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On conjugation families

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1. Introduction.

In his paper [2], Goldschmidt has proved a generalization of Alperin's theorem in [1]. The purpose of this paper is to give another proof of his result in [2], namely, to show that his family defined in [2] is a conjugation family.

Let p be a prime, G be a finite group, $Syl_p(G)$ denote the set of Sylow p-subgroups of G, and P be an element of $Syl_p(G)$. Let $\mathscr{S}$ be the set of all pairs (H, T) such that H is a nontrivial subgroup in P and T is a subgroup in $N_G(H)$.

Our notation corresponds to that of Alperin [1], Goldschmidt [2], and Glauberman [3]. Let $\mathscr{F}$ be a subset of $\mathscr{S}$, H be a subgroup of P, and L be a finite group.

a) Suppose that A and B are nonempty subsets of P and $g\in G$. We say that A is $\mathscr{F}$-conjugate to B via g if there exist elements $(H_1, T_1), \cdots, (H_n, T_n)$ in $\mathscr{F}$ and $g_1, \cdots, g_n$ in G such that $g_i\in T_i$ $(i=1, \cdots, n)$, $A^g=B$, where $g=g_1\cdots g_n$, and $A\subseteq H$ and $A^{g_1\cdots g_i}\subseteq H_{i+1}$ $(i=1, \cdots, n-1)$.

b) We say that $\mathscr{F}$ is a conjugation family (for P in G) if it has the following property: whenever A and B are nonempty subsets of P and $g\in G$ and $A^g=B$, then A is $\mathscr{F}$-conjugate to B via g.

c) We say that H is a tame intersection (in P) if $H=P\cap Q$ for some $Q\in Syl_p(G)$ and $N_P(H)\in Syl_p(N_G(H))$. In particular, there is a Sylow p-subgroup R of G such that $N_R(H)\in Syl_p(N_G(H))$ and $P\cap R=H$.

d) We say that L is p-isolated if, for some $S\in Syl_p(L)$, $\langle N_L(E)\colon 1\neq E\leq S\rangle$ is a nontrivial proper subgroup of L. In particular, if L is p-isolated, then there exists S_1 in $Syl_p(L)$ such that $S\cap S_1=1$.

THEOREM A.

For each $(H, N_G(H))\in\mathscr{S}$, *we assign a normal subgroup* K_H *of* $N_G(H)$. *Let* $\mathscr{F}$ *be the set of all pairs* $(H, T)\in\mathscr{S}$ *satisfying the following conditions* i), ii), iii), *and* iv).

i) *H is a tame intersection in P.*

ii) *$H=P$ or the factor group $N_G(H)/H$ is p-isolated.*

iii) *If* $K_H \cap P \not\leq H$, *then* $T \leq K_H$.

iv) *For each element* x *in* $T \cap N_P(H) - H$, $\langle x^T \rangle = T$.

Then $\mathcal{F}$ *is a conjugation family.*

As corollaries of Theorem A, we can have several conjugation families.

COROLLARY 1 (Goldschmidt's family in [2]).

Let $\mathcal{F}_1$ *be the set of all pairs* $(H, T) \in \mathcal{S}$ *satisfying the conditions* i) *and* ii) *in Theorem A and the following condition* iii)$'$:

iii)$'$ *If either* $O_{p',p}(N_G(H)) \cap P \neq H$ *or* $C_P(H) \not\leq H$ *holds, then* $T \leq C_G(H)$. *Then* $\mathcal{F}_1$ *is a conjuation family.*

COROLLARY 2.

For each $(H, N_G(H)) \in \mathcal{S}$, *we assign a normal series* $H \geq H_1 \gneqq H_2 \gneqq \cdots \gneqq H_n$ *of* $N_G(H)$. *Let* $\mathcal{F}_2$ *be the set of all pairs* $(H, T) \in \mathcal{S}$ *satisfying the conditions* i) *and* ii) *in Theorem A and the following condition* iii)$''$:

iii)$''$ *If there exists an element* t *in* P–H *with the property that* $[t, H_i] \subseteq H_{i+1}$ $(i=1, \cdots, n-1)$, *then* $T \leq C_{N_G(H)}(H_1/H_n)$.

Then $\mathcal{F}_2$ *is a conjugation family.*

COROLLARY 3.

Let $\mathcal{F}_3$ *be the set of all pairs* $(H, T) \in \mathcal{S}$ *satisfying the conditions* i) *and* ii) *in Theorem A and the following conditions* iii)$'''$:

iii)$'''$ *If* $C_P(\Omega_1(Z(H))) \not\leq H$, *then* $T \leq C_G(Z(H)) \cap N_G(H)$.

Then $\mathcal{F}_3$ *is a conjugation family.*

MAIN THEOREM.

For each subgroup H *in* P, *we assign a normal subgroup* K_H *of* $N_G(H)$. *For an arbitrary conjugation family* $\mathcal{F}^*$, *we define a family* $\mathcal{F}$ *to be the set of all pairs* $(H, T) \in \mathcal{S}$ *satisfying the conditions* a), b), c), d), *and* e).

a) H *is a tame intersection in* P.

b) $H=P$ *or the factor group* $N_G(H)/H$ *is* p*-isolated.*

c) *If* $K_H \cap P \not\leq H$, *then* $T \leq K_H$.

d) *For each element* x *in* $T \cap N_P(H) - H$, $\langle x^T \rangle = T$.

e) $(H^g, L) \in \mathcal{F}^*$ *for some element* g *in* G *and some subgroup* L *in* $N_G(H^g)$.

Then $\mathcal{F}$ *is a conjugation family.*

2. Proof of Main Theorem.

Let $\mathcal{S}_1$ be the set of all pairs $(H, T) \in \mathcal{S}$ satisfying the conditions that

$(H^s, L)\in\mathscr{F}^*$ for some element s in G and a subgroup L in $N_G(H^s)$. Clearly, $\mathscr{S}_1$ contains $\mathscr{F}^*$ and $\mathscr{F}$ and we have that $\mathscr{S}_1$ is a conjugation family.

Suppose that $\mathscr{F}$ is not a conjugation family. Therefore, there are a subset A in P and an element g in G such that $A^g\subseteq P$ and A is not $\mathscr{F}$-conjugate to A^g via g. Choose such a pair (A, g) with maximal order $|\langle A\rangle|$. Set $B=A^g$ and let $\sum=\{\mathscr{T}|\mathscr{S}_1\supseteq\mathscr{T}\supseteq\mathscr{F}$ and A is $\mathscr{T}$-conjugate to B via $g\}$.

We introduce an order on Σ as follows. First, we define an order $(\gtrapprox)$ on $\mathscr{S}$: $(H_1, T_1)>(H_2, T_2)$ if one of the following conditions holds;

a) $|H_1|\gneqq|H_2|$.

b) $|H_1|=|H_2|$ and $|N_P(H_1)|\gneqq|N_P(H_2)|$.

c) $|H_1|=|H_2|$, $|N_P(H_1)|=|N_P(H_2)|$, and $|T_1K_{H_1}|\lneqq|T_2K_{H_2}|$.

d) $|H_1|=|H_2|$, $|N_P(H_1)|=|N_P(H_2)|$, $|T_1K_{H_1}|=|T_2K_{H_2}|$, and $|T_1|\lneqq|T_2|$.

And $(H_1, T_1)\approx(H_2, T_2)$ if $|H_1|=|H_2|$, $|N_P(H_1)|=|N_P(H_2)|$, $|T_1K_{H_1}|=|T_2K_{H_2}|$, and $|T_1|=|T_2|$.

We define an order $(\gg)$ on Σ. For $\mathscr{T}_1, \mathscr{T}_2\in\Sigma$, we choose minimal elements (H_1, T_1) and (H_2, T_2) in $\mathscr{T}_1-\mathscr{F}$ and $\mathscr{T}_2-\mathscr{F}$, respectively, with respect to the above order $(\gtrapprox)$ and we define $\mathscr{T}_1\gg\mathscr{T}_2$ if either $(H_1, T_1)>(H_2, T_2)$ or $(H_1, T_1)\approx(H_2, T_2)$ and the number of minimal elements of $\mathscr{T}_1-\mathscr{F}$ is fewer than that of $\mathscr{T}_2-\mathscr{F}$. Note that $\Sigma\neq\phi$, since $\mathscr{S}_1$ is a conjugation family, and $\mathscr{T}-\mathscr{F}\neq\phi$ for every $\mathscr{T}\in\Sigma$, since $\mathscr{F}\notin\Sigma$. Let $\mathscr{T}$ be a maximal element of Σ with respect to the order $(\gg)$. To prove Main Theorem, we will show that $\mathscr{T}\subseteq\mathscr{F}$, which is a contradiction. Let (H, T) be a minimal element of $\mathscr{T}-\mathscr{F}$ with respect to the order $(\lessapprox)$.

(1) $|H|\geq|\langle A\rangle|$.

Suppose false, then H contains no conjugates of A. Set $\mathscr{T}^*=\mathscr{T}-(H, T)$, then A is $\mathscr{T}^*$-conjugate to B via g. This contradicts the choice of $\mathscr{T}$.

(2) $N_P(H)\in Syl_p(N_G(H))$.

Suppose false, then P contains a conjugate L of H which satisfies $N_P(L)\in Syl_p(N_G(L))$. Then there is an element x in G such that $N_P(H)^x\gneqq N_P(L)\leq P$ and $H^x=L$. Since $|N_P(H)|\gneqq|H|\geq|A|$, we have that $N_P(H)$ is $\mathscr{F}$-conjugate to $N_P(H)^x$ via x by the choice of A. Thus, we may use $\{(H^x, T^x)\}\cup\mathscr{F}$ in place of $\{(H, T)\}$, namely, if X is $\{(H, T)\}$-conjugate to X^y via y, then X is $\mathscr{F}$-conjugate to X^x via x and X^x is $\{(H^x, T^x)\}$-conjugate to X^{yx} via y^x and X^{yx} is $\mathscr{F}$-conjugate to X^y via x^{-1}, (see the diagram I). Set $\mathscr{T}^*=\{\mathscr{T}-(H, T)\}\cup\{(H^x, T^x)\}\cup\mathscr{F}$. Then A is $\mathscr{T}^*$-conjugate to B via g, which implies that $\mathscr{T}^*\in\Sigma$. However, since $|N_P(H^x)|\gneqq|N_P(H)|$,

$$\begin{tikzcd} & H^x \arrow[r, "T^x"] & H^x \arrow[d, "\mathscr{F}"] & \\ {} \arrow[r] & H \arrow[u, "\mathscr{F}"] \arrow[r, "T"] & H \arrow[r] & {} \end{tikzcd}$$

(diagram I)

we have $(H^x, T^x)>(H, T)$. Thus, we have that $\mathscr{T}^* \gg \mathscr{T}$, a contradiction.

(3) $P=H$ or $N_G(H)/H$ is p-isolated.

Suppose false, then $N_G(H)=\langle N_{N_G(H)}(P_i): N_P(H)\geq P_i \gneqq H\rangle$. Thus, if H is $\{(H, N_G(H))\}$-conjugate to H^t via t, then there are elements t_i in $N_{N_G(H)}(P_i)$ such that $t=t_1\cdots t_k$ and H is $\{(P_1, N_{N_G(H)}(P_1))\}$-conjugate to H^{t_1} via t_1 and $H^{t_1\cdots t_{i-1}}$ is $\{(P_i, N_{N_G(H)}(P_i))\}$-conjugate to $H^{t_1\cdots t_i}$ via t_i. Hence, H is $\{(P_i, N_{N_G(H)}(P_i)): N_P(H)\geq P_i \gneqq H\}$-conjugate to H^t via t. Therefore, we may use $\{(P_i, N_{N_G(H)}(P_i)): N_P(H)\geq P_i \gneqq H\}$ in place of $\{(H, T)\}$. Since $|P_i| \gneqq |H| \geq |A|$ for every P_i, we may use $\mathscr{F}$ in place of $\{(P_i, N_{N_G(H)}(P_i)): N_P(H) \geq P_i \gneqq H\}$, by the choice of A. Thus we may use $\mathscr{F}$ in place of $\{(H, T)\}$. Set $\mathscr{T}^*=\mathscr{T}-(H, T)$. Then A is $\mathscr{T}^*$-conjugate to B via g, which contradicts the choice of $\mathscr{T}$.

(4) H is a tame intersection.

Since $P=H$ or $N_G(H)/H$ is p-isolated, there is $Q \in Syl_p(G)$ such that $N_Q(H) \in Syl_p(N_G(H))$ and $P \cap N_Q(H)=H$ by the definition (d). Thus $P \cap Q =H$. On the other hand, $N_P(H) \in Syl_p(N_G(H))$ by (2). Thus H is a tame intersection in P.

(5) If $K_H \cap P \nleq H$, then $T \leq K_H$.

Suppose false, then $P_0=K_H H \cap P \gneqq H$ and $T \nleq K_H$. Then $N_G(H)= N_{N_G(H)}(P_0)K_H$ by the Frattini argument. Thus we may use $\{(P_0, N_{N_G(H)}(P_0))\} \cup \{H, K_H)\}$ in place of $\{(H, T)\}$. Since $|P_0| \gneqq |H| \geq |A|$, we may use $\mathscr{F}$ in place of $\{(P_0, N_{N_G(H)}(P_0))\}$. Thus we may use $\mathscr{F} \cup \{(H, K_H)\}$ in place of $\{(H, T)\}$. Set $\mathscr{T}^*=\{\mathscr{T}-(H, T)\} \cup \{(H, K_H)\}$. Then A is $\mathscr{T}^*$-conjugate to B via g. Thus, we have $\mathscr{T}^* \in \Sigma$. But, since $|TK_H/K_H|>1$, we have $\mathscr{T}^* \gg \mathscr{T}$, a contradiction.

(6) For each element x in $T \cap N_P(H)-H$, $\langle x^T\rangle=T$.

Suppose false, then $T_0=\langle x^T\rangle$ is a proper normal subgroup of T and $T_0 \cap P \nleq H$. Let Q_1 be a Sylow p-subgroup of T contains $T \cap P$ and Q be a Sylow p-subgroup of $N_G(H)$ contains Q_1. Then $Q \cap N_P(H) \nleq H$. Since $N_G(H)/H$ is p-isolated, there is an element z in $N_G(H)$ such that $Q^z= N_P(H)$ and $z \in \langle N_{N_G(H)}(P_i): N_P(H) \geq P_i \gneqq H\rangle$. Thus H is $\mathscr{F}$-conjugate to

H^z via z by the choice of A. Hence, we may assume that $P_0=T_0\cap P\in Syl_p(T_0)$. Then $T=N_T(P_0H)T_0$ by the Frattini argument. Thus, we may use $\{(P_0H, N_T(P_0H))\}\cup\{(H, T_0)\}$ in place of $\{(H, T)\}$. Since $|P_0H|\gneq|H|\geq|A|$, we may use $\mathscr{F}$ in place of $\{(P_0H, N_T(P_0H))\}$ by the choice of A. Set $\mathscr{T}^*=\{\mathscr{T}-(H, T)\}\cup\{(H, T_0)\}$. Then $\mathscr{T}^*\in\Sigma$ and $|T|>|T_0|$, we have that $\mathscr{T}^*\ll\mathscr{T}$, a contradiction.

This completes the proof of Main Theorem.

3. Proof of Corollaries.

PROOF of Theorem A.

By Alperin's theorem in [1], $\mathscr{S}$ is a conjugation family. Therefore, Theorem A is a corollary of Main Theorem.

PROOF of Corollary 1.

Let $\mathscr{F}$ be a conjugation family defined in Theorem A by taking $K_H=C_G(H)O_{p',p}(N_G(H))$. Let $(H, T)\in\mathscr{F}$. If $K_H\cap P\nleq H$, then $T\leq C_G(H)O_{p',p}(N_G(H))$. Since $O_{p',p}(N_G(H))\leq C_G(H)N_P(H)$ and $(P, N_G(P))\in\mathscr{F}$, $\{\mathscr{F}-(H, T)\{\cup\{(H, C_G(H))\}$ is a conjugation family. By repeating these steps, we have that $\mathscr{F}_1$ is a conjugation family.

PROOF of Corollary 2.

Let $\mathscr{F}$ be a conjugation family defined in Theorem A by taking $K_H=N_G(H)\cup(\bigcap_{i=1}^{n-1}C_G(H_i/H_{i+1}))$. Let $(H, T)\in\mathscr{F}$. If $P\cap(\bigcap_{i=1}^{n-1}C_G(H_i/H_{i+1}))\nleq H$, then $K_H\cap P\nleq H$. Thus, $T\leq\bigcap_{i=1}^{n-1}C_G(H_i/H_{i+1})\cap N_G(H)$ by Theorem A. Since $\bigcap_{i=1}^{n-1}C_G(H_i/H_{i+1})\cap N_G(H)\leq N_P(H)O^p(\bigcap_{i=1}^{n-1}C_G(H_i/H_{i+1}))$ and $O^p(\bigcap_{i=1}^{n-1}C_G(H_i/H_{i+1}))\leq C_G(H_1/H_n)$ and $(P, N_G(P))\in\mathscr{F}$, $\{\mathscr{F}-(H, T)\}\cup\{(H, C_{N_G(H)}(H_1/H_n)\}$ is a conjugation family. By repeating these steps, we have that $\mathscr{F}_2$ is a conjugation family.

PROOF of Corollary 3.

Let $\mathscr{F}$ be a conjugation family defined in Theorem A by taking $K_H=C_G(\Omega_1(Z(H)))\cap N_G(H)$. Let $(H, T)\in\mathscr{F}$. If $K_H\cap P\nleq H$, then $T\leq K_H$ by Theorem A. On the other hand, $K_H\leq O^p(G_G(\Omega_1(Z(H)))\cap N_G(H))N_P(H)$ and $O^p(C_G(\Omega_1(Z(H)))\cap N_G(H))\leq C_G(Z(H))\cap N_G(H)$. Since $(P, N_G(P))\in\mathscr{F}$, $\{\mathscr{F}-(H, T)\}\cup\{(H, C_G(Z(H)))\}$ is a conjugation family. By repeating these steps, we have that $\mathscr{F}_3$ is a conjugation family.

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