^{(*)}(-x_0^0, x^{0\prime\prime}, -\eta^0, \sigma^0)=0$.
Then it follows from Lemma 9.2 in [11] that $L(x^0, \eta^0, \sigma^0)=0$. In what follows we shall restrict ourselves to the case where $\det P^0(x^0, \eta^0, \sigma^0, \lambda)$ has no double real zero, because the other one also may be deal with similarly. Put

$$S_0(x', \tau, \sigma)=(h_{\mathrm{I}}^+, h_{\mathrm{III}}^+, h_{\mathrm{I}}^-, h_{\mathrm{III}}^-)(x', \tau, \sigma).$$

Furthermore set

$$(B_+, B_2)(x', \tau, \sigma)=B(x')S_0(x', \tau, \sigma),$$

$$(B_2', B_+')(x', \tau, \sigma)=B'(x')\left(S_0^{-1}(x', \tau, \sigma)\right)^*,$$

where B_+, B_2, B_2' and B_+' are $m\times m$ matrices. Then

$$L(x', \tau, \sigma)=\det B_+(x', \tau, \sigma)$$

and it follows from (9.8) in [11] that

$$L^{(*)}(-x_0, x'', -\bar{\tau}, \sigma)=\det B_+'(x', \tau, \sigma).$$

Therefore it suffices to prove that

$$\operatorname{rank} B_+'(x^0, \eta^0, \sigma^0)=m-1.$$

(See (II) α) at P. 149 in [11]). Hereafter by B_+ etc. we denote $B_+(x^0, \eta^0, \sigma^0)$ etc. respectively. Then from the hypotheses we have

$$\operatorname{rank} B_+=m-1 \quad \text{and} \quad \operatorname{rank}(B_+, B_2)=m.$$

Hence there is an orthogonal matrix T which exchanges a column of B_+ and one of B_2 so that the first m columns of BS_0T are linearly independent. Therefore it is enough to show that the last m columns of $B'(S_0^{-1})^*T$ are so. Put

$$S_0T=\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \quad \text{and} \quad (S_0^{-1})^*T=\begin{bmatrix} T_{11}' & T_{12}' \\ T_{21}' & T_{22}' \end{bmatrix}.$$

Then it follows from (6.10) that

$$BS_0T=(T_{11}, T_{12}), \quad B'(S_0^{-1})^*T=(T_{21}', T_{22}')$$

and

$$T_{11}T_{21}'^*+T_{12}T_{22}'^*=0,$$

where T_{11} is nonsingular. Hence we have

$$T'_{21}=-T'_{22}(T_{11}^{-1}T_{12})^*.$$

Thus we find that rank $T'_{22}=m$, since rank $(T'_{21}, T'_{22})=m$. This completes the proof.

The following criterion is useful in applications. (See the proof of Corollary 7.2).

COROLLARY 6.2. *Let the constant coefficients problem* $(P^0, B)_{x'}$ *be* L^2-*well posed for every* $x'\in\Gamma$. *Furthermore suppose that* $Q(x', \eta, \sigma)$ *is real. Then necessary and sufficient conditions for* (P, B) *to be* L^2-*well posed are that for such point* (x^0, η^0, σ^0) *as in Lemma* 3.2 *there are indexes* $i, j\in \mathrm{III}$ *satisfying* (3.1) *and, say* $i=j=m$,

$$(6.11)\qquad \left(|k_{\mathrm{I}m}|^2+|k_{m\mathrm{I}}|^2\right)(x', \eta, \sigma)\leqq C_1 \operatorname{Im}\nu(x', \sigma'),\ \textit{if}\ \eta=\operatorname{Re}\nu(x', \sigma),$$

and that near such point (x^0, η^0, σ^0) *as in Lemma* 3.3

$$(6.12)\qquad \left(|k_{\mathrm{I\,II}}|^2+|k_{\mathrm{II\,I}}|^2\right)(x', \eta, \sigma)\leqq \mp C_2 Q(x', \eta, \sigma),\ \textit{if}\ \eta=\theta(x', \sigma)$$

according to the case (*a*) *or* (*b*) *of* (2.4) *respectively. Here* $k_{\mathrm{I}m}$, $k_{m\mathrm{I}}$ *and* $k_{\mathrm{I\,II}}$, $k_{\mathrm{II\,I}}$ *are the vectors defined by* (4.6), (4.11) *and* (5.8), (5.9) *respectively and* C_1, C_2 *are positive constants.*

PROOF. We first remark that under the first hypothesis the left hand side of (6.11) is still well-defined. (See the proofs of Lemma 3.1 and 3.2).

Necessity. The main theorem mplies that the condition (C) holds. Therefore (6.11) follows from (4.21). Next we see from (3.6), (3.12) and (5.13) that for some constant $C>0$

$$\begin{aligned}&\left(|k_{\mathrm{I\,II}}|^2+|k_{\mathrm{II\,I}}|^2\right)\left(x', \theta(x', \sigma), \sigma\right)\\ &\quad\leqq 2\left(|k_{\mathrm{I\,II}}|^2+|k_{\mathrm{II\,I}}|^2\right)\left(x', \rho(x', \sigma), \sigma\right)+O\left(|\rho(x', \sigma)-\theta(x', \sigma)|^2\right)\\ &\quad\leqq C\left|D(x', \sigma)\right|.\end{aligned}$$

On the other hand it follows from (2.16) and the realness of $Q(x', \eta, \sigma)$ that $L(x', \eta, \sigma)$ is real for η with $\eta\geqq\theta(x', \sigma)$ modulo nonzero factor. Hence we find by the implicit function theorem that $D(x', \sigma)$ is real. Furthermore it is known that under the hypotheses of the corollary $Q(x', \theta(x', \sigma), \sigma)$ is nonpositive. (See the proof of Lemma 6.5 in [11]). Thus we obtain (6.12) by (2.18).

Sufficiency. As seen in the proof of the implication (C)→(A), the L^2-well posedness of (P, B) follows from the a priori estimates for the problem and its adjoint problem $(P^{(*)}, B^{(*)})$ which are derived from (4. 21) and (5. 22) for those problems as well as Conditions (*i*) and (*ii*). Notice that the two conditions are also fulfilled by $(P^{(*)}, B^{(*)})$. Therefore it is enough to show that (4. 21) and (5. 22) are valid for both the problems.

We now see easily that (6. 11) implies (4. 21). On the other hand it follows from (6. 12) that for some constant $C>0$

$$\begin{aligned}&\left(|k_{\mathrm{I\,II}}|^2+|k_{\mathrm{II\,I}}|^2\right)\left(x', \rho(x', \sigma), \sigma\right)\\&\quad\leqq -CQ\left(x', \rho(x', \sigma), \sigma\right)+0\left(|\rho(x', \sigma)-\theta(x', \sigma)|\right).\end{aligned}$$

Hence we obtain (5. 22) by (2. 15). Therefore it suffices to show that (6. 11), (6. 12) and the hypotheses are fulfilled by $(P^{(*)}, B^{(*)})$.

Now the L^2-well posedness of $(P^0, B)_{x'}$ implies that of the adjoint problem. (See Theorem 2 in [8]). Furthermore from (2. 1), (5. 8), (5. 9), (6. 5) and (6. 6) we obtain

$$(B''^{(*)-1}B'^{(*)})(x', \eta, \sigma)=-\left((B''^{-1}B')(-x_0, x'', -\eta, \sigma)\right)^*$$

(see also the proof of Lemma 9. 1 in [11]), so

$$\begin{aligned}k^{(*)}_{\mathrm{I\,II}}(x', \eta, \sigma)&=-\left(k_{\mathrm{II\,I}}(-x_0, x'', -\eta, \sigma)\right)^*,\\ k^{(*)}_{\mathrm{II\,I}}(x', \eta, \sigma)&=-\left(k_{\mathrm{I\,II}}(-x_0, x'', -\eta, \sigma)\right)^*\end{aligned}$$

and

$$Q^{(*)}(x', \eta, \sigma)=\overline{-Q(-x_0, x'', -\eta, \sigma)}.$$

Thus (6. 12) and the hypotheses for $(P^{(*)}, B^{(*)})$ follow from those for (P, B). (See also (6. 7)). Similarly we obtain (6. 11) for $(P^{(*)}, B^{(*)})$, since (2. 11) and (6. 9) imply that

$$\begin{aligned}\operatorname{Re}\nu^{(*)}(x', \sigma)&=-\operatorname{Re}\nu(-x_0, x'', \sigma),\\ \operatorname{Im}\nu^{(*)}(x', \sigma)&=\operatorname{Im}\nu(-x_0, x'', \sigma).\end{aligned}$$

The proof is complete.

REMARK 6. 3. Condition (*iii*) has been used only in deriving (3. 13) from (3. 6). Therefore if the simple real zeros are absent then it may be weakened so that

$$\operatorname{Re} D(x', \sigma)\leqq 0 \quad \text{or} \quad \operatorname{Im} D(x', \sigma)\geqq 0$$

according to the case (*a*) or (*b*) of (2. 4) respectively.

REMARK 6. 4. In the case where $P(x, D)$ is a differential operator of higher order we can also obtain the analogous results by reducing the problem in question to a system with (tangential) pseudo-differential operators. (See the proof of Corollary 7.2 below).

§ 7. Applications

Consider the following problem for a strictly x_0-hyperbolic operator $\tilde{P}$:

$$(\tilde{P}, \tilde{B}_1, \cdots, \tilde{B}_m) \begin{cases} \tilde{P}(x, D)\, u = f & \text{in } \Omega, \\ \tilde{B}_j(x', D)\, u = g_j & \text{on } \Gamma, j = 1, \cdots, m. \end{cases}$$

Here the principal symbols $\tilde{P}^0(x, \tau, \sigma, \lambda)$ and $\tilde{B}^0_j(x', \tau, \sigma, \lambda)$ of $\tilde{P}$ and $\tilde{B}_j$ have the following forms:

$$\begin{aligned} \tilde{P}^0 &= P^0_1 \cdots P^0_m, \\ \tilde{B}^0_1 &= B^0_1, \\ \tilde{B}^0_2 &= B^0_2 P^0_1, \\ \tilde{B}^0_3 &= B^0_3 P^0_2 P^0_1, \\ &\vdots \\ \tilde{B}^0_m &= B^0_m P^0_{m-1} \cdots P^0_1, \end{aligned}$$

where $P^0_j(x, D)$, $j = 1, \cdots, m$ are homogeneous x_0-hyperbolic operators of second order whose normal cones cut by $\tau = 1$ do not intersect each other and are bounded surfaces in the (σ, λ) space for every fixed $x \in \Gamma$. Furthermore $B^0_j(x', D), j = 1, \cdots, m$ are homogeneous boundary differential operators at most of first order such that Γ is noncharacteristic for B^0_j. All the coefficients are assumed to be real. (See [10], [2, 3]). Then we have

COROLLARY 7.1. ([9]). *The problem* $(\tilde{P}, \tilde{B}_1, \cdots, \tilde{B}_m)$ *is L^2-well posed if and only if every constant coefficients problem* $(\tilde{P}^0, \tilde{B}^0_1, \cdots, \tilde{B}^0_m)_{x'}$, $x' \in \Gamma$ *is L^2-well posed and the constants C in* (1.1) *with respect to these problems are independent of the parameter* x'.

PROOF. Let $(\tilde{P}^0, {}^0_1\tilde{B}, \cdots, \tilde{B}^0_m)_{x'}$ be L^2-well posed for every $x' \in \Gamma$. Then in view of Theorem it is enough to show that Conditions (i), (ii) and (iii) are fulfilled. We first see by the assumptions with respect to the normal cones that Condition (i) holds and for every $j = 1, \cdots, m$ $P^0_j(x, \tau, \sigma, \lambda)$ has zeros $\lambda^+_j(x, \tau, \sigma)$ and $\lambda^-_j(x, \tau, \sigma)$ with positive imaginary part and negative one when $\operatorname{Im} \tau < 0$ respectively. Let $\tilde{L}$ and $L_j, j = 1, \cdots, m$ be the Lopatinskii determinants of $(\tilde{P}^0, \tilde{B}^0_1, \cdots, \tilde{B}^0_m)$ and (P^0_j, B^0_j). Then it follows from (3.2) and the proof of Theorem 1 in [2] respectively that

$$\tilde{L} = L_1 \cdots L_m \cdot (\text{nonzero factor}) \tag{7.1}$$

and that every constant coefficients problem $(P^0_j, B^0_j)_{x'}$ $(x' \in \Gamma, j = 1, \cdots, m)$ is L^2-well posed. Hence we find by virtue of Theorem 3 in [16] and the realness of the coefficients of B_j^0 that $\tilde{L}(x', \tau, \sigma)$ vanishes at a point $(x^0, \tau^0, \sigma^0) \in \Gamma \times \Sigma$ if and only if $\operatorname{Im} \tau^0 = 0$ and there is an index l such taht $L_l(x^0, \tau^0, \sigma^0) = 0$ (so B_l^0 is of first order) and $P_l^0(x^0, \tau^0, \sigma^0, \lambda)$ has a double real zero. Furthermore we see by means of Condition (i) that such l is uniquely determined by the point (x^0, τ^0, σ^0) in question. Therefore from (7.1) we have

$$\tilde{L}(x', \tau, \sigma) = L_l(x', \tau, \sigma) \cdot (\text{nonzero factor})$$

near (x^0, τ^0, σ^0). Hence it suffices to show that L_l fulfilles Conditions (ii) and (iii). Now the former follows from (2.4) with $\lambda_{\mathrm{II}}^+ = \lambda_l^+$, since

$$L_l(x', \tau, \sigma) = B_l^0\left(x', \tau', \sigma', \lambda_l^+(x', \tau', \sigma')\right).$$

Therefore we find from the proof of the Corollary 6.2 that $D(x', \sigma)$ is nonpositive. This completes the proof.

In particular let

$$P_j^0(x, D) = D_0^2 - a_j(x)^2 \sum_{k=1}^{n} D_k, \; j=1, \cdots, m, \; 0 < a_m < \cdots < a_1$$

and

$$B_j^0(x', D) = D_n - \sum_{k=1}^{n-1} b_{jk}(x') D_k - c_j(x') D_0, \; j=1, \cdots, m.$$

Then Corollary in [9] is a special case of the following which is a consequence of Corollary 6.2.

Corollary 7.2. Necessary and sufficient conditions for the problem $(\tilde{P}, \tilde{B}_1, \cdots, \tilde{B}_m)$ to be L^2-well posed are that

$$\left(\sum_{k=1}^{n-1} b_{jk}(x')^2\right)^{\frac{1}{2}} \leqq a_j(x') c_j(x'), \; x' \in \Gamma, j=1, \cdots, m \tag{7.2}$$

and that if $L_l(x^0, \eta^0, \sigma^0) = 0$ and $P_l^0(x^0, \eta^0, \sigma^0, \lambda)$ has a double real zero, i.e.,

$$\eta^0 = \pm a_l(x^0)|\sigma^0| \quad \text{and} \quad \sum_{k=1}^{n-1} b_{lk}(x^0) \sigma_k^0 + c_l(x^0) \eta^0 = 0 \tag{7.3}$$

for some $l (1 \leqq l < m)$ and $(x^0, \eta^0, \sigma^0) \in \Gamma \times \partial \Sigma$, then

$$\left(\sum_{k=1}^{n-1} b_{jk}(x') \sigma_k + c_j(x') \eta\right)^2 \leqq \pm C\left(\sum_{k=1}^{n-1} b_{lk}(x') \sigma_k + c_l(x') \eta\right) \tag{7.4}$$

for $j = l+1, \cdots, m$ and $(x', \eta, \sigma) \in \Gamma \times \Sigma_0$ with $\eta = \theta_l(x', \sigma)$, according to $\eta^0 = \pm a_l(x^0)|\sigma^0|$ respectively, where $\theta_l(x', \sigma) = \pm a_l(x')|\sigma|$ and C is a positive constant.

Proof. It is known that for every fixed $x' \in \Gamma$ $(\tilde{P}^0, \tilde{B}_1^0, \cdots, \tilde{B}_m^0)_{x'}$ is L^2-well posed if and only if (7.2) is valid and

$$\left(\sum_{k=1}^{n-1} b_{lk}(x')^2\right)^{\frac{1}{2}} = a_l(x') c_l(x') \qquad (1 \leqq l < m) \tag{7.5}$$

implies

(7.6) $$b_{jk}(x')=c_j(x')=0 \qquad (j=l+1, \cdots, m, k=1, \cdots, n-1).$$

(See Theorem 1 in [3] and Lemma 4.1 in [2]).

Now it follows from (7.5) for fixed x', say x^0, that (7.3) is valid for some (η^0, σ^0). Moreover (7.2), (7.3) and (7.4) give (7.6) for $x'=x^0$, since $0<a_j(x^0)<a_l(x^0)$ for $j>l$. Therefore every constant coefficients problem $(\tilde{P}^0, \tilde{B}_1^0, \cdots, \tilde{B}_m^0)_{x'}$ may be assumed to be L^2-well posed.

Next to examine (7.4) let $(x^0, \eta^0, \sigma^0)\in \Gamma\times\partial\Sigma$ be a point satisfying (7.3) and reduce $(\tilde{P}, \tilde{B}_1, \cdots, \tilde{B}_m)$ to a system by the transformation:

$$V={}^t(\Lambda^{2m-1}u, \Lambda^{2m-2}D_n u, \cdots, D_n^{2m-1}u).$$

Notice that the Lopatinskii determinant of the system thus obtained is equal to that of the original problem modulo nonzero factor. Then we find from §2 in [3] (in particular (2.27) and (2.28)) that $k_{\mathrm{I\,II}}$, $k_{\mathrm{II\,I}}$ and Q defined by (5.9) (with $Q=k_{\mathrm{II\,II}}$) are as follows:

(7.7) $$Q(x', \tau, \sigma)=-\alpha_l(x', \tau', \sigma'),$$

(7.8) $$k_{\mathrm{I\,II}}=0,\ k_{\mathrm{II\,I}}=(k_{lj}^l\,;\ j\to l+1, \cdots, m).$$

Here

(7.9) $$\alpha_j(x', \tau, \sigma)=\sum_{k=1}^{n-1} b_{jk}(x')\sigma_k+c_j(x')\tau, j=1, \cdots, m,$$

(7.10) $$k_{lj}^l(x', \tau, \sigma)=\left|(B_p^0Q_p^0)(\lambda_j^-), (B_p^0Q_p^0)(\lambda_q^+)\,;\ p\downarrow l, \cdots, j, q\to l+1, \cdots, j\right|(x', \tau', \sigma')$$

where $Q_1^0=1$, $Q_p^0=P_1^0P_2^0\cdots P_{p-1}^0, p=2, \cdots, m$, $Q_p^0(\lambda_q^+)$ etc. are abbreviations for $Q_p^0(x', \tau, \sigma, \lambda_q^+(x', \tau, \sigma))$ etc. respectively and we have omitted a nonzero factor in (7.10). Since (7.7) and (7.9) imply that $Q(x', \eta, \sigma)$ is real, we find by virtue of (7.7), (7.8) and Corollary 6.2 that $(\tilde{P}, \tilde{B}_1, \cdots, \tilde{B}_m)$ is L^2-well posed if and only if for some constant $C>0$

(7.11) $$\sum_{j=l+1}^{m}\left|k_{lj}^l(x', \eta, \sigma)\right|^2\leqq C\alpha_l(x', \eta, \sigma) \quad \text{for} \quad \eta=a_l(x')|\sigma|.$$

Furthermore by a simple calculation we see from (7.10) that for $j=l+1, \cdots, m$

$$\begin{aligned} &k_{lj}^l(x', \eta, \sigma)\\ &=2(-1)^{j-l+1}\alpha_j Q_j^0(\lambda_j^-)\prod_{q=l+1}^{j}Q_l^0(\lambda_q^+)\lambda_q^+\begin{vmatrix}1 & \cdots 1\\ P_l^0(\lambda_{l+1}^+) & \cdots P_l^0(\lambda_j^+)\\ \vdots & \\ (P_l^0\cdots P_{j-2}^0)(\lambda_{l+1}^+) & \cdots (P_l^0\cdots P_{j-2}^0)(\lambda_j^+)\end{vmatrix}\\ &\quad+O\left(|\alpha_l|+\cdots+|\alpha_{j-1}|\right)\end{aligned}$$

$$=\alpha_j(x', \eta', \sigma')\cdot(\text{nonzero factor})+O\left(|\alpha_l|+\cdots+|\alpha_{j-1}|\right)$$

Thus (7. 11) is equivalent to (7. 4) by (7. 9).

REMARK 7. 3. In [13] Sakamoto was concerned with another iterated problem $(\tilde{P}, \hat{B}_1, \cdots, \hat{B}_m)$ such that $\hat{B}_j^0=B_j^0\prod_{k\neq j}P_k^0$, where $\hat{B}_j^0$ is the principal symbol of $\hat{B}_j$. For convenience let P_j^0 and B_j^0 be the same ones as in Corollary 7. 2. Then we see that both k_{III} and k_{III} in (7. 8) vanish. Thus we find from the preceding argument that $(\tilde{P}, \hat{B}_1, \cdots, \hat{B}_m)$ is L^2-well posed if and only if $(P_j^0, B_j^0), j=1, \cdots, m$ are L^2-well posed.

References

[1] R. AGEMI: Remarks on L^2-well-posed mixed problems for hyperbolic equations of second order, Hokkaido Math. J., Vol. 2, 214–230 (1973).

[2] R. AGEMI: Iterated mixed problems for d'Alembertians, ibid, Vol. 3, 104–128 (1974).

[3] R. AGEMI: Iterated mixed problems for d'Alembertians II, ibid., Vol. 4, 281–294 (1975).

[4] R. AGEMI: On a characterization of L^2-well posed mixed problems for hyperbolic equations of second order, Proc. Japan Acad., Vol. 51, 247–252 (1975).

[5] R. AGEMI and T. SHIROTA: On necessary and sufficient conditions for L^2-well posedness of mixed problems for hyperbolic equations, J. Fac. Sci. Hokkaido Univ., Ser. I, Vol. 21, 133–151 (1970).

[6] M. S. AGRANOVICH: Boundary value problems for systems with a parameter, Math. USSR Sbornik, Vol. 13, 25–64 (1971).

[7] H. O. KREISS: Initial boundary value problems for hyperbolic systems, Comm. Pure Appl. Math., Vol. 23, 277–298 (1970).

[8] K. KUBOTA: Remarks on boundary value problems for hyperbolic equations, Hokkaido Math. J., Vol. 2, 202–213 (1973).

[9] K. KUBOTA: A characterization of L^2-well posedness for iterations of hyperbolic mixed problems of second order, Proc. Japan Acad., Vol. 52, 492–495 (1976).

[10] S. MIZOHATA: Queques problèmes au bord, du type mixte pour des équations hyperboliques, Collège de France 23–60 (1966–1967).

[11] T. OHKUBO and T. SHIROTA: On structures of L^2-well-posed mixed problems for hyperbolic systems of first order, Hokkaido Math. J., Vol. 4, 82–158 (1975).

[12] J. V. RALSTON: Note on a paper of Kreiss, Comm. Pure Appl. Math., Vol. 24, 759–762 (1971).

[13] R. SAKAMOTO: Iterated hyperbolic mixed problems, Publ. Res. Inst. Math. Sci., Kyoto Univ., Vol. 6, 1–42 (1970).

[14] R. SAKAMOTO: Mixed problems for hyperbolic equations I, J. Math. Kyoto Univ., Vol. 10, 349–373 (1970).

[15] S. SATO and T. SHIROTA: Remarks on modified symmetrizers for 2×2 hyperbolic mixed problems, Hokkaido Math. J., Vol. 5, 120–138 (1976).

[16] T. SHIROTA: On the propagation speed of hyperbolic operator with mixed boundary conditions, J. Fac. Sci. Hokkaido Univ., Ser. I, Vol. 22, 25–31 (1972).

ADDED IN PROOF: The statement of Corollary 7.1 is also valid in the case where the coefficients of $B_j^0(x', D)$ are complex-valued, if Condition (*ii*) is satisfied and $P_j^0(x, D)$, $j=1, \cdots, m$ are the same ones as in Corollary 7.2.

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