Hokkaido Mathematical Journal Vol. 6 (1977) p. 31-38

σ-hypersurfaces of generalized complex space forms

Lieven VANHECKE

(Received April 14, 1976)

In [9] we proved the following

THEOREM *Let M be a σ-submanifold of a generalized complex space form $\tilde{M}(\mu, \alpha)$. The following conditions are equivalent if the real codimension is >2:*

(i) *the normal connection of M is trivial;*

(ii) *$\mu=\alpha$ and M is totally geodesic in $\tilde{M}$.*

$\mu=\alpha$ means that the ambient space is a space of constant curvature. On the other hand, the following theorem of Chern [1] is well-known:

THEOREM *Let M (with real dimension >3) be a Kähler-Einstein hypersurface of a complex space form $\tilde{M}(\mu)$. Then M is totally geodesic in $\tilde{M}$ or $\mu>0$.*

The main purpose of this paper is to extend the first theorem to the case of a *real codimension* 2 and nonpositive α. Therefore we prove the following extension of Chern's theorem. We suppose that $\tilde{M}(\mu, \alpha)$ is *locally symmetric.*

THEOREM *Let M (with real dimension $n>3$) be a σ-hypersurface of a generalized complex space form $\tilde{M}(\mu, \alpha)$. If M is Einstein, then M is totally geodesic in $\tilde{M}$ or $\mu-\alpha+n\nu>0$, ν denoting the antiholomorphic sectional curvature.*

Finally, we give some considerations about σ-submanifolds and in particular we give a new proof of the fact that S^6, considered with the usual nearly Kähler structures, has no holomorphic hypersurfaces [3].

1. Let $\tilde{M}$ be an m-dimensional C^∞ Riemannian manifold with Levi-Civita connection $\tilde{\nabla}$. Then the curvature tensor $\tilde{R}$ of $\tilde{M}$ is given by $\tilde{R}(X, Y)=\tilde{\nabla}_{[X,Y]}-[\tilde{\nabla}_X, \tilde{\nabla}_Y]$ for any $X, Y\in\mathscr{X}(\tilde{M})$ where $\mathscr{X}(\tilde{M})$ is the Lie algebra of C^∞ vector fields on $\tilde{M}$. Further, let $\{E_i\}$ be a local orthonormal frame field on $\tilde{M}$. Then the Ricci tensor $\tilde{S}(X, Y)$ is defined by

$$\tilde{S}(X, Y)=\sum_{i=1}^{m}\tilde{R}(X, E_i, Y, E_i)$$

where $\tilde{R}(X, E_i, Y, E_i)=\tilde{g}(\tilde{R}(X, E_i)Y, E_i)$ and $\tilde{g}$ is the metric tensor of $\tilde{M}$.

Let M be an n-dimensional submanifold immersed in $\tilde{M}$. Then we have

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y)$$

where $\nabla_X Y$ denotes the component of $\tilde{\nabla}_X Y$ tangent to M and $X, Y \in \mathscr{X}(M)$. σ is a symmetric covariant tensor field of degree 2 with values in $\mathscr{X}(M)^{\perp}$. We have further

$$\tilde{\nabla}_X N = -A_N X + D_X N$$

where N is a normal vector field. $-A_N X$(resp. $D_X N$) denotes the tangential (resp. normal) component of $\tilde{\nabla}_X N$. D is the linear connection in the normal bundle (called the *normal connection*) $T(M)^{\perp}$ and A is a cross-section of a vector bundle $\mathrm{Hom}(T(M)^{\perp}, S(M))$ where $S(M)$ is the bundle whose fibre at each point is the space of symmetric linear transformations of $T_x(M) \to T_x(M)$, $x \in M$, i.e. for any normal vector $N \in T_x(M)^{\perp}$, $A_N : T_x(M) \to T_x(M)$. We have

$$\tilde{g}(\sigma(X, Y), N) = g(A_N X, Y) = g(X, A_N Y)$$

where g denotes the induced metric tensor on M. σ and A are called both the *second fundamental form*.

$H = \frac{1}{n}$ trace σ is the *mean curvature vector* of M in $\tilde{M}$ and the submanifold is *minimal* if $H=0$. If the second fundamental form σ vanishes identically, M is called a *totally geodesic* submanifold.

Let $R^{\perp}$ be the curvature tensor associated with D, *i.e.* $R^{\perp}(X, Y) = D_{[X,Y]} - [D_X, D_Y]$. Then the normal connection is *flat* (or *trivial*) if $R^{\perp}$ vanishes identically. Note that the normal connection is flat if the (real) codimension is one and if the (real) codimension is higher, then the normal connection is not flat in general.

The *equations of Gauss* and *Ricci* are given respectively by

$$\begin{aligned}\tilde{R}(X, Y, Z, W) &= R(X, Y, Z, W) \\ &\quad + \tilde{g}\{\sigma(X, W), \sigma(Y, Z) - \tilde{g}\{\sigma(X, Z), \sigma(Y, W)\end{aligned} \tag{1}$$

and

$$\{\tilde{R}(X, Y)N\}^{\perp} = R^{\perp}(X, Y)N - \sigma(A_N X, Y) + \sigma(X, A_N Y) \tag{2}$$

or

$$\tilde{R}(X, Y, N, N') = R^{\perp}(X, Y, N, N') + g([A_N, A_{N'}]X, Y) \tag{2$'$}$$

where $X, Y, Z, W \in \mathscr{X}(M)$ and $N, N' \in \mathscr{X}(M)^{\perp}$.

2. Let $(\tilde{M}, g, J)$ be a C^{∞} Riemannian manifold which is *almost Hermitian*, that is, the tangent bundle has an almost complex structure J and a

Riemannian metric g such that $g(JX, JY)=g(X, Y)$ for all $X, Y \in \mathscr{X}(\tilde{M})$. Then $\dim \tilde{M}=m=2p$ and $\tilde{M}$ is orientable.

A subspace T' of $T_x(\tilde{M})$ is holomorphic if $JT' \subset T'$ and a submanifold M immersed in $\tilde{M}$ is a *holomorphic submanifold* if each tangent space of M is holomorphic. The sectional curvature of $\tilde{M}$ restricted to a holomorphic 2-plane is called the *holomorphic sectional curvature.*

Studying almost Hermitian manifolds $\tilde{M}$ we find for some classes nice identities for the curvature tensor [5], [6]. One of these is of particular interest, namely

$$\tilde{R}(X, Y, Z, W)=\tilde{R}(JX, JY, JZ, JW)$$

for all $X, Y, Z, W \in \mathscr{X}(\tilde{M})$. A manifold such that the curvature tensor satisfies this identity is called an *RK-manifold* [8].

We say further that an almost Hermitian manifold is of *constant type* [4], [8] at $x \in \tilde{M}$ provided that for all $X \in T_x(\tilde{M})$ we have

$$\lambda(X, Y)=\lambda(X, Z) \tag{3}$$

with

$$\lambda(X, Y)=\tilde{R}(X, Y, X, Y)-\tilde{R}(X, Y, JX, JY)$$

whenever the planes defined by X, Z and X, Y are antiholomorphic and $g(Y, Y)=g(Z, Z)$. We recall that a subspace T' of $T_x(\tilde{M})$ is *antiholomorphic* if $JT' \subset T'^{\perp}$, where $T'^{\perp}$ is the orthogonal space of T' in $T_x(\tilde{M})$. If the condition (3) holds for all $x \in \tilde{M}$, we say that $\tilde{M}$ has (*pointwise*) *constant type.* Finally, if $X, Y \in \mathscr{X}(\tilde{M})$ with $g(X, Y)=g(JX, Y)=0$, $\lambda(X, Y)$ is constant whenever $g(X, X)=g(Y, Y)=1$, then $\tilde{M}$ is said to have *global constant type.*

We proved in [8] that an RK-manifold has (pointwise) constant type if and only if there exists a C^∞ function α such that

$$\lambda(X, Y)=\alpha\{g(X, X)\, g(Y, Y)-g(X, Y)^2-g(JX, Y)^2\} \tag{4}$$

for all $X, Y \in \mathscr{X}(\tilde{M})$. Furthermore, $\tilde{M}$ has global constant type if and only if (4) holds with a constant function α. This α is called the *constant type* of $\tilde{M}$.

An RK-manifold with constant holomorphic sectional curvature μ and constant type α is called a *generalized complex space form* [11]. The class formed by such manifolds contains the well-known complex space forms but it contains for example also S^6 with the usual almost complex structure. The curvature tensor of such a manifold is given by [9]

$$\tilde{R}(X, Y, Z, W)=\nu\{g(X, Z)\, g(Y, W)-g(X, W)\, g(Y, Z)\}+$$
$$\frac{1}{4}(\mu-\alpha)\{g(X, JZ)\, g(Y, JW)-g(X, JW)\, g(Y, JZ)+2g(X, JY)\, g(Z, JW)\} \tag{5}$$

where ν denotes the antiholomorphic sectional curvature and this is given by

$$4\nu=\mu+3\alpha. \tag{6}$$

Note that a generalized complex space form is a *space of constant curvature* if and only if $\mu=\alpha$.

3. Let M be an n-dimensional holomorphic submanifold immersed in an almost Hermitian manifold $\tilde{M}$. M is called a *σ-submanifold* [10] if the second fundamental form σ is *complex bilinear*, i.e.

$$\sigma(JX, Y)=\sigma(X, JY)=J\sigma(X, Y) \tag{7}$$

and this is equivalent with

$$A_{JN}X=JA_NX, \qquad A_NJX+JA_NX=0 \tag{8}$$

for all $X, Y\subset\mathscr{X}(M)$. It is easy to verify that a *σ-submanifold is a minimal submanifold.*

Let ∇' be the covariant differentiation with respect to the connection in (tangent bundle) $\oplus$ (normal bundle). Hence

$$(\nabla'_X\sigma)(Y, Z)=D_X\sigma(Y, Z)-\sigma(\nabla_XY, Z)-\sigma(Y, \nabla_XZ)$$

for all $X, Y, Z\in\mathscr{X}(M)$. Suppose further that $\{E_a\}$ is a local orthonormal frame field of $\tilde{M}$ such that E_i, $i, j=1, 2, \cdots, n$ are tangent to M and E_λ, λ, $\mu=n+1, \cdots, m$ are normal to M. If we set $A_\lambda=A_{E_\lambda}$ and

$$h^\lambda_{ij}=g(A_\lambda E_i, E_j) \quad \text{or} \quad \sigma(E_i, E_j)=\sum_\lambda h^\lambda_{ij}E_\lambda,$$

then we have the following formula for a *minimal* submanifold M of $\tilde{M}$ [2]:

$$\begin{aligned}\frac{1}{2}\Delta\|\sigma\|^2=&\|\nabla'\sigma\|^2+\sum tr(A_\lambda A_\mu-A_\mu A_\lambda)^2-\sum(tr\ A_\lambda A_\mu)^2\\&-\sum(4\tilde{R}^\lambda_{\mu ij}h^\lambda_{jk}h^\mu_{ik}-\tilde{R}^\lambda_{k\mu k}h^\lambda_{ij}h^\mu_{ij}+2\tilde{R}^i_{jkj}h^\lambda_{il}h^\lambda_{kl}+2\tilde{R}^i_{jkl}h^\lambda_{il}h^\lambda_{jk}).\end{aligned} \tag{9}$$

Here Δ denotes the Laplacian and $\|\sigma\|$ is the length of the second fundamental form σ.

Based on this formula (9) it is a straightforward calculation from (5) to obtain, if $\tilde{M}(\mu, \alpha)$ is *locally symmetric* (and this will be supposed in section 4 and 5),

LEMMA 1 *Let M be an n-dimensional σ-submanifold of an m-dimensional generalized complex space form $\tilde{M}(\mu, \alpha)$. Then*

$$\begin{aligned}\frac{1}{2}\Delta\|\sigma\|^2=&\|\nabla'\sigma\|^2+\sum tr(A_\lambda A_\mu-A_\mu A_\lambda)^2\\&-\sum(tr\ A_\lambda A_\mu)^2+\frac{1}{4}\{(n+4)(\mu-\alpha)+4n\alpha\}\|\sigma\|^2.\end{aligned} \tag{10}$$

The last therm may be replaced by $(\mu-\alpha+n\nu)\|\sigma\|^2$.

4. We shall prove now a partial generalization of a well-known theorem of Chern [1]. Therefore we wish to consider an *Einstein space M,* i. e. there exists a function ρ on M such that

$$S(X, Y)=\frac{\rho}{n}g(X, Y)$$

for all $X, Y\in\mathscr{X}(M)$. It is well-known that ρ is a constant function if $n\geqq 3$.

THEOREM 2 *Let M^n be an n-dimensional σ-hypersurface of a generalized complex space form $M^m(\mu, \alpha)$ $(m=n+2)$. If $n>3$ and M is Einstein, then either M is totally geodesic in $\tilde{M}$ or $\mu-\alpha+n\nu>0$.*

PROOF Since M is minimal it follows from (1)

$$S(X, Y)=\sum_{i=1}^{n}\tilde{R}(X, E_i, Y, E_i)-\sum_{i=1}^{n}g\{\sigma(X, E_i), \sigma(Y, E_i)\}.$$

Further we get by (5) and (6)

$$\sum_{i=1}^{n}\tilde{R}(X, E_i, Y, E_i)=\left\{(n-1)\nu+\frac{3}{4}(\mu-\alpha)\right\}g(X, Y).$$

If we take

$$E_{n+2}=JE_{n+1},$$

then we obtain from (7) or (8)

$$\sum_{i=1}^{n}g\{\sigma(X, E_i), \sigma(Y, E_i)\}=2g(A_{n+1}X, A_{n+1}Y).$$

Since M is Einstein, we obtain from these formulas

$$g(A_{n+1}X, A_{n+1}Y)=\frac{1}{2}\left\{(n-1)\nu+\frac{3}{4}(\mu-\alpha)-\frac{\rho}{n}\right\}g(X, Y).$$

On the other hand, it follows easily

$$\rho=n\left\{(n-1)\nu+\frac{3}{4}(\mu-\alpha)\right\}-\|\sigma\|^2$$

which gives that $\|\sigma\|$ is constant and

$$g(A_{n+1}X, A_{n+1}Y)=\frac{1}{2n}\|\sigma\|^2g(X, Y)$$

or

$$A_{n+1}^2=\frac{1}{2n}\|\sigma\|^2I. \tag{11}$$

Suppose $n=2q$ and take an orthonormal frame such that

$$E_{i^*}=JE_i, \quad i=1, 2, \cdots, q, \quad i^*=q+1, \cdots, n.$$

It follows from (11) and since M is minimal that, for a suitable choice of $\{E_i\}$, we can assume

$$A_{n+1}=\begin{pmatrix} \lambda & & & & & \\ & \ddots & & & 0 & \\ & & \lambda & & & \\ & & & -\lambda & & \\ & 0 & & & \ddots & \\ & & & & & -\lambda \end{pmatrix} \tag{12}$$

where $\lambda=\dfrac{1}{\sqrt{2n}}\|\sigma\|$.

If $\|\sigma\|=0$, then M is totally geodesic in $\tilde{M}$. From now on we assume therefore $\|\sigma\|\neq 0$.

It is not difficult to see that (12) implies

$$\nabla'\sigma=0 \tag{13}$$

(see for example [7]). On the other hand, since $\|\sigma\|$ is constant, it follows from Lemma 1

$$\|\nabla'\sigma\|^2=\|\sigma\|^2\left\{\frac{n+4}{2n}\|\sigma\|^2-(\mu-\alpha+n\nu)\right\}.$$

This, together with (13) implies that $\dfrac{n+4}{2n}\|\sigma\|^2=\mu-\alpha+n\nu$ holds only if $\mu-\alpha+n\nu>0$ and this proves the theorem.

REMARK The theorem is also valid for $n=2$ if the scalar curvature ρ is constant.

5. In [9] we proved the following

LEMMA 3 *Let M be a σ-submanifold of a generalized complex space form $\tilde{M}(\mu,\alpha)$. If the normal connection is trivial then*

(i) *$\mu\leqslant\alpha$ and equality holds if and only if M is totally geodesic;*

(ii) *M is an Einstein manifold and the scalar curvature is constant.*

The proof is based on the fact that if N is normal vector, then

$$g(A_NX,A_NY)=-\frac{1}{4}(\mu-\alpha)\,g(X,Y). \tag{14}$$

From this lemma we obtain

THEOREM 4 *Let M be an n-dimensional σ-hypersurface of a generalized complex space form $\tilde{M}(\mu,\alpha)$ with $\alpha\leqslant 0$. Then, the following statements are equivalent*

(i) *the normal connection is trivial;*

(ii) *$\mu=\alpha$ and M is totally geodesic in $\tilde{M}$.*

PROOF From (2) and (5) it is clear that (ii) implies (i).

Therefore suppose that the normal connection is flat. Hence by Lemma 3 and Theorem 2 we have only to consider the case

$$\frac{n+4}{2n}\|\sigma\|^2=\mu-\alpha+n\nu>0\,. \tag{15}$$

It follows from (11) and (14)

$$\|\sigma\|^2=-\frac{n}{2}(\mu-\alpha)$$

and with (15) this implies

$$(n+4)(\mu-\alpha)+2n\alpha=0\,.$$

So α is positive and this gives a contradiction. Hence M is totally geodesic in $\widetilde{M}$.

REMARK If $\alpha>0$, then the theorem is true if $\mu=\alpha$ but the case $\mu\neq\alpha$ is unsolved.

6. Let $\widetilde{M}(\mu, \alpha)$ be a generalized complex space form and M a σ-submanifold with real dimension >2. It follows easily from the definition, from (5) and (7) that M is an RK-manifold with constant type α (see also [9]).

On the other hand, a Kähler manifold ($\nabla J=0$) has always vanishing constant type. Hence, we obtain the following

THEOREM 5 *Let M^n $(n>2)$ be a σ-submanifold of a generalized complex space form $\widetilde{M}(\mu, \alpha)$ with $\alpha>0$. Then, M cannot be a Kähler manifold with respect to the induced almost complex structure.*

A nearly Kähler manifold $\widetilde{M}$ [4] is an almost Hermitian manifold such that for all $X\in\mathscr{X}(\widetilde{M})$

$$\widetilde{\nabla}_X(J)X=0\,.$$

We proved in [9] that any holomorphic submanifold of a nearly Kähler manifold is a σ-submanifold. Further, a nearly Kähler manifold with constant holomorphic sectional curvature which is not a Kähler manifold is locally isometric to S^6 [5]. S^6 is a generalized complex space form with $\mu=\alpha>0$ and A. Gray proved in [3] that a 4-dimensional holomorphic submanifold of S^6 is Kähler. Hence we have

COROLLARY 6 *S^6 has no 4-dimensional holomorphic submanifolds.*

This corollary is also proved in a different way by A. Gray [3]. Note that we consider S^6 with the usual nearly Kähler structures.

Bibliography

[1] S. S. CHERN: *Einstein hypersurfaces in a Kählerian manifold of constant holomorphic curvature*, J. of Diff. Geom., 1 (1967), 21–31.

[2] S. S. CHERN, M. P. DO CARMO and S. KOBAYASHI: *Minimal submanifolds of a sphere with second fundamental form of constant length*, Functional Analysis and Related Fields, Springer, 1970, 59–75.

[3] A. GRAY: *Almost complex submanifolds of the six sphere*, Proc. Amer. Math. Soc., 20 (1969), 277–279.

[4] A. GRAY: *Nearly Kähler manifolds*, J. of Diff. Geom., 4 (1970), 283–309.

[5] A. GRAY and L. VANHECKE: *Almost Hermitian manifolds with constant holomorphic sectional curvature* to appear.

[6] A. GRAY: *Curvature identities for Hermitian and almost Hermitian manifolds*, to appear.

[7] K. OGIUE: *Differential geometry of Kaehler submanifolds*, Advances in Math., 13 (1974), 73–114.

[8] L. VANHECKE: *Almost Hermitian manifolds with J-invariant Riemann curvature tensor*, to appear in Rend. Sem. Matem. Torino.

[9] L. VANHECKE: *On immsersions with trivial normal connection in some almost Hermitian manifolds*, to appear in Renol. Moctem. Roma.

[10] L. VANHECKE: *Submanifolds of almost Hermitian manifolds and nomal connection*, to appear in Rivista Matem. Univ. Parma.

[11] L. VANHECKE: *Antiholomorphic submanifolds of generalized complex space forms*, Bull. Inst. Math., Acad. Sinica, Taiwan, 4 (1976), 127–140.

Department of Mathematics
Katholieke Universiteit Leuven
Celestijnenlaan 200 B
B-3030 HEVERLEE (Belgium)