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A certain congruence theorem for closed submanifolds of codimension 2 in a space of constant curvature

(The continuation of the previous paper [1][1])

Dedicated to Professor Tominosuke Ōtuki on his 60th birthday

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Introduction.

We consider an $(m+2)$-dimensional orientable Riemannian space S^{m+2} with constant curvature of class $C^{\nu}(\nu\geqq 3)$ which admits a continuous differentiable one-parameter group G of $1-1$ mappings T_{τ} of S^{m+2} onto itself (the group parameter τ, $-\infty<\tau<+\infty$, is assumed to be always canonic i.e., $T_{\tau_1}\cdot T_{\tau_2}=T_{\tau_1+\tau_2}$). We assume that the orbits (or streamlines) of the points of S^{m+2} produced by T_{τ} are regular curves covering S^{m+2} simply.

The purpose of the present paper is to generalize the following theorem given by H. Hopf and the present author [2] for two orientable closed submanifolds of codimension 2.

THEOREM. *Let W^{m+1} and $\overline{W}^{m+1}$ be two orientable closed hypersurfaces in S^{m+2} and p and $\bar{p}$ be the corresponding points of these hypersurfaces along an orbit, and $H_r(p)$ and $\overline{H}_r(p)$ be the r-th mean curvature at these points respectively. Assume that the set of points in which the orbit is tangent to W^{m+1} or $\overline{W}^{m+1}$ has no inner point and that the second fundamental form of $W^{m+1}(t)\overset{\mathrm{def}}{=}(1-t)W^{m+1}+t\overline{W}^{m+1}$, $0\leqq t\leqq 1$ is positive definite. If G is a group of isometries of S^{m+2} and if the relation $H_r(p)=\overline{H}_r(p)$ holds for each point $p\in W^{m+1}$, then W^{m+1} and $\overline{W}^{m+1}$ are congruent mod G.*

Especially, in case of $r=m$, that is, the generalized theorem relating to the Gauss curvature was already proved in the previous paper [1].

§ 1. Generalized theorem.

We suppose an $(m+2)$-dimensional orientable Riemannian space S^{m+2} with constant curvature of class $C^{\nu}(\nu\geqq 3)$ which admits an infinitesimal isometric transformation

$$(1.1)\qquad \hat{x}^i = x^i + \xi^i(x)\,\delta\tau$$

1) Numbers in brackets refer to the references at the end of the paper.

where x^i are local coordinates in S^{m+2} and ξ^i are the components of a contravariant vector ξ. We assume that orbits of the transformations generated by ξ cover S^{m+2} simply and that ξ is everywhere continuous and $\neq 0$. Let us choose a coordinate system such that the orbits of transformations are new x^1-coordinate curves, that is, a coordinate system in which the vector ξ^i has components $\xi^i = \delta^i_1$, where the symbol δ^i_j denotes Kronecker's delta; then (1. 1) becomes as follows

$$\hat{x}^i = x^i + \delta^i_1 \delta\tau \tag{1. 2}$$

and S^{m+2} admits a one-parameter continuous group G of transformations which are $1-1$-mappings of S^{m+2} onto itself and are given by the expression $\hat{x}^i = x^i + \delta^i_1 \tau$ in the new special coordinate system ([3]).

Now we consider two orientable closed submanifolds W^m and $\overline{W}^m$ of codimension 2 (of class C^ν) imbedded in S^{m+2} which are given as follows

$$\begin{cases} W^m: & x^i = x^i(u^\alpha) \\ \overline{W}^m: & \bar{x}^i = x^i(u^\alpha) + \delta^i_1 \tau(u^\alpha) \end{cases} \qquad i = 1, \cdots, m+2, \ \alpha = 1, \cdots, m \tag{1. 3}$$

where u^α are local coordinates of W^m and τ is a continuous function attached to each point of the submanifold W^m. We shall henceforth confine ourselves to Latin indices running from 1 to $m+2$ and Greek indices from 1 to m, and we assume that the set of points in which the orbit is tangent to W^m or $\overline{W}^m$ does not have an inner point.

Then we can take the family of the submanifolds

$$W^m(t) = (1-t)\, W^m + t\overline{W}^m, \qquad 0 \leqq t \leqq 1$$

generated by W^m and $\overline{W}^m$ whose points correspond along the orbits of the transformations where W^m and $\overline{W}^m$ mean $W^m(0)$ and $W^m(1)$ respectively. Thus according to (1. 3), $W^m(t)$ is given by the expression

$$W^m(t): \ x^i(u^\alpha, t) = (1-t)\, x^i(u^\alpha) + t\bar{x}^i(u^\alpha) \qquad 0 \leqq t \leqq 1, \tag{1. 4}$$

and (1. 4) may be rewritten as follows

$$W^m(t): \ x^i(u^\alpha, t) = x^i(u^\alpha) + \delta^i_1 t\tau(u^\alpha) \qquad 0 \leqq t \leqq 1. \tag{1. 5}$$

We shall indicate by $\underset{P}{n}^i(t)$ $(P=1, 2)$ the contravariant unit vectors normal $W^m(t)$ and suppose that they are mutually orthogonal. For a point on $W^m(t)$ such that the orbit is not tangent to $W^m(t)$, let $\underset{1}{n}^i$ be in the vector space spanned by $m+1$ independent vectors $\partial x^i(u^\alpha, t)/\partial u^\alpha$ $(\alpha = 1, \cdots, m)$ and δ^i_1 be the unit vector normal $W^m(t)$. Then we can determine $\underset{2}{n}^i$ from the vectors $x^i_\alpha(u, t)$ and $\underset{1}{n}^i$, where $x^i_\alpha(u, t)$ means $\partial x^i(u, t)/\partial u^\alpha$.

Let us denote the metric tensor of S^{m+2} by g_{ij} and the derivative with respect to t by the dash. Throughout this paper repeated lower case Latin indices call for summation from 1 to $m+2$ and repeated lower case Greek indices for summation from 1 to m. Then differentiating the following relations with respect to t

$$g_{ij}\underset{1}{n^i}(t)\,x^j_\alpha(t)=0, \quad g_{ij}\underset{1}{n^i}(t)\,\underset{1}{n^j}(t)=1, \quad g_{ij}\underset{1}{n^i}(t)\,\underset{2}{n^j}(t)=0$$

since the transformation group G is isometric, that is, $\partial g_{ij}/\partial x^1=0$, we have

$$g_{ij}\underset{1}{n'^i}(t)\,x^j_\alpha+g_{ij}\underset{1}{n^i}\,x'^j_\alpha(t)=0, \tag{1.6}$$

$$g_{ij}\underset{1}{n'^i}(t)\,\underset{1}{n^j}(t)=0, \tag{1.7}$$

$$g_{ij}\underset{1}{n'^i}(t)\,\underset{2}{n^j}(t)+g_{ij}\underset{1}{n^i}(t)\,\underset{2}{n'^j}(t)=0, \tag{1.8}$$

where we can express as follows

$$\underset{1}{n^i}(t)=\rho\delta^i_1+\varphi^\alpha x^i_\alpha(t), \tag{1.9}$$

then we have

$$\underset{1}{n'^i}(t)=\rho'\,\delta^i_1+\varphi'^\alpha x^i_\alpha(t)+\varphi^\alpha x'^i_\alpha(t),$$

and from (1.5), putting $\partial\tau/\partial u^\alpha=\tau_\alpha$, we obtain

$$x'^i_\alpha(t)=\tau_\alpha\delta^i_1, \tag{1.10}$$

therefore we have

$$\underset{1}{n'^i}(t)=(\rho'+\varphi^\alpha\tau_\alpha)\,\delta^i_1+\varphi'^\alpha x^i_\alpha(t).$$

Consequently we get

$$g_{ij}\underset{1}{n'^i}\underset{2}{n^j}=0. \tag{1.11}$$

From (1.6), (1.7) and (1.11), we have

$$g_{ij}\underset{1}{n'^i}x^j_\alpha(t)\,g^{\alpha\beta}(t)\,x^k_\beta(t)=-g_{ij}\underset{1}{n^i}\delta^j_1\tau_\alpha g^{\alpha\beta}(t)\,x^k_\beta(t),$$

$$g_{ij}\underset{1}{n'^i}\underset{1}{n^j}\underset{1}{n^k}=0,$$

$$g_{ij}\underset{1}{n'^i}\underset{2}{n^j}\underset{2}{n^k}=0,$$

where $g^{\alpha\beta}(t)$ is the contravariant metric tensor of $W^m(t)$. Since

$$g^{jk}=x^j_\alpha(t)\,x^k_\beta(t)\,g^{\alpha\beta}(t)+\underset{1}{n^j}(t)\,\underset{1}{n^k}(t)+\underset{2}{n^j}(t)\,\underset{2}{n^k}(t)$$

we have

$$g_{ij}\underset{1}{n'^i}g^{jk}=-g_{ij}\underset{1}{n^i}(t)\,\delta^j_1\tau_\alpha g^{\alpha\beta}(t)\,x^k_\beta(t).$$

Thus we get

(1. 12) $$\underset{1}{n}'^{i} = -g^{\alpha\beta}(t)\,\tau_\alpha \delta_1^k \underset{1}{n}_k(t)\, x_\beta^i(t).$$

Furthermore differentiating the following relations with respect to t

$$g_{ij}\underset{2}{n}^i(t)\, x_\alpha^j(t) = 0, \qquad g_{ij}\underset{2}{n}^i(t)\,\underset{2}{n}^j(t) = 1$$

we have

$$g_{ij}\underset{2}{n}'^i(t)\, x_\alpha^j(t) = -g_{ij}\underset{2}{n}^i(t)\,\tau_\alpha \delta_1^j,$$

$$g_{ij}\underset{2}{n}'^i(t)\,\underset{2}{n}^j(t) = 0,$$

since from (1. 9) $\underset{2}{n}_i(t)\delta_1^i = 0$ and from (1. 8) and (1. 11) $g_{ij}\underset{2}{n}'^i(t)\,\underset{1}{n}^j(t) = 0$, we get

$$g_{ij}\underset{2}{n}'^i(t)\, x_\alpha^j(t)\, g^{\alpha\beta}(t)\, x_\beta^k(t) = 0,$$

$$g_{ij}\underset{2}{n}'^i(t)\,\underset{2}{n}^j(t)\,\underset{2}{n}^k(t) = 0,$$

$$g_{ij}\underset{2}{n}'^i(t)\,\underset{1}{n}^j(t)\,\underset{1}{n}^k(t) = 0,$$

then we have

(1. 13) $$g_{ij}\underset{2}{n}'^i(t)\, g^{jk} = \underset{2}{n}'^k(t) = 0.$$

Let us denote the operation of D-symbol due to van der Waerden-Bortolotti ([4] p. 254) by the symbol ";" and the symbol δv by $v_{;\alpha} du^\alpha$. Then as well-known on the theory of a submanifold in a Riemannian space, we have

(1. 14) $$x^i_{\alpha;\beta} = \frac{\partial x_\alpha^i}{\partial u^\beta} + \Gamma^i_{jk} x_\alpha^j x_\beta^k - {}^*\Gamma^\gamma_{\alpha\beta} x^i_\gamma,$$

(1. 15) $$\underset{P}{n}{}_{;\alpha}^{i} = \frac{\partial \underset{P}{n}^i}{\partial u^\alpha} + \Gamma^i_{jk}\underset{P}{n}^j x_\alpha^k - \sum_Q {}^{**}\Gamma^Q_{P\alpha}\underset{Q}{n}^i,$$

where Γ^i_{jk} and ${}^*\Gamma^\gamma_{\alpha\beta}$ are the christoffel symbols with respect to g_{ij} and the metric tensor $g_{\alpha\beta}$ of W^m respectively, and

$${}^{**}\Gamma^Q_{P\alpha} \overset{\text{def}}{\equiv} \frac{\partial \underset{P}{n}^i}{\partial u^\alpha}\underset{Q}{n}_i + \Gamma^i_{jk}\underset{P}{n}^j x_\alpha^k \underset{Q}{n}_i.$$

Since $\underset{P}{n}^i \underset{Q}{n}_i = \delta_{PQ}$, ${}^{**}\Gamma^Q_{P\alpha}$ is anti-symmetric with respect to the indices P and Q. Consequently we have

$${}^{**}\Gamma^P_{P\alpha} = 0 \quad \text{for any} \quad P.$$

Calculating $(\delta \underset{1}{n}^i)'$, we have

$$\delta \underset{1}{n}^i = d\underset{1}{n}^i + \Gamma^i_{jk}\underset{1}{n}^j x_\gamma^k du^\gamma - {}^{**}\Gamma^2_{1\alpha}\underset{2}{n}^i du^\alpha,$$

$$(\delta \underset{1}{n}^i)' = (d\underset{1}{n}^i)' + (\Gamma^i_{jk})' \underset{1}{n}^j x^k_\tau du^\tau + \Gamma^i_{jk} \underset{1}{n}'^j x^k_\tau du^\tau$$
$$+ \Gamma^i_{jk}\underset{1}{n}^j (x^k_\tau)' du^\tau - ({}^{**}\Gamma^2_{1\alpha})' \underset{2}{n}^i du^\alpha - {}^{**}\Gamma^2_{1\alpha} (\underset{2}{n}^i)' du^\alpha,$$

since G is isometric, that is, $\partial g_{ij}/\partial x^1=0$, we have $\partial \Gamma^i_{jk}/\partial x^1=0$, and we have $\underset{2}{n}'^i=0$ from (1. 13). Consequently we obtain the following relation between $\delta\underset{1}{n}'^i$ and $(\delta\underset{1}{n}^i)'$

(1. 16) $$(\delta \underset{1}{n}^i)' = \delta\underset{1}{n}'^i + \Gamma^i_{j1}\underset{1}{n}^j(t)\,\tau_\tau du^\tau - ({}^{**}\Gamma^2_{1\alpha})'\,\underset{2}{n}^i(t)\,du^\alpha.$$

We claim that the following theorem holds

THEOREM. *Let W^m and $\overline{W}^m$ be two orientable closed submanifolds of codimension 2 in S^{m+2} and p and $\bar{p}$ be the corresponding points of these submanifolds along an orbit, and $\underset{1}{H}_r(p)$ and $\underset{1}{\overline{H}}_r(p)$ be the r-th mean curvatures with respect to $\underset{1}{n}$ at these points respectively. Assume that the set of points in which the orbit is tangent to W^m or $\overline{W}^m$ has no inner point and that the second fundamental form with respect to $\underset{1}{n}$ of $W^m(t)$, $0\leqq t\leqq 1$ is positive definite. If G is a group of isometries of S^{m+2} and if the relation $\underset{1}{H}_r(p)=\underset{1}{\overline{H}}_r(p)$ holds for each $p\in W^m$, then W^m and $\overline{W}^m$ are congruent mod G.*

PROOF. We consider the following differential form of degree $m-1$ attached to each point p on the submanifold $W^m(t)$

$$\big((\underset{1}{n}', \underset{2}{n}, \delta_1\tau, \underbrace{\delta\underset{1}{n}, \cdots, \delta\underset{1}{n}}_{r-1}, dx, \cdots, dx)\big)$$

(1. 17) $$\overset{\text{def}}{\equiv} \sqrt{g}\,(\underset{1}{n}', \underset{2}{n}, \delta_1\tau, \delta\underset{1}{n}, \cdots, \delta\underset{1}{n}, dx, \cdots, dx)$$
$$= (-1)^{r-1}\sqrt{g}\,(\underset{1}{n}', \underset{2}{n}, \delta_1\tau, x_{\alpha_1}, \cdots, x_{\alpha_{r-1}}, x_{\beta_r}, \cdots, x_{\beta_{m-1}})$$
$$\times\ \underset{1}{b}^{\alpha_1}_{\beta_1}(t)\cdots \underset{1}{b}^{\alpha_{r-1}}_{\beta_{r-1}}(t)\, du^{\beta_1}\wedge du^{\beta_2}\wedge\cdots\wedge du^{\beta_{r-1}}\wedge du^{\beta_r}\wedge\cdots\wedge du^{\beta_{m-1}}$$

where g is the determinant of the metric tensor g_{ij} of S^{m+2}, the symbol ( ) means a determinant of order $m+2$ whose columns are the components of respective vectors, $\underset{P}{b}_{\alpha\beta}(t)$ is the second fundamental tensor with respect to $\underset{P}{n}$ and $\underset{P}{b}^{\beta}_{\alpha}(t)$ denotes $\underset{P}{b}_{\alpha\gamma}(t)\,g^{\beta\gamma}(t)$.

Then the exterior differential of the differential form (1. 17) becomes as follows

$$
\begin{aligned}
&d\Big(\big(\underset{1}{n}', \underset{2}{n}, \delta_1\tau, \underset{1}{\delta n}, \cdots, \underset{1}{\delta n}, dx, \cdots, dx\big)\Big)\\
&\quad =\Big(\big(\underset{1}{\delta n}', \underset{2}{n}, \delta_1\tau, \underset{1}{\delta n}, \cdots, \underset{1}{\delta n}, dx, \cdots, dx\big)\Big)\\
(1.18)\qquad &\qquad +\Big(\big(\underset{1}{n}', \underset{2}{\delta n}, \delta_1\tau, \underset{1}{\delta n}, \cdots, \underset{1}{\delta n}, dx, \cdots, dx\big)\Big)\\
&\qquad +\Big(\big(\underset{1}{n}', \underset{2}{n}, \delta(\delta_1)\tau, \underset{1}{\delta n}, \cdots, \underset{1}{\delta n}, dx, \cdots, dx\big)\Big)\\
&\qquad +\Big(\big(\underset{1}{n}', \underset{2}{n}, \delta_1 d\tau, \underset{1}{\delta n}, \cdots, \underset{1}{\delta n}, dx, \cdots, dx\big)\Big),
\end{aligned}
$$

because since S^{m+2} is a space of constant curvature, we have

$$\Big(\big(\underset{1}{n}', \underset{2}{n}, \delta_1\tau, \delta\underset{1}{\delta n}, \underset{1}{\delta n}, \cdots, \underset{1}{\delta n}, dx, \cdots, dx\big)\Big)=0$$

and also we have

$$\Big(\big(\underset{1}{n}', \underset{2}{n}, \delta_1\tau, \underset{1}{\delta n}, \cdots, \underset{1}{\delta n}, \delta dx, \cdots, dx\big)\Big)=0\,.$$

Let $\varepsilon_{i_1\cdots i_{m+2}}$ and $\varepsilon_{\alpha_1\cdots\alpha_m}$ be the ε-symbol of S^{m+2} and $W^m(t)$ respectively,

$$\varepsilon_{i_1\cdots i_{m+2}} \overset{\text{def}}{\equiv} \sqrt{g}\, e_{i_1\cdots i_{m+2}}, \qquad \varepsilon_{\alpha_1\cdots\alpha_m} \overset{\text{def}}{\equiv} \sqrt{g^*(t)}\, e_{\alpha_1\cdots\alpha_m}$$

the symbol $e_{i_1\cdots i_{m+2}}$ meaning plus one or minus one, depending on whether the indices $i_1\cdots i_{m+2}$ denote an even permutation of $1, 2, \cdots, m+2$ or odd permutation, and zero when at least any two indices have the same value, and also the symbol $e_{\alpha_1\cdots\alpha_m}$ meaning similarly for the indices $\alpha_1, \cdots, \alpha_m$ running from 1 to m, where $g^*(t)$ means the determinant of $g_{\alpha\beta}(t)$.

Then making use of the relation

$$\underset{2}{n}_i(\mathrm{t})\,\varepsilon_{\alpha\alpha_1\cdots\alpha_{m-1}} = \varepsilon_{jii_1\cdots i_m}\underset{1}{n}^j x_\alpha^{i_1} x_{\alpha_1}^{i_2}\cdots x_{\alpha_{m-1}}^{i_m},$$

we have

$$
\begin{aligned}
&\Big(\big(\underset{1}{n}', \delta_1\tau, x_\alpha, x_{\alpha_1}, \cdots, x_{\alpha_{m-1}}\big)\Big)\\
&\qquad = \underset{1}{n}_i(t)\,\delta_1^i\tau\,\Big(\big(\underset{1}{n}', \underset{1}{n}, x_\alpha, x_{\alpha_1}, \cdots, x_{\alpha_{m-1}}\big)\Big)\\
&\qquad = -\tau\,\underset{1}{n}_l(t)\,\delta_1^l \underset{2}{n}_i \underset{1}{n}'^i(t)\,\varepsilon_{\alpha\alpha_1\cdots\alpha_{m-1}}\\
&\qquad = 0
\end{aligned}
$$

from (1.11), consequently we get

$$
\begin{aligned}
&\Big(\big(\underset{1}{n}', \underset{2}{\delta n}, \delta_1\tau, \underset{1}{\delta n}, \cdots, \underset{1}{dn}, dx, \cdots, dx\big)\Big)\\
(1.19)\quad &= -\Big(\big(\underset{1}{n}', \delta_1\tau, x_\alpha, x_{\alpha_1}, \cdots, x_{\alpha_{r-1}}, x_{\beta_r}, \cdots, x_{\beta_{m-1}}\big)\Big)\\
&\times(-1)^r \underset{2}{b}{}_\beta^\alpha(t)\,\underset{1}{b}{}_{\beta_1}^{\alpha_1}(t)\cdots \underset{1}{b}{}_{\beta_{r-1}}^{\alpha_{r-1}}(t)\, du^\beta\wedge du^{\beta_1}\wedge\cdots\wedge du^{\beta_{r-1}}\wedge du^{\beta_r}\wedge\cdots\wedge du^{\beta_{m-1}}\\
&= 0.
\end{aligned}
$$

Next making use of the relation

$$\underset{1}{n}_i(t)\,\varepsilon_{\alpha_1\cdots\alpha_m}=\varepsilon_{iji_1\cdots i_m}\,\underset{2}{n}^j(t)\,x^{i_1}_{\alpha_1}\cdots x^{i_m}_{\alpha_m},$$

we have

$$\underset{1}{n}_i(t)\,\delta^i_1\,\varepsilon_{\alpha_1\cdots\alpha_m}=\varepsilon_{iji_1\cdots i_m}\,\delta^i_1\,\underset{2}{n}^j\,x^{i_1}_{\alpha_1}\cdots x^{i_m}_{\alpha_m},$$

since G is isometric, we have

$$\begin{aligned}\left(\underset{1}{n}_i(t)\,\delta^i_1\,\varepsilon_{\alpha_1\cdots\alpha_m}\right)' &= \varepsilon_{iji_1\cdots i_m}\,\delta^i_1\,\underset{2}{n}'^j\,x^{i_1}_{\alpha_1}\cdots x^{i_m}_{\alpha_m}\\ &+\varepsilon_{iji_1i_2\cdots i_m}\,\delta^i_1\,\underset{2}{n}^j\,x'^{i_1}_{\alpha_1}x^{i_2}_{\alpha_2}\cdots x^{i_m}_{\alpha_m}\\ &+\cdots\cdots\cdots\\ &+\varepsilon_{iji_1\cdots i_m}\,\delta^i_1\,\underset{2}{n}^j\,x^{i_1}_{\alpha_1}\cdots x^{i_{m-1}}_{\alpha_{m-1}}x'^{i_m}_{\alpha_m},\end{aligned}$$

on substituting (1. 10) and (1. 13) into the terms of the right-hand member of the above equation, we have

$$\left(\underset{1}{n}_i(t)\,\delta^i_1\,\varepsilon_{\alpha_1\cdots\alpha_m}\right)'=0.$$

Therefore the quantity $\underset{1}{n}_i(t)\,\delta^i_1\sqrt{g^*(t)}$ is independent of t, then we have

$$\begin{aligned}r\left(((\underset{1}{\delta n})',\ \underset{2}{n},\ \delta_1\tau,\ \underset{1}{\delta n},\ \cdots,\ \underset{1}{\delta n},\ dx,\ \cdots,\ dx)\right)\\ =(-1)^{r-1}m!\,\underset{1}{H'_r}\,\underset{1}{n}_i\,\delta^i_1\,\tau\,dA(t)\end{aligned}\tag{1. 20}$$

where $dA(t)$ is the area element of $W^m(t)$, and using (1. 12), we obtain

$$\begin{aligned}&\left((\underset{1}{n}',\ \underset{2}{n},\ \delta_1 d\tau,\ \underset{1}{\delta n},\ \cdots,\ \underset{1}{\delta n},\ dx,\ \cdots,\ dx)\right)\\ &\quad=-\left((\delta_1 d\tau,\ \underset{2}{n},\ \underset{1}{n}',\ \underset{1}{\delta n},\ \cdots,\ \underset{1}{\delta n},\ dx,\ \cdots,\ dx)\right)\\ &\quad=(-1)^{r+1}\left((\delta_1\tau_\theta,\ \underset{2}{n},\ \underset{1}{n}_i\,\delta^i_1\,\tau_\alpha\,g^{\alpha\gamma}x_\gamma,\ x_{\alpha_1}\cdots x_{\alpha_{r-1}},x_{\beta_r}\cdots,\ x_{\beta_{m-1}})\right)\\ &\qquad\times\underset{1}{b}^{\alpha_1}_{\beta_1}\cdots\underset{1}{b}^{\alpha_{r-1}}_{\beta_{r-1}}du^\theta\wedge du^{\beta_1}\wedge\cdots\wedge du^{\beta_{r-1}}\wedge du^{\beta_r}\wedge\cdots\wedge du^{\beta_{m-1}}\\ &\quad=(-1)^{r+1}(\underset{1}{n}_i\,\delta^i_1)^2\,\varepsilon_{\gamma\alpha_1\cdots\alpha_{r-1}\beta_r\cdots\beta_{m-1}}\,g^{\gamma\alpha}\,\underset{1}{b}^{\alpha_1}_{\beta_1}\cdots\underset{1}{b}^{\alpha_{r-1}}_{\beta_{r-1}}\varepsilon^{\theta\beta_1\cdots\beta_{m-1}}\tau_\alpha\tau_\theta\,dA(t).\end{aligned}$$

Putting

$$(m-1)!\ \underset{1}{c}^{\alpha\beta}_{(r)}=\varepsilon^{\alpha}_{\alpha_1\cdots\alpha_{r-1}\alpha_r\cdots\alpha_{m-1}}\,\varepsilon^{\beta\beta_1\cdots\beta_{r-1}\beta_r\cdots\alpha_{m-1}}\,\underset{1}{b}^{\alpha_1}_{\beta_1}\cdots\underset{1}{b}^{\alpha_{r-1}}_{\beta_{r-1}},$$

we get

$$\begin{aligned}\left((\underset{1}{n}',\ \underset{2}{n},\ \delta_1 d\tau,\ \underset{1}{\delta n},\ \cdots,\ \underset{1}{\delta n},\ dx,\ \cdots,\ dx)\right)\\ =(-1)^{r-1}(m-1)!\ \underset{1}{c}^{\alpha\beta}_{(r)}\,\tau_\alpha\tau_\beta\ (\underset{1}{n}_i\,\delta^i_1)^2\,dA.\end{aligned}\tag{1. 21}$$

By making use of (1. 16), (1. 18), (1. 19), (1. 20), (1. 21) and the relation

$$\delta(\delta_1^i)=\Gamma^i_{j1}x^j_\gamma du^\gamma,$$

we have

$$d\Big((\underset{1}{n}', \underset{2}{n}, \delta_1\tau, \underset{1}{\delta n}, \cdots, \underset{1}{\delta n}, dx, \cdots, dx)\Big)$$
$$=\frac{(-1)^{r-1}m!}{r}\Big\{\underset{1}{H}'_r \underset{1}{n}_i\delta_1^i\tau\, dA+\frac{r}{m}\underset{1}{c}^{\alpha\beta}_{(r)}(\underset{1}{n}_i\delta_1^i)^2\tau_\alpha\tau_\beta dA\Big\}$$
$$+\Big((\underset{1}{n}', \underset{2}{n}, \tau\Gamma_{j1}x^j_\gamma du^\gamma, \underset{1}{\delta n}, \cdots, \underset{1}{\delta n}, dx, \cdots, dx)\Big)$$
$$-\Big((\Gamma_{j1}\underset{1}{n}^j\tau_\gamma du^\gamma, \underset{2}{n}, \delta_1\tau, \underset{1}{\delta n}, \cdots, \underset{1}{\delta n}, dx, \cdots, dx)\Big).$$

After some calculations, we have

$$\Big((\underset{1}{n}', \underset{2}{n}, \tau\Gamma_{j1}x^j_\gamma du^\gamma, \underset{1}{\delta n}, \cdots, \underset{1}{\delta n}, dx, \cdots, dx)\Big)$$
$$=(-1)^{r-1}\tau(m-1)!\,\underset{1}{n}_i(t)\delta_1^i\Gamma_{jk1}\underset{1}{n}^k(t)x^j_\alpha\tau_\beta\underset{1}{c}^{\alpha\beta}_{(r)}dA(t),$$

since $\underset{1}{c}^{\alpha\beta}_{(r)}$ is the symmetric tensor, using the symbol $(\alpha\beta)$ for the symmetric part for the indices α and β, we get

$$\text{(1. 22)}\quad \Big((\underset{1}{n}', \underset{2}{n}, \tau\Gamma_{j1}x^j_\gamma du^\gamma, \underset{1}{\delta n}, \cdots, \underset{1}{\delta n}, dx, \cdots, dx)\Big)$$
$$=(-1)^{r-1}\tau(m-1)!\,\underset{1}{n}_i(t)\delta_1^i\Gamma_{jk1}\underset{1}{n}^k(t)x^j_{(\alpha}\tau_{\beta)}\underset{1}{c}^{\alpha\beta}_{(r)}dA(t),$$

where Γ_{jk1} means $g_{kl}\Gamma^l_{j1}$.

On the other hand, since G is isometric, that is, $\partial g_{ij}/\partial x^1=0$, we obtain

$$\Gamma^i_{j1}\underset{1}{n}^j(t)\underset{1}{n}_i(t)=\frac{1}{2}g^{ik}\Big(\frac{\partial g_{kj}}{\partial x^1}+\frac{\partial g_{1k}}{\partial x^j}-\frac{\partial g_{j1}}{\partial x^k}\Big)\underset{1}{n}^j(t)\underset{1}{n}_i(t)$$
$$=\frac{1}{2}\frac{\partial g_{kj}}{\partial x^1}\underset{1}{n}^j(t)\underset{1}{n}^k(t)=0.$$

Making use of the vector δ_1^i by the expression

$$\delta_1^i=\underset{1}{n}_l(t)\delta_1^l\underset{1}{n}^i(t)+\phi^\beta x^i_\beta(t),$$

we get

$$\text{(1. 23)}\quad -\Big((\Gamma_{j1}\underset{1}{n}^j(t)\tau_\gamma du^\gamma, \underset{2}{n}, \delta_1\tau, \underset{1}{\delta n}, \cdots, \underset{1}{\delta n}, dx, \cdots, dx)\Big)$$
$$=(-1)^{r-1}\tau(m-1)!\,\underset{1}{n}_i(t)\delta_1^i\Gamma_{jk1}\underset{1}{n}^j x^k_{(\alpha}\tau_{\beta)}\underset{1}{c}^{\alpha\beta}_{(r)}dA(t).$$

Thus from (1. 22), (1. 23) and $\Gamma_{ij1}+\Gamma_{ji1}=\partial g_{ij}/\partial x^1=0$, we can see the following result

$$\begin{aligned}&\Big((\underset{1}{n'}, \underset{2}{n}, \tau\Gamma_{j1}x^j_\gamma du^\gamma, \underset{1}{\delta n}, \cdots, \underset{1}{\delta n}, dx, \cdots, dx)\Big)\\ &\qquad -\Big((\Gamma_{j1}\underset{1}{n^j}(t)\tau_\gamma du^\gamma, \underset{2}{n}, \delta_1\tau, \underset{1}{\delta n}, \cdots, \underset{1}{\delta n}, dx, \cdots, dx)\Big)\\ &\qquad =(-1)^{r-1}(m-1)!\,\tau \underset{1}{n_i}(t)\delta^i_1(\Gamma_{jk1}+\Gamma_{kj1})\underset{1}{n^j}(t)x^k_{(\alpha}\tau_{\beta)}\underset{1}{c^{\alpha\beta}_{(r)}}\,dA(t)\\ &\qquad =0.\end{aligned}$$

Finally we have

$$d\Big((\underset{1}{n'}, \underset{2}{n}, \delta_1\tau, \underset{1}{\delta n}, \cdots, \underset{1}{\delta n}, dx, \cdots, dx)\Big)$$

$$\text{(1. 24)}\qquad =\frac{(-1)^{r-1}m!}{r}\Big\{\underset{1}{H'_r}\underset{1}{n_i}(t)\delta^i_1\tau\, dA+\frac{r}{m}\underset{1}{c^{\alpha\beta}_{(r)}}(\underset{1}{n_i}(t)\delta^i_1)^2\tau_\alpha\tau_\beta\, dA\Big\}.$$

Integrating both members of (1. 24) over the interval $0\leqq t\leqq 1$, and putting

$$\underset{1}{C^{\alpha\beta}_{(r)}}=g^*(0)^{1/2}\int_0^1 g^*(t)^{-1/2}\underset{1}{c^{\alpha\beta}_{(r)}}\,dt,$$

we have

$$m(\underset{1}{\bar{H}_r}-\underset{1}{H_r})\underset{1}{n_i}(0)\delta^i_1\tau\, dA(0)+r\underset{1}{C^{\alpha\beta}_{(r)}}\tau_\alpha\tau_\beta(\underset{1}{n_i}(0)\delta^i_1)^2dA(0)$$

$$\text{(1. 25)}\qquad =\frac{r(-1)^{r-1}}{(m-1)!}d\int_0^1\Big((\underset{1}{n'}, \underset{2}{n}, \delta_1\tau, \underset{1}{\delta n}, \cdots, \underset{1}{\delta n}, dx, \cdots, dx)\Big)dt.$$

Furthermore integrating both members of (1. 25) and applying Stokes' theorem

$$\begin{aligned}&\frac{m}{r}\iint_{W^m}(\underset{1}{\bar{H}_r}-\underset{1}{H_r})\underset{1}{n_i}(0)\delta^i_1\tau\, dA(0)+\iint_{W^m}(\underset{1}{n_i}(0)\delta^i_1)^2\underset{1}{C^{\alpha\beta}_{(r)}}\tau_\alpha\tau_\beta\, dA(0)\\ &\qquad =\frac{(-1)^{r-1}}{(m-1)!}\int_{\partial W^m}\int_0^1\Big((\underset{1}{n'}, \underset{2}{n}, \delta_1\tau, \underset{1}{\delta n}, \cdots, \underset{1}{\delta n}, dx, \cdots, dx)\Big)dt.\end{aligned}$$

Since W^m is closed, we have

$$\frac{m}{r}\iint_{W^m}(\underset{1}{\bar{H}_r}-\underset{1}{H_r})\underset{1}{n_i}(0)\delta^i_1\tau\, dA(0)+\iint_{W^m}\Big(\underset{1}{n_i}(0)\,\delta^i_1\Big)^2\underset{1}{C^{\alpha\beta}_{(r)}}\tau_\alpha\tau_\beta\, dA(0)=0,$$

using the hypotheses $\underset{1}{\bar{H}_r}=\underset{1}{H_r}$, we obtain

$$\iint_{W^m}\Big(\underset{1}{n_i}(0)\,\delta^i_1\Big)^2\underset{1}{C^{\alpha\beta}_{(r)}}\tau_\alpha\tau_\beta\, dA(0)=0.$$

On the other hand, from that the second fundamental form with respect to $\underset{1}{n}$ of $W^m(t)$ is positive definite everywhere in $W^m(t)$, $0\leqq t\leqq 1$, the quantity $\underset{1}{C^{\alpha\beta}_{(r)}}v_\alpha v_\beta$ becomes positive definite. From that the set of points in which the orbit of transformation is tangent to W^m or $\bar{W}^m$ has no inner point, a point on W^m such that $\underset{1}{n}_i(0)\delta_1^i=0$ must be an isolate point. Moreover since τ is a continuous function of W^m, then we have

$$\tau = \text{constant}$$

for all points of W^m. Consequently we can arrive at the following result

$$W^m \equiv \overline{W}^m \quad \text{mod} \quad G.$$

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