Hokkaido Mathematical Journal Vol. 6 (1977) p. 313-344

Construction of a parametrix for the Cauchy problem of some weakly hyperbolic equation I.

By Atsushi YOSHIKAWA

(Received September 20, 1976)

§ 0. Introduction

Consider the partial differential operator

(0.0) $$P=D_t^2-t^2\sum_{j=1}^{n}D_j^2+a(t, x)D_t+\sum_{j=1}^{n}b_j(t, x)D_j+c(t, x)$$

in $\boldsymbol{R}^{n+1}$. Here a, $b_1, \cdots, b_n$, c are C^{∞} functions of $(t, x)=(t, x_1, \cdots, x_n)\in\boldsymbol{R}\times\boldsymbol{R}^n$, and

$$D_t=-i\partial/\partial t, \quad D_j=-i\partial/\partial x_j, \quad (j=1, \cdots, n)$$

$i^2=-1$ as usual. We are going to construct a parametrix for the Cauchy problem associated to the operator P:

(0.1) $$(Pu)(t, x)=0, \ t>0, \ x\in\boldsymbol{R}^n,$$

(0.2) $$u(0, x)=f(x), \ D_tu(0, x)=g(x), \ x\in\boldsymbol{R}^n,$$

f, g being distributions in $\mathcal{E}'(\boldsymbol{R}^n)$.

For simplicity, we shall assume that

(0.3) $$|\mathrm{Im}\sum_{j=1}^{n}b_j(0, x)\xi_j| \ \textit{be uniformly bounded}$$

for all $x\in\boldsymbol{R}^n$ and $\xi=(\xi_1, \cdots, \xi_n)$ on the unit sphere $\boldsymbol{S}^{n-1}$.

Let

(0.4) $$m(\sigma)=-\frac{1}{4}+\frac{1}{4}\sup\left\{\sigma\,\mathrm{Im}\sum_{j=1}^{n}b_j(0, x)\xi_j\right\}, \quad \sigma^2=1,$$

the supremum being taken over $(x, \xi)\in\boldsymbol{R}^n\times\mathrm{S}^{n-1}$.

We then have the following

THEOREM. *There exist symbols*

(0.5) $$\begin{cases} p_{\sigma}(t, x, \xi)\in S^{m(\sigma)+\varepsilon, 2m(\sigma)+2\varepsilon}, \\ \tilde{p}_{\sigma}(t, x, \xi)\in S^{m(\sigma)+\varepsilon}, \\ q_{\sigma}(t, x, \xi)\in S^{m(\sigma)+\varepsilon-1/2, 2m(\sigma)+2\varepsilon}, \\ \tilde{q}_{\sigma}(t, x, \xi)\in S^{m(\sigma)+\varepsilon-1/2}, \ \sigma^2=1, \ (t, x, \xi)\in\overline{\boldsymbol{R}}_+\times\boldsymbol{R}^n\times\boldsymbol{R}^n \end{cases}$$

such that

(0.6)
$$E(f, g)(t, x) =$$
$$= \sum_{\sigma=\pm 1}(2\pi)^{-n}\iint_{R^n\times R^n} e^{i\{\langle x-y,\xi\rangle+\sigma t^2|\xi|/2\}}\, p_\sigma(t, x, \xi) f(y)\, dy\, d\xi$$
$$+ \sum_{\sigma=\pm 1}(2\pi)^{-n}\iint_{R^n\times R^n} e^{i\{\langle x-y,\xi\rangle+\sigma t^2|\xi|/2\}}\, \tilde{p}_\sigma(t, x, \xi) f(y) dy\, d\xi$$
$$+ \sum_{\sigma=\pm 1}(2\pi)^{-n}\iint_{R^n\times R^n} e^{i\{\langle x-y,\xi\rangle+\sigma t^2|\xi|/2\}}\, q_\sigma(t, x, \xi)\, g(y) dy\, d\xi$$
$$+ \sum_{\sigma=\pm 1}(2\pi)^{-n}\iint_{R^n\times R^n} e^{i\{\langle x-y,\xi\rangle+\sigma t^2|\xi|/2\}}\, \tilde{q}_\sigma(t, x, \xi)\, g(y)\, dy\, d\xi$$

gives a parametrix for the Cauchy problem (0.1) (0.2) *in* $\boldsymbol{R}_+\times\boldsymbol{R}^n$. *Namely, for* $u=E(f, g)$, (0.1) *holds in* $\boldsymbol{R}_+\times\boldsymbol{R}^n$ *modulo* $C^\infty(\overline{\boldsymbol{R}}_+\times\boldsymbol{R}^n)$ *and* (0.2) *on* $\boldsymbol{R}^n$ *modulo* $C^\infty(\boldsymbol{R}^n)$. *The integrals in* (0.6) *are taken as oscillatory ones of Hörmander* [4]. *In* (0.5), ε *is an arbitrary positive number and can be dropped if* $n=1$ *and* $b_1(0, x)$ *is independent of* x.

In the above statement, S^m and $S^{m,\kappa}$ for real m, κ are respectively a variant of the symbol class of Hörmander $S^m_{1,0}$, e.g., [4], and one of the class of Boutet de Monvel [1]. They will be difined by Definitions 3.7 and 3.9 below.

Note that $\sum_{j=1}^n b_j(0, x)\xi_j$ coincides with the value of the subprincipal symbol of the operator P at the intersection of the surface $t=0$ and the characterisitic variety of P. As will be clear from the discussions below (in particular, proofs of Propositions 3.3 and 3.5), the requirement (0.3) is inessential. We only need to restrict supports of f and g and localize $m(\sigma)$ accordingly.

We remark that the quantity $m(\sigma)$ appears in the explicit computation of Chi [2] for the simplest case. Related discussions are also found as the index of well-posedness in Ivrii and Petkov [7].

For weakly hyperbolic operators, the investigations of the necessary conditions of well-posedness have progressed much in recent years. See in particular the expository article of Ivrii and Petkov [7] (also [5]). As to the sufficient conditions, several authors, especially Oleinik [9], Ivrii [6], Menikoff [8], obtained a priori estimates for classes of operators including the operator P treated here. However, our construction allows to derive quite explicit informations of general nature concerning the solutions. In this respect, our method may provide some insight though it has yet many shortcomings for the time being.

The main step in the proof of Theorem is the construction of "taming" terms $p_\sigma(t, x, \xi)$ and $q_\sigma(t, x, \xi)$. The initial condition is absorbed in these terms. Here the asymptotic behaviors of the solutions of certain second order ordinary differential equations play important rôles. These are explained in §§1 to 3. The operator being tamed, the remaining constructions of $\tilde{p}_\sigma(t, x, \xi)$ and $\tilde{q}_\sigma(t, x, \xi)$ are much simpler. These constructions will be done in §4. Thus our program will be:

1. Heuritsics;
2. Asymptotic properties of the solutions of the equation:
$$(D_t^2+B-t^2)\,u(t) = e^{\pm it^2/2}h(t), \quad (B\in \boldsymbol{C});$$
3. Construction of the taming terms;
4. Completion of the proof of Theorem.

The method used here is also applicable to the inhomogeneous case and to a more general class of operators. These extensions are actually done in our articles scheduled to appear in the next issue.

§1. Heuristics.

Let

$$(1.1)\qquad E_0(f)(t, x) = \\ = \sum_{\sigma=\pm 1}(2\pi)^{-n}\iint e^{i\{\langle x-y,\xi\rangle+\sigma t^2|\xi|/2\}}p_\sigma(t, x, \xi)f(y)\,dyd\xi$$

and

$$(1.2)\qquad E_1(g)(t, x) = \\ = \sum_{\sigma=\pm 1}(2\pi)^{-n}\iint e^{i\{\langle x-y,\xi\rangle+\sigma t^2|\xi|/2\}}q_\sigma(t, x, \xi)\,g(y)\,dyd\xi\,.$$

We begin by a speculation that E_0 is given formally by the integral:

$$(1.3)\qquad E_0(f)(t, x) = (2\pi)^{-n}\iint e^{i\langle x-y,\xi\rangle}p(t, x, y, \xi)f(y)\,dyd\xi\,.$$

Here $p(t, x, y, \xi)$ is the symbol to be specified.
Now apply the operator P formally to E_0. Then

$$(1.4)\qquad PE_0(f)(t, x) = (2\pi)^{-n}\iint e^{i\langle x-y,\xi\rangle}P^\wedge p(t, x, y, \xi)f(y)\,dyd\xi\,,$$

where

$$(1.5)\qquad P^\wedge = P^\wedge(t, x, \xi;\ D_t, D_x) = \\ = D_t^2-t^2|\xi|^2-2t^2\sum_{j=1}^n\xi_jD_j-t^2\sum_{j=1}^nD_j^2+a(t, x)\,D_t \\ +\sum_{j=1}^nb_j(t, x)\,\xi_j+\sum_{j=1}^nb_j(t, x)\,D_j+c(t, x)\,.$$

Assume that $p(t, x, y, \xi)$ be given by a formal sum:

(1.6) $$p(t, x, y, \xi) = \sum_{k=0}^{\infty} p_k(t, x, y, \xi)$$

Here, for each k, $p_k(t, x, y, \xi)$ is semi-homogeneous of degree $-k/2$, that is,

$$p_k(\lambda^{-1/2}t, x, y, \lambda\xi) = \lambda^{-k/2} p_k(t, x, y, \xi)$$

for $\lambda > 0$.

On the other hand, we expand $a(t, x)$, $b_j(t, x)$ and $c(t, x)$ in Taylor series so that

$$a(t, x) = \sum_{k=0}^{\infty} a_k(x)\, t^k,$$
$$B(t, x, \xi) = \sum_{j=1}^{n} b_j(t, x)\, \xi_j = \sum_{k=0}^{\infty} B_k(x, \xi)\, t^k,$$
$$B(t, x, D_x) = \sum_{j=1}^{n} b_j(t, x)\, D_j = \sum_{k=0}^{\infty} B_k(x, D_x)\, t^k,$$
$$c(t, x) = \sum_{k=0}^{\infty} c_k(x)\, t^k.$$

Then we obtain the following expansion of $P^\frown$ respecting semihomogeneity:

(1.7) $$P^\frown = \sum_{j=0}^{\infty} P_j^\frown,$$

each $P_j^\frown$ being semi-homogeneous of degree $-j/2+1$. More precisely,

(1.8) $$\begin{cases} P_0^\frown = D_t^2 - t^2|\xi|^2 + B_0(x, \xi), \\ P_1^\frown = a_0(x) D_t + t B_1(x, \xi), \\ P_2^\frown = t a_1(x) D_t + t^2 B_2(x, \xi) - 2t^2 \sum_{j=1}^{n} \xi_j D_j + B_0(x, D) + c_0(x), \\ P_3^\frown = t^2 a_2(x) D_t + t^3 B_3(x, \xi) + t B_1(x, D) + t c_1(x), \\ P_4^\frown = -t^2 \sum_{j=1}^{n} D_j^2 + t^3 a_3(x) D_t + t^4 B_4(x, \xi) + t^2 B_2(x, D) + t^2 c_2(x), \end{cases}$$

and

(1.9) $$P_j^\frown = t^{j-1} a_{j-1}(x) D_t + t^j B_j(x, \xi) + t^{j-2} B_{j-2}(x, D) + t^{j-2} c_{j-2}(x)$$

for $j \geqq 5$.

Therefore,

(1.10) $$P^\frown p(t, x, y, \xi) = \sum_{l=0}^{\infty} \sum_{j+j=l} P_k^\frown p_j(t, x, y, \xi)$$

gives an expansion of $P^\frown p(t, x, y, \xi)$ respecting semi-homogeneity, $\sum_{k+j=l} P_k^\frown p_j$ being semi-homogeneous of degree $1-l/2$.

Hence, by solving the equations

(1.11) $$\sum_{k+j=l} P_k^\frown p_j(t, x, y, \xi) = 0, \quad l = 0, 1, 2, \cdots.$$

successively, we can determine each $p_k(t, x, y, \xi)$. The initial conditions to be posed should be

(1.12) $$p_0(0, x, y, \xi) = 1, \quad D_t p_0(0, x, y, \xi) = 0$$

and

(1.13) $$p_j(0, x, y, \xi) = 0, \quad D_t p_j(0, x, y, \xi) = 0$$

for $j \geqq 1$.

We proceed to $E_1(g)$ in a similar manner. Set

(1.14) $$E_1(g)(t, x) = (2\pi)^{-n} \iint e^{i\langle x-y, \xi\rangle} q(t, x, y, \xi) g(y) dy d\xi,$$

where

(1.15) $$q(t, x, y, \xi) = \sum_{k=0}^{\infty} q_k(t, x, y, \xi),$$

each q_k being semi-homogeneous of degree $-k/2-1/2$. Then by the equations

(1.16) $$\sum_{j+k=l} P_j^{\wedge} q_k(t, x, y, \xi) = 0, \quad l = 0, 1, \cdots,$$

with the initial conditions

(1.17) $$q_0(0, x, y, \xi) = 0, \quad D_t q_0(0, x, y, \xi) = 1$$

and

(1.18) $$q_j(0, x, y, \xi) = 0, \quad D_t q_j(0, x, y, \xi) = 0$$

for $j \geqq 1$, we can successively determine every $q_k(t, x, y, \xi)$.

We shall show in § 3 that $p_k(t, x, y, \xi)$ and $q_k(t, x, y, \xi)$ can be expressed in the forms

(1.19) $$p_k(t, x, y, \xi) = \sum_{\sigma=\pm 1} e^{i\sigma t^2|\xi|/2} p_{k\sigma}(t, x, \xi)$$

and

(1.20) $$q_k(t, x, y, \xi) = \sum_{\sigma=\pm 1} e^{i\sigma t^2|\xi|/2} q_{k\sigma}(t, x, \xi).$$

Here $p_{k\sigma}(t, x, \xi)$ and $q_{k\sigma}(t, x, \xi)$ are respectively semi-homogeneous of degree $-k/2$ and $-k/2-1/2$. Furthermore they satisfy appropriate estimates by the results of § 2 (cf. Proposition 3.5). This allows us to find symbols $p_\sigma(t, x, \xi)$ and $q_\sigma(t, x, \xi)$ with

(1.21) $$p_\sigma(t, x, \xi) \sim \sum_{k=0}^{\infty} p_{k\sigma}(t, x, \xi)$$

and

(1.22) $$q_\sigma(t, x, \xi) \sim \sum_{k=0}^{\infty} q_{k\sigma}(t, x, \xi)$$

in a well-defined manner (cf. Corollary 3.14).

However, the equations corresponding to (1.11) and (1.16) are not exactly satisfied. Instead, we have

(1.23) $$P^{\wedge} \sum_{\sigma=\pm 1} e^{i\sigma t^2|\xi|/2} p_\sigma(t, x, \xi) = \sum_{\sigma=\pm 1} e^{i\sigma t^2|\xi|/2} \tilde{r}_\sigma(t, x, \xi)$$

and

$$P^{\wedge}\sum_{\sigma=\pm1} e^{i\sigma t^2|\xi|/2} q_\sigma(t, x, \xi) = \sum_{\sigma=\pm1} e^{i\sigma t^2|\xi|/2} \tilde{s}_\sigma(t, x, \xi). \tag{1.24}$$

Here $\tilde{r}_\sigma(t, x, \xi)$ and $\tilde{s}_\sigma(t, x, \xi)$ are, though not smoothing, essentially flat at $t=0$ (cf. Proposition 3.15). In this sense, we call $p_\sigma(t, x, \xi)$ and $q_\sigma(t, x, \xi)$ taming terms. In fact, as will be shown in § 4, this flatness allows us to solve ordinary transport equations determined by the operator P. We can thus determine the remaining $\tilde{p}_\sigma(t, x, \xi)$ and $\tilde{q}_\sigma(t, x, \xi)$.

§ 2. Asymptotic properties of the solutions of the equation $(D_t^2+B-t^2)\,u(t)=e^{\pm it^2/2}h(t)$, $(B\in \boldsymbol{C})$.

We first determine a fundamental pair of the solutions of the equation

$$(D_t^2+B-t^2)\,u(t) = 0, \quad B\in \boldsymbol{C}. \tag{2.1}$$

This is essentially equivalent to determine a fundamental system of the solutions of the equation

$$(D_t^2+2\sigma tD_t-4i\sigma\mu_\sigma)\,v(t) = 0, \tag{2.2}$$

where

$$\mu = \frac{1}{4}+\frac{\sigma Bi}{4}, \quad \sigma^2=1; \tag{2.3}$$

for we have

$$(D_t^2+B-t^2)\,\{e^{i\sigma t^2/2}w(t)\} = e^{i\sigma t^2/2}(D_t^2+2\sigma tD_t-4i\sigma\mu_\sigma)\,w(t). \tag{2.4}$$

PROPOSITION 2.1. *A fundamental system of the solutions of the equation* (2.1) *is given by*

$$\phi^1(t) = \sum_{\sigma=\pm1} C_\sigma^1 e^{\sigma it^2/2}\psi_\sigma(t, B), \tag{2.5}$$

$$\phi^2(t) = \sum_{\sigma=\pm1} C_\sigma^2 e^{\sigma it^2/2}\psi_\sigma(t, B); \tag{2.6}$$

here

$$C_\sigma^1 = \frac{\Gamma(1/2)}{\Gamma(\mu_{-\sigma})}\, e^{-\frac{\sigma\pi i}{2}\left(\frac{1}{2}-\mu_{-\sigma}\right)}, \tag{2.7}$$

$$C_\sigma^2 = \frac{\Gamma(3/2)}{\Gamma(1-\mu_\sigma)}\, e^{-\frac{\sigma\pi i}{2}\left(\frac{1}{2}+\mu_\sigma\right)}, \tag{2.8}$$

$$\psi_\sigma(t, B) = e^{-\frac{\sigma\pi i}{2}\mu_\sigma}\,\Psi\left(\mu_\sigma, \frac{1}{2};\; -i\sigma t^2\right), \tag{2.9}$$

$\sigma^2=1$; *for complex* a, c, $\Psi(a, c; z)$ *is a confluent hypergeometric function*:

(2.10) $\Psi(a, c; z)$

$$= \frac{\Gamma(1-c)}{\Gamma(a-c+1)}\Phi(a, c; z) + \frac{\Gamma(c-1)}{\Gamma(a)} z^{1-c}\Phi(a-c+1,\ 2-c;\ -z),$$

$|\arg z| < 3\pi/2$, z^{1-c} *being the principal branch with* Φ *Humbert's symbol*

$$\Phi(a, c; z) = {}_1F_1(a, c; z).$$

PROOF: We determine $\phi^1(t)$ and $\phi^2(t)$ by requiring

$$\phi^1(0) = 1, \qquad \phi^{1\prime}(0) = 0; \tag{2.11}$$

$$\phi^2(0) = 0, \qquad \phi^{2\prime}(0) = 1. \tag{2.12}$$

Then by (2.2) and (2.4), $\psi^j(t) = e^{-it^2/2}\phi^j(t)$, $j=1, 2$, form a fundamental system of the solutions of (2.2) for $\sigma=1$. By (2.11) and (2.2), we see that $\psi^1(t)$ is an even function of t on the real axis. Thus, letting $\psi^1(t) = U(t^2)$, we see that $U(0)=1$, and

$$-4t^2U''(t^2) - (4t^2i+2)\,U'(t^2) - 4i\mu_1U(t^2) = 0.$$

Therefore, if

$$\begin{cases} -4zU''(z) - (4zi+2)U'(z) - 4i\mu_1U(z) = 0, \\ U(0)=1, \end{cases} \tag{2.13}$$

then $\psi^1(t) = U(t^2)$. Let $U(z) = \tilde{U}(-iz)$. Then, from (2.13), we have

$$-iz\tilde{U}''(-iz) + \left(\frac{1}{2} + iz\right)\tilde{U}'(-iz) - \mu_1\tilde{U}(-iz) = 0.$$

In particular, if $\tilde{U}(z)$ satisfies

$$\begin{cases} z\tilde{U}''(z) + \left(\frac{1}{2} - z\right)\tilde{U}'(z) - \mu_1\tilde{U}(z) = 0, \\ \tilde{U}(0) = 1 \end{cases} \tag{2.14}$$

then

$$\psi^1(t) = \tilde{U}(-it^2). \tag{2.15}$$

As is well-known (e.g., Erdélyi et al. [3]), the solution of (2.14) is given by

$$\tilde{U}(z) = \Phi\left(\mu_1,\ \frac{1}{2};\ z\right). \tag{2.16}$$

In a similar way, since $\psi^2(t)$ is an odd function of t on the real axis, we have

$$\psi^2(t) = t\tilde{V}(-it^2) \tag{2.17}$$

with

(2.18)
$$\begin{cases} z\tilde{V}''(z)+\left(\frac{3}{2}-z\right)\tilde{V}'(z)-\left(\mu_1+\frac{1}{2}\right)\tilde{V}(z)=0\,, \\ \tilde{V}(0)=1\,, \end{cases}$$

or

(2.19)
$$\tilde{V}(z)=\Phi\left(\mu_1+\frac{1}{2},\ \frac{3}{2};\ z\right).$$

Note that $\Phi(a, c;\ z)$ is an entire function of z. Now we recall the following well-known relation (e. g., [3]):

(2.20) $\Phi(a, c;\ z)$
$$=\frac{\Gamma(c)}{\Gamma(c-a)}e^{i\varepsilon a\pi}\Psi(a, c;\ z)+\frac{\Gamma(c)}{\Gamma(a)}e^{i\varepsilon(a-c)\pi}e^{z}\Psi(c-a, c;\ -z)\,,$$

a, c being complex numbers and $\varepsilon=\operatorname{sgn}\operatorname{Im} z$. Therefore, for real t,

(2.21)
$$\varphi^1(t)=\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma(\mu_{-1})}e^{-\pi i\mu_1}\Psi\left(\mu_1,\ \frac{1}{2};\ -it^2\right)$$
$$+\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma(\mu_1)}e^{\pi i\mu_{-1}}e^{-it^2}\Psi\left(\mu_{-1},\ \frac{1}{2};\ it^2\right),$$

and

(2.22)
$$\varphi^2(t)=\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma(1-\mu_1)}e^{-\pi i(\mu_1+\frac{1}{2})}t\Psi\left(\mu_1+\frac{1}{2},\ \frac{3}{2};\ -it^2\right)$$
$$+\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma(1-\mu_{-1})}e^{\pi i(\mu_{-1}+\frac{1}{2})}e^{-it^2}t\Psi\left(\mu_{-1}+\frac{1}{2},\ \frac{3}{2};\ it^2\right).$$

Meanwhile, from (2.10), we have, for complex a,

$$\Psi\left(a,\ \frac{1}{2};\ -i\sigma t^2\right)$$
$$=\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(a+\frac{1}{2}\right)}\Phi\left(a,\ \frac{1}{2};\ -i\sigma t^2\right)+\frac{\Gamma\left(-\frac{1}{2}\right)}{\Gamma(a)}(-i\sigma t^2)^{\frac{1}{2}}\,\Phi\left(a+\frac{1}{2},\ \frac{3}{2};\ -i\sigma t^2\right)$$

with $|\arg(-i\sigma t^2)|<3\pi/2$. We take an entire branch

$$(-i\sigma t^2)^{\frac{1}{2}}=e^{-\frac{\pi i\sigma}{4}}t$$

and see

$$(2.23)\quad \Psi\left(a, \frac{1}{2}; -i\sigma t^2\right)$$
$$= \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(a+\frac{1}{2}\right)} \Phi\left(a, \frac{1}{2}; -i\sigma t^2\right) + \frac{\Gamma\left(-\frac{1}{2}\right)}{\Gamma(a)} e^{-\frac{\pi i\sigma}{4}} t\Phi\left(a+\frac{1}{2}, \frac{3}{2}; -i\sigma t^2\right)$$

is an entire funcition of t by the right hand side. Similarly,

$$(2.24)\quad t\Psi\left(a+\frac{1}{2}, \frac{3}{2}; -i\sigma t^2\right)$$
$$= \frac{\Gamma\left(-\frac{1}{2}\right)}{\Gamma(a)} t\Phi\left(a+\frac{1}{2}, \frac{3}{2}; -i\sigma t^2\right) + \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(a+\frac{1}{2}\right)} e^{\frac{\pi i\sigma}{4}} \Phi\left(a, \frac{1}{2}; -i\sigma t^2\right)$$

is an entire function of t by the right hand side. From (2.23) and (2.24), we deduce

$$(2.25)\quad \Psi\left(a, \frac{1}{2}; -i\sigma t^2\right) = e^{-\frac{\pi i\sigma}{4}} t\Psi\left(a+\frac{1}{2}, \frac{3}{2}; -i\sigma t^2\right).$$

Of course, this is a particular case of the well-known relation (e.g., [3]):

$$\Psi(a, c; z) = z^{1-c}\Psi(a-c+1, 2-c; z).$$

Rewriting (2.21) and (2.22) by using (2.25), we get

$$(2.26)\quad \phi^1(t) = e^{it^2/2}\psi^1(t)$$
$$= \sum_{\sigma=\pm1} e^{\sigma it^2/2} \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma(\mu_{-\sigma})} e^{-\sigma\pi i\mu_\sigma}\Psi\left(\mu_\sigma, \frac{1}{2}; -i\sigma t^2\right)$$

and

$$(2.27)\quad \phi^2(t) = e^{it^2/2}\psi^2(t)$$
$$= \sum_{\sigma=\pm1} e^{\sigma it^2/2} \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma(1-\mu_\sigma)} e^{-\sigma\pi i(\frac{1}{4}+\mu_\sigma)}\Psi\left(\mu_\sigma, \frac{1}{2}; -\sigma t^2\right).$$

Then by (2.7), (2.8), (2.9) and (2.26), (2.27), we have (2.5) and (2.6). This completes the proof of the proposition.

COROLLARY 2.2. *Let*

$$\tilde{\phi}_\sigma(t) = e^{i\sigma t^2/2}\psi_\sigma(t, B), \quad \sigma^2 = 1.$$

Then

(2.28) $$\tilde{\phi}_\sigma(t) = -2i\sigma C^2_{-\sigma}\phi^1(t) + 2i\sigma C^1_{-\sigma}\phi^2(t),$$

$\sigma^2=1$, *and they form a fundamental pair of the solutions of the equation* (2.1).

PROOF. By (2.7) and (2.8),

(2.29) $$C^1_1 C^2_{-1} - C^1_{-1} C^2_1 = \frac{1}{2}\left(e^{\pi i\mu_{-1}}\sin\pi\mu_{-1} - e^{-\pi i\mu_1}\sin\pi\mu_1\right) = \frac{i}{2}.$$

COROLLARY 2.3.

(2.30) $$\begin{aligned}\psi_\sigma(-t, B) &= e^{-2\sigma\pi i\mu_\sigma}\psi_\sigma(t, B)\\ &+ \frac{2\pi}{\Gamma(\mu_\sigma)\Gamma(1-\mu_{-\sigma})}e^{-\sigma\pi i(\mu_\sigma-\frac{1}{4})}e^{-i\sigma t^2}\psi_{-\sigma}(t, B), \quad \sigma^2 = 1.\end{aligned}$$

PROOF. Since the equation (2.1) is invariant under a rotation by an integral multiple of π, we see that $\tilde{\phi}_\sigma(-t)$, $\sigma^2=1$, satisfy the equation (2.1). Thus from Corollary 2.2, we have

$$\tilde{\phi}_\sigma(-t) = \sum_{\sigma'=\pm1} A^\sigma_{\sigma'}\,\tilde{\phi}_\sigma(t).$$

Then by (2.29), (2.7), (2.8), we have

$$A^\sigma_\sigma = \frac{C^1_\sigma C^2_{-\sigma} + C^1_{-\sigma}C^2_\sigma}{C^1_\sigma C^2_{-\sigma} - C^1_{-\sigma}C^2_\sigma} = e^{-2\sigma\pi i\mu_\sigma},$$

$$A^\sigma_{-\sigma} = \frac{2C^1_{-\sigma}C^2_{-\sigma}}{C^1_\sigma C^2_{-\sigma} - C^1_{-\sigma}C^2_\sigma} = \frac{2\pi e^{-\sigma\pi i(\mu_\sigma-\frac{1}{4})}}{\Gamma(\mu_\sigma)\Gamma(1-\mu_{-\sigma})}.$$

COROLLARY 2.4. *Let* $\phi^1(t,\tau)$, $\phi^2(t,\tau)$ *be solutions of the equation* (2.1) *with*

(2.31) $$\phi^1(t,\tau) = 1, \quad \phi^{1\prime}(t,\tau) = 0 \text{ when } t = \tau,$$

(2.32) $$\phi^2(t,\tau) = 0, \quad \phi^{2\prime}(t,\tau) = 1 \text{ when } t = \tau.$$

Then

(2.33) $$\phi^1(t,\tau) = -\sum_{\sigma=\pm1}\frac{\sigma i}{2}\left\{i\sigma\tau\psi_{-\sigma}(\tau, B) - \psi'_{-\sigma}(\tau, B)\right\}e^{i\sigma(t^2-\tau^2)/2}\psi_\sigma(t, B),$$

(2.34) $$\phi^2(t,\tau) = -\sum_{\sigma=\pm1}\frac{\sigma i}{2}\psi_{-\sigma}(\tau, B)e^{i\sigma(t^2-{}^2)/2}\psi_\sigma(t, B).$$

PROOF. We know that $\phi^j(t, 0)=\phi^j(t)$, $j=1, 2$. Let

(2.35) $$\phi^j(t,\tau) = \sum_{k=1}^2 a^j_k(\tau)\phi^k(t), \quad j = 1, 2.$$

Then we have from (2.31) and (2.32)

$$\phi^1(t, \tau) = \phi^{2\prime}(\tau)\phi^1(t) - \phi^{1\prime}(\tau)\phi^2(t), \tag{2.36}$$

$$\phi^2(t, \tau) = -\phi^2(\tau)\phi^1(\tau) + \phi(\tau)\phi^2(t). \tag{2.37}$$

In particular,

$$\phi^1(\mathrm{t}, \tau) = -\frac{\partial}{\partial \tau}\phi^2(t, \tau). \tag{2.38}$$

Now,

$$\begin{aligned}\phi^2(t, \tau) &= -\left(\sum_{\sigma=\pm 1} C_\sigma^2 e^{i\sigma\tau^2/2}\psi_\sigma(\tau, B)\right)\left(\sum_{\sigma'=\pm 1} C_{\sigma'}^1 e^{i\sigma' t^2/2}\psi_{\sigma'}(t, B)\right)\\ &\quad + \left(\sum_{\sigma=\pm 1} C_\sigma^1 e^{i\sigma\tau^2/2}\psi_\sigma(\tau, B)\right)\left(\sum_{\sigma'=\pm 1} C_{\sigma'}^2 e^{i\sigma' t^2/2}\psi_{\sigma'}(t, B)\right)\\ &= -(C_1^1 C_{-1}^2 - C_{-1}^1 C_1^2)\sum_{\sigma=\pm 1}\sigma e^{i\sigma(t^2-\tau^2)/2}\psi_{-\sigma}(\tau, B)\psi_\sigma(t, B),\end{aligned}$$

which is nothing but (2. 34). From (2. 34) and (2. 38) we get (2. 33).

In the following section, we shall mainly be concerned with the case where B depends on parameters. Taking this into account, we shall assume in the remaining part of this section that B is a C^∞ function on a C^∞ manifold Ω:

$$B \in C^\infty(\Omega). \tag{2.39}$$

For the sake of simplicity, we make a further assumption that

(2. 40) Im B *is uniformly bounded on* Ω;

thus, by (2. 3), so are $\mathrm{Re}\mu_\sigma$, $\sigma^2=1$.

Let Σ be an open sector in $\boldsymbol{C}\setminus\{0\}$ or in some Riemann surface over $\boldsymbol{C}\setminus\{0\}$. It is convenient to introduce the following notations.

DEFINITION 2. 5. For real ν, we denote by $O^\nu(\Sigma, \Omega)$ the space of all functions $f(z, \omega)$, holomorphic in $z \in \Sigma$ and C^∞ in $\omega \in \Omega$, such that for any differential operator Λ on Ω, we have

$$|\Lambda f(z, \omega)| \leqq C(1+|z|)^\nu \tag{2.41}$$

for $z \in \Sigma'$, $|z| \geqq 1$, and $\omega \in K$. Here Σ' is any closed subsector of Σ, K any compact subset of Ω and C a positive constant depending on Σ', K and Λ.

All the functions appearing in the remaining part of this section are smooth on Ω. We generally omit the variables ω of Ω in the expressions of these functions.

Let $f(z) = f(z, \omega)$ be a function which is holomorphic in Σ and C^∞ on Ω, and $f^*(z) = f^*(z, \omega)$ a formal series of the form:

$$f^*(z) = \sum_{j=0}^{J}(\log z)^j z^{\nu_j}\sum_{k=0}^{\infty} f_{jk} z^{-k}, \tag{2.42}$$

ν_j, $f_{jk}\in C^\infty(\Omega)$. If, for any N,

$$f(z)-\sum_{j=0}^{J}(\log z)^j z^{\nu_j}\sum_{k=0}^{N} f_{jk}z^{-k}\in O^{\varepsilon_N}(\Sigma, \Omega) \tag{2.43}$$

with $\lim_{N\to\infty}\varepsilon_N=-\infty$, then we say that $f^*(z)$ is the asymptotic expansion of $f(z)$ in the sector Σ, and write

$$f(z)\sim f^*(z). \tag{2.44}$$

PROPOSITION 2.6. *Let*

$$\phi_\sigma^*(t, B)=t^{-2\mu_\sigma}\sum_{k=0}^{\infty}a_k t^{-2k}, \tag{2.45}$$

with

$$a_k=\frac{(\mu_\sigma)_k\left(\mu_\sigma+\frac{1}{2}\right)_k}{k!}(i\sigma)^{-k}, \quad k=0, 1, 2, \cdots. \tag{2.46}$$

Here

$$(a)_k=\Gamma(a+k)/\Gamma(a), \quad k=0, 1, 2, \cdots.$$

for complex a. Then

$$\phi_\sigma(t, B)\sim\phi_\sigma^*(t, B) \tag{2.47}$$

in the sector

$$S=\left\{t\in \boldsymbol{C}\backslash\{0\}\,;\,|\arg t|<\frac{\pi}{4}\right\}. \tag{2.48}$$

PROOF. Let

$$\Psi^*\left(\mu_\sigma, \frac{1}{2}\,;\,z\right)=z^{-\mu_\sigma}\sum_{k=0}^{\infty}(i\sigma)^{-k}a_k z^{-k}. \tag{2.49}$$

We claim

$$\Psi\left(\mu_\sigma, \frac{1}{2}\,;\,z\right)\sim\Psi^*\left(\mu_\sigma, \frac{1}{2}\,;\,z\right) \tag{2.50}$$

in the sector $\Sigma=\{|\arg z|<3\pi/2\}$. Then since $t\in S$ implies $-i\sigma t^2\in\Sigma$ and

$$\Psi^*\left(\mu_\sigma, \frac{1}{2}\,;\,-i\sigma t^2\right)=(-i\sigma)^{-\mu_\sigma}\phi_\sigma^*(t, B),$$

(2.47) follows from (2.50) and (2.9) in view of

$$(-i\sigma)^{-\mu_\sigma}=(e^{-\sigma\pi i/2})^{-\mu_\sigma}=e^{\sigma\pi i\mu_\sigma/2}.$$

To prove (2.50), we employ the following integral representation for the function Ψ (cf. [3]):

(2.51)
$$\Psi\left(\mu_\sigma, \frac{1}{2}; z\right) = \frac{1}{2\pi i} e^{-\mu_\sigma \pi i} \Gamma(1-\mu_\sigma) \int_{\infty e^{i\theta}}^{(0+)} e^{-z\zeta} \zeta^{\mu_\sigma - 1} (1+\zeta)^{-\mu_\sigma - 1/2} d\zeta.$$

Here $-\pi<\theta<\pi$, $-\frac{1}{2}\pi<\theta+\arg z<\frac{1}{2}\pi$, and the contour of integration is a loop starting (and ending) at $t=\infty e^{i\theta}$ and encircling 0 counter-clockwise. Let

(2.52)
$$\rho_N(\zeta, \omega) = (1+\zeta)^{-\mu_\sigma - 1/2} - \sum_{k=0}^{N} (-1)^k (\mu_\sigma + 1/2)_k \zeta^k / k!$$

and

(2.53)
$$R_N(z, \omega) = \frac{1}{2\pi i} e^{-\mu_\sigma \pi i} \Gamma(1-\mu_\sigma) \int_{\infty e^{i\theta}}^{(0+)} e^{-z\zeta} \zeta^{\mu_\sigma - 1} \rho_N(\zeta, \omega) d\zeta.$$

Substituting (2.52) to (2.51), we have

(2.52)
$$R_N(z, \omega) = \Psi\left(\mu_\sigma, \frac{1}{2}; z\right) - z^{-\mu_\sigma} \sum_{k=0}^{N} (i\sigma)^{-k} a_k z^{-k}.$$

Now $\rho_N(\zeta, \omega) = \zeta^{N+1} \tilde{\rho}^N(\zeta, \omega)$ if

$$\tilde{\rho}_N(\zeta, \omega) = (-1)^{N+1} (\mu_\sigma + 1/2)_{N+1} \int_0^1 (1-s)^N (1+s\zeta)^{-\mu_\sigma - 3/2 - N} ds / N!.$$

Thus, by (2.53),

$$R_N(z, \omega) = \frac{1}{2\pi i} e^{-\mu_\sigma \pi i} \Gamma(1-\mu_\sigma) z^{-\mu_\sigma - N - 1} \times \int_{\infty e^{i(\theta+\arg z)}}^{(0+)} e^{-\zeta} \zeta^{\mu_\sigma + N} \tilde{\rho}_N(z^{-1}\zeta, \omega)\, d\zeta.$$

Hence, it is readily seen, by virtue of (2.40), that

$$R_N(z, \omega) \in O^{\varepsilon_N}(\Sigma, \Omega),$$

where

$$\varepsilon_N = \sup_\Omega(-\mathrm{Re}\,\mu_\sigma) - N - 1 + \varepsilon,$$

ε being an arbitrary positive number.

REMARK 2.7. Let

$$u^*(t) = t^\mu \sum_{k=0}^{\infty} b_k t^{-k}, \quad b_0 = 1.$$

Then a formal substitution of $u^*(t)$ into (2.2) shows $\mu = -2\mu_\sigma$, $b_{2k} = a_k$, $b_{2k+1} = 0$, $k = 0, 1, 2, \cdots$. That is, $\phi_\sigma^*(t, B)$ is a formal solution of (2.2).

PROPOSITION 2. 8. *Let*

(2. 55) $$g(t)=g(t,\omega)\in O^{-N}(s,\Omega)$$

for sufficiently large $N>0$. *Then the equation*

(2. 56) $$(D_t^2+2\sigma tD_t-4i\sigma\mu_\sigma)v(t)=g(t)$$

has a solution

(2. 57) $$v(t)=v(t,\omega)\in O^{-N+\varepsilon}(S,\Omega),$$

ε *being an arbitrary positive number.*

PROOF. By (2. 4) and Proposition 2. 1, $\tilde{\phi}^j(t)=e^{-i\sigma t^2/2}\phi^j(t)$, $j=1,2$, form a fundamental pair of the solutions of the equation (2. 2). Thus

(2. 58) $$v(t)=\int_\infty^t W(\tau)^{-1}\{-\tilde{\phi}^1(t)\tilde{\phi}^2(\tau)+\tilde{\phi}^1(\tau)\tilde{\phi}^2(t)\}\,g(\tau)\,d\tau$$

gives a particular solution of (2. 56). Here

(2. 59) $$W(\tau)=\tilde{\phi}^1(\tau)\tilde{\phi}^{2\prime}(\tau)-\tilde{\phi}^2(\tau)\tilde{\phi}^{1\prime}(\tau)$$

is the wronskian. Note that

$$W(0)=1$$

and

$$W'(\tau)=-2\sigma\tau iW(\tau),$$

whence

(2. 60) $$W(\tau)=e^{-i\sigma\tau^2}$$

On the other hand,

$$\begin{aligned}
&-\tilde{\phi}^1(t)\tilde{\phi}^2(\tau)+\tilde{\phi}^1(\tau)\tilde{\phi}^2(t)\\
&=e^{-i\sigma t^2/2-i\sigma\tau^2/2}\{-\phi^1(t)\phi^2(\tau)+\phi^1(\tau)\phi^2(t)\}\\
&=e^{-i\sigma(t^2+\tau^2)/2}\phi^2(t,\tau)\\
&=-\frac{i}{2}e^{-i\sigma(t^2+\tau^2)/2}\textstyle\sum_{\sigma'=\pm1}\sigma e^{i\sigma'(t^2-\tau^2)/2}\psi_{-\sigma'}(\tau,B)\,\psi_{\sigma'}(t,B).\\
&=-\frac{\sigma i}{2}e^{-i\sigma\tau^2}\psi_{-\sigma}(\tau,B)\psi_\sigma(t,B)+\frac{\sigma i}{2}e^{-i\sigma t^2}\psi_\sigma(\tau,B)\,\psi_{-\sigma}(t,B)
\end{aligned}$$

by (2. 37) and (2. 34).
Hence, by (2. 58) (2. 59) (2. 60), we have

(2. 61) $$\begin{aligned}v(t)=&\frac{-i\sigma}{2}\int_\infty^t\psi_{-\sigma}(\tau,B)\,g(\tau)\,d\tau\psi_\sigma(t,B)\\
&+\frac{i\sigma}{2}\int_\infty^t e^{-i\sigma(t^2-\tau^2)}\psi_\sigma(\tau,B)\,g(\tau)d\tau\psi_{-\sigma}(t,B)=v_1(t)+v_2(t).\end{aligned}$$

Now by Proposition 2.6 and since $\mu_1+\mu_{-1}=\frac{1}{2}$, we see that

$$v_1\in O^{-N+\varepsilon}(S, \Omega), \tag{2.26}$$

ε any positive number. When t lies in any closed subsector of S, we can choose a path of intergration in S in such a way that

$$\operatorname{Im} \sigma t^2 \leqq \operatorname{Im} \sigma\tau^2,$$

that is,

$$\arg t<\arg\tau<\frac{\pi}{4} \text{ for } \sigma=1$$

or

$$-\frac{\pi}{4}<\arg\tau<\arg t \text{ for } \sigma=-1.$$

Thus again by Proposition 2.6, we have

$$v^2\in O^{-N+\varepsilon}(S, \Omega) \tag{2.63}$$

ε any positive number. (2.57) now follows from (2.62) and (2.63).

PROPOSITION 2.9. *Let $h(t)=h(t, \omega)$ be entire analytic in t with the following asymptotic expansion in the sector S*:

$$h(t)\sim t^\nu \textstyle\sum_{k=0}^\infty h_k t^{-k}, \quad h_0\neq 0. \tag{2.64}$$

If $\nu+2\mu_\sigma$ is not a non-negative integer, then the solution of the equation

$$(D_t^2+B-t^2)\, u(t)=e^{-i\sigma t^2/2}h(t) \tag{2.65}$$

has the following asymptotic expansion in the sector S modulo a homogeneous solution:

$$e^{-i\sigma t^2/2}u(t)\sim t^\nu \textstyle\sum_{k=0}^\infty u_k t^{-k}, \tag{2.66}$$

where

$$u_k=\frac{1}{2\sigma i(k-\nu-2\mu_\sigma)}\left\{h_k+(\nu-k+2)(\nu-k+1)u_{k-2}\right\}, \tag{2.67}$$

$$u_{-2}=u_{-1}=0.$$

PROOF. Let

$$h^*(t)=t^\nu \textstyle\sum_{k=0}^\infty h_k t^{-k}$$

and

$$v^*(t)=t^\nu \textstyle\sum_{k=0}^\infty u_k t^{-k}.$$

We first show that $e^{i\sigma t^2/2}v^*(t)$ is a formal solution of

$$(D_t^2+B-t^2)\{e^{i\sigma t^2/2}v^*(t)\}=e^{i\sigma t^2/2}h^*(t),$$

or equivalently

$$(D_t^2+2\sigma tD_t-4i\sigma\mu_\sigma)v^*(t)=h^*(t). \tag{2.68}$$

But then we see that (2.68) is equivalent to

$$-(\nu-k+2)(\nu-k+1)u_{k-2}-2\sigma i(\nu+2\mu_\sigma-k)u_k=h_k,$$
$$u_k=0 \text{ for } k<0.$$

or just (2.67). Now let

$$h(t)=t^\nu\sum_{k=0}^N h_k t^{-k}+h_N(t).$$

Then

$$h_N\in O^{\varepsilon_N}(S,\Omega), \tag{2.69}$$

$\varepsilon_N\to-\infty$ as $N\to\infty$.

Therefore, if we set

$$v(t)=e^{-i\sigma t^2/2}u(t)=t^\nu\sum_{k=0}^N u_k t^{-k}+v_N(t),$$

then

$$(D_t^2+2\sigma tD_t-4i\sigma\mu_\sigma)v_N(t)$$
$$=h_N(t)+(\nu-N+1)(\nu-N)u_{N-1}t^{\nu-N-1}+(\nu-N)(\nu-N-1)u_N t^{\nu-N-2}.$$

Hence, by Proposition 2.8 and (2.69),

$$v_N\in O^{\varepsilon'_N}(S,\Omega),$$

$\varepsilon'_N=\max(\varepsilon_N, \operatorname{Re}\nu-N-1)+\varepsilon$, $\varepsilon>0$.

Just in the same way, we have the following two corollaries.

COROLLARY 2.10. *Let $h(t)$ be as in Poposition 2.9. If $m=\nu+2\mu_\sigma$ is a non-negative integer, then the solution of the equation (2.65) has the following asymptotic expansion in the sector S modulo a homogeneous solution:*

$$e^{-i\sigma t^2/2}u(t)$$
$$\sim t^\nu\sum_{\substack{k=0\\k\neq m}}^{\infty}u_k t^{-k}+u_m\psi_\sigma^*(t,B)\log t; \tag{2.70}$$

here $\psi_\sigma^*(t,B)$ is defined by (2.45) and

$$u_k=0 \ for \ k<0,$$

$$u_k=\frac{1}{2\sigma i(k-m)}\left\{h_k+(\nu-k+2)(\nu-k+1)u_{k-2}\right\} \ for \ k\leqq m-1,$$

$$u_m=-\frac{1}{2\sigma i}\left\{h_m+(\nu-m+2)(\nu-m+1)u_{m-2}\right\},$$

$$u_{j+m}=\frac{1}{2\sigma ij}\left\{h_{m+j}+(1-\delta_{j2})(\nu-m-j+2)(\nu-m-j+1)u_{m+j-2}\right.$$

$$\left.-u_m\left[b_{j-2}+2(\nu-m-j+1)b_{j-1}+2\sigma ib_j\right]\right\}$$

for $j\geqq 1$ *with* $b_{2l}=a_l$ *in* (2.46) *and* $b_{2l+1}=0$, $l=0,1,2,\cdots$, *and* δ_{j2} *the Kronecker symbol.*

PROOF. Let

$$u_0^*(t)=t^\nu\sum_{\substack{k=0\\k\neq m}}^\infty u_kt^{-k},$$

and

$$u^*(t)=u_0^*(t)+u_m\phi_\sigma^*(t,B)\log t$$

Then

(2.71) $$(D_t^2+2\sigma tD_t-4i\sigma\mu_\sigma)u^*(t)$$
$$=(D_t^2+2\sigma tD_t-4i\sigma\mu_\sigma)u_0^*(t)+u_m(-2\sigma i+t^{-2}-2it^{-1}D_t)\phi_\sigma^*(t,B)$$
$$=h^*(t)$$

by the choice of u_k.

COROLLARY 2.11. *Let* $h_l^*(t)$, $l=0,1,\cdots,L$, *be formal series of the form*

(2.72) $$h_l^*(t)=t^{\nu_l}\sum_{k=0}^\infty h_{lk}t^{-k}.$$

Assume that $h(t)$ *be entire analytic in* t *and have the following asymptotic expansion in the sector* S:

(2.73) $$h(t)\sim\sum_{l=0}^L(\log t)^lh_l^*(t).$$

Then the solution $u(t)$ *of the equation* (2.65) *has the following asymptotic expansion in the sector* S *modulo a homogeneous solution*:

(2.74) $$e^{-i\sigma t^2/2}u(t)\sim\sum_{l=0}^L\sum_{j=0}^{l+1}c_{l+1-j}^l(\log t)^j\phi_\sigma^*(t,B)$$
$$+\sum_{l=0}^L\sum_{j=0}^l(\log t)^jt^{\nu_l}\sum_{k=0}^\infty w_{l-j,k}^lt^{-k}$$

$c_{l+1-j}^l=0$ *when* $\nu_l+2\mu_\sigma$ *is not a non-negative integer.*

PROOF. We only need to show that

(2.75) $$(D_t^2+2\sigma tD_t-4i\sigma\mu_\sigma)v^*(t)=\sum_{l=0}^L(\log t)^lh_l^*(t)$$

has the formal solution of the form

$$v^*(t)=\sum_{l=0}^{L}\sum_{j=0}^{l+1}c_{l+1-j}(\log t)^j\phi_\sigma^*(t,B)+\sum_{l=0}^{L}\sum_{j=0}^{l}(\log t)^j t^{\nu_l}\sum_{k=0}^{\infty}w_{l-j,k}^{l}t^{-k}. \tag{2.76}$$

Let

$$\tilde{h}^*(t)=t^{\tilde{\nu}}\sum_{k=0}^{\infty}h_k t^{-k}.$$

We examine the formal solution of

$$(D_t^2+2\sigma tD_t-4i\sigma\mu_\sigma)\tilde{v}^*(t)=(\log t)^l\tilde{h}^*(t). \tag{2.77}$$

Let $\tilde{v}^*(t)=(\log t)^l\tilde{w}^*(t)$. Then

$$\begin{aligned}(D_t^2+2\sigma tD_t-4i\sigma\mu_\sigma)&\left\{(\log t)^l\tilde{w}^*(t)\right\}\\ &=(\log t)^l(D_t^2+2\sigma tD_t-4i\sigma\mu_\sigma)\tilde{w}^*(t)\\ &+(\log t)^{l-1}(-2l\sigma i-2lit^{-1}D_t+lt^{-2})\tilde{w}^*(t)\\ &+(\log t)^{l-2}\left\{-l(l-1)t^{-2}\tilde{w}^*(t)\right\}.\end{aligned} \tag{2.78}$$

Hence, if $\tilde{\nu}+2\mu_\sigma$ is not a non-negative integer, we can solve $\tilde{w}_0{}^*(t)$ from

$$(D_t^2+2\sigma tD_t-4i\sigma\mu_\sigma)\tilde{w}_0^*(t)=\tilde{h}^*(t) \tag{2.79}$$

in the form

$$\tilde{w}_0^*(t)=t^{\tilde{\nu}}\sum_{k=0}^{\infty}\tilde{w}_{0k}t^{-k}. \tag{2.80}$$

Thus, setting $\tilde{v}^*(t)=(\log t)^l\tilde{w}_0^*(t)+\tilde{v}_1^*(t)$, we see from (2.77) (2.78) (2.79) that $\tilde{v}_1^*(t)$ satisfies

$$\begin{aligned}(D_t^2+2\sigma tD_t-4i\sigma\mu_\sigma)\tilde{v}_1^*(t)\\ =(\log t)^{l-1}\tilde{h}_1^*(t)+(\log t)^{l-2}\tilde{h}_2^*(t)\end{aligned} \tag{2.81}$$

with

$$\tilde{h}_1^*(t)=l[-2\sigma i-2it^{-1}D_t+t^{-2}]\tilde{w}_0^*(t)$$

and

$$\tilde{h}_2^*(t)=-l(l-1)t^{-2}\tilde{w}_0^*(t).$$

Therefore, we can solve $\tilde{v}_1^*(t)$ in a similar manner so that we finally obtain the formal solution of (2.77) in the form

$$\tilde{v}^*(t)=\sum_{j=0}^{l}(\log t)^j\tilde{w}_{l-j}^*(t), \tag{2.82}$$

$$\tilde{w}_{l-j}^*(t)=t^{\tilde{\nu}}\sum_{j=0}^{\infty}\tilde{w}_{l-j,k}t^{-k}. \tag{2.83}$$

On the other hand, if $\tilde{\nu}+2\mu_\sigma=\tilde{m}$ is a non-negative integer, we set

(2.84) $$\tilde{w}_0^*(t)=\tilde{w}_{00}^*(t)+\tilde{c}_0\log t\cdot\phi_\sigma^*(t,B)$$

as in (2.71). Here

(2.85) $$\tilde{w}_{00}^*(t)=t^{\tilde{\nu}}\sum_{\substack{k=0\\k\neq\tilde{m}}}^{\infty}\tilde{w}_{00k}t^{-k}$$

Then substituting $\tilde{v}^*(t)=(\log t)^l\tilde{w}_0^*(t)+\tilde{v}_1^*(t)$ in (2.77), we can determine $\tilde{w}_{00}^*$ and $\tilde{c}_0$ so that $\tilde{v}_1^*(t)$ satisfies (2.81) with

$$\tilde{h}_1^*(t)=l\left\{(-2\sigma i-2it^{-1}D_t+t^{-2})\tilde{w}_{00}^*(t)-(l+1)\tilde{c}_0t^{-2}\phi_\sigma^*(t,B)\right\}$$

and

$$\tilde{h}_2^*(t)=-l(l-1)t^{-2}\tilde{w}_{00}^*(t).$$

Thus, we can solve $\tilde{v}_1^*(t)$ in a similar manner so that we finally obtain the formal solution of (2.77) in the form

(2.86) $$\begin{aligned}\tilde{v}^*(t)=&\sum_{j=0}^{l+1}\tilde{c}_{l+1-j}(\log t)^j\phi_\sigma^*(t,B)\\&+\sum_{j=0}^{l}(\log t)^j\tilde{w}_{l-j,0}^*(t),\end{aligned}$$

(2.87) $$\tilde{w}_{l-j,0}^*(t)=t^{\tilde{\nu}}\sum_{\substack{k=0\\k\neq\tilde{m}}}^{\infty}\tilde{w}_{l-j,0,k}t^{-k}.$$

Now (2.82), (2.83). (2.86), (2.87) show that the formal solution of (2.75) is given in the form (2.76).

REMARK 2.12. In order to estimate $p_{k\sigma}(t,x,\xi)$ and $q_{k\sigma}(t,x,\xi)$ of (1.19) and (1.20), we shall use Corollary 2.11 in the case when all $\nu_l+2\mu_\sigma$ are non-negative integers.

REMARK 2.13. Corollary 2.11 shows that roughly the equation

$$(D_t^2+B-t^2)u(t)=e^{\pm it^2/2}O(t^{\nu+\varepsilon}),\quad \varepsilon>0,$$

has a solution

$$u(t)=e^{\pm it^2/2}O(t^{\nu+\varepsilon})+\sum_{\sigma=\pm1}e^{i\sigma t^2/2}O(t^{-2\mu\sigma+\varepsilon})$$

as $t\to\infty$ in the sector S. Here ε reflects possible logarithmic terms in the asymptotic expansions. The second term in the right hand side represents the modification by a homogeneous solution and those additional logarithmic terms appearing in (2.74).

REMARK 2.14 A similar discussion also applies to the equation of the form

(2.88) $$(D_t^2+B-t^2)u(t)=e^{i\sigma(t^2-\tau^2)/2}g(\tau)h(t),$$

$\sigma^2=1$. We first solve $v(t)$ from

$$(D_t^2+2\sigma tD_t-4i\sigma\mu_\sigma)v(t)=h(t)$$

with a good asymptotic property. Then

$$u(t)=e^{i\sigma(t^2-\tau^2)/2}g(\tau)\,v(t)$$

is a particular solution of (2.88). Thus

$$\tilde{u}(t)=e^{i\sigma(t^2-\tau^2)/2}g(\tau)\,v(t)-g(\tau)\,v(\tau)\,\phi^1(t,\tau)$$
$$-g(\tau)\left\{v'(\tau)+i\sigma\tau v(\tau)\right\}\phi^2(t,\tau)$$

is a solution of (2.88) with

$$\tilde{u}(\tau)=0,\quad \tilde{u}'(\tau)=0.$$

Thus, in particular, if $h(t)=O(t^{\nu+\varepsilon})$, $g(\tau)=O(\tau^{\rho+\varepsilon'})$ in the sector S, then

$$\tilde{u}(t)=e^{i\sigma(t^2-\tau^2)/2}O(t^{\nu+\varepsilon})\,O(\tau^{\rho+\varepsilon'})$$
$$+\sum_{\sigma'=\pm1}e^{i\sigma'(t^2-\tau^2)/2}O(\tau^{\rho+\nu+1-2\mu_{-\sigma}+\varepsilon+\varepsilon'})\,O(t^{-2\mu_{\sigma'}})$$

if $\mathrm{Re}\,\nu\geqq-2\mathrm{Re}\,\mu_\sigma$. Here ε and ε' should be interpreted as in Remark 2.13.

§ 3. Construction of the taming terms

We proceed to solve the equations (1.11) and (1.16) with the initial conditions (1.12), (1.13) and (1.17) (1.18). As we have mentioned in § 1, the solutions $p_k(t,x,y,\xi)$ and $q_k(t,x,y,\xi)$ are to be in the forms (1.19) and (1.20).

PROPOSITION 3.1. *Let* $|\xi|\geqq1$. *The initial value problem*:

(3.1) $$(D_t^2+B_0(x,\xi)-t^2|\xi|^2)p_0(t,x,y,\xi)=0,$$

(3.2) $$p_0(0,x,y,\xi)=1,\quad D_tp_0(0,x,y,\xi)=0,$$

is solved by

(3.3) $$p_0(t,x,y,\xi)=\sum_{\sigma=\pm1}e^{i\sigma t^2|\xi|/2}p_{0\sigma}(t,x,\xi);$$

here

(3.4) $$p_{0\sigma}(t,x,\xi)=C_\sigma^1\phi_\sigma\left(|\xi|^{1/2}t,B_0(x,\xi/|\xi|)\right)$$

with C_σ^1, ϕ_σ *as defined by* (2.7) *and* (2.9).

PROOF. By the semi-homogeneity, $p_0(t,x,y,\xi)=p_0(|\xi|^{1/2}t,x,y,\xi/|\xi|)$. Then $p_0(t,x,y,\xi/|\xi|)$ satisfies (2.1) and (2.11) with $B=B_0(x,\xi/|\xi|)$.

In a similar way, we have

PROPOSITION 3.2. *Let* $|\xi|\geqq1$. *The initial value problem*:

(3.5) $$(D_t^2+B_0(x,\xi)-t^2|\xi|^2)q_0(t,x,y,\xi)=0,$$

(3.6) $$q_0(0, x, y, \xi) = 0, \quad D_t q_0(0, x, y, \xi) = 1,$$

is solved by

(3.7) $$q_0(t, x, y, \xi) = \sum_{\sigma=\pm 1} e^{i\sigma t^2|\xi|/2} q_{0\sigma}(t, x, \xi);$$

here

(3.8) $$q_{0\sigma}(t, x, \xi) = i|\xi|^{-1/2} C_t^2 \psi_\sigma\left(|\xi|^{1/2}t, B_0(x, \xi/|\xi|)\right)$$

with C_σ^2 *and* ψ_σ *as defined by* (2.8) *and* (2.9).

By applications of Proposition 2.6 and Corollary 2.11, we have the following estimates.

PROPOSITION 3.3. *Let* K *be any compact subset of* $\boldsymbol{R}^n$. *For any non-negative integer* l, *and non-negative integral multi-indices* $\alpha=(\alpha_1, \cdots, \alpha_n)$, $\beta=(\beta_1, \cdots, \beta_n)$,

(3.9) $$|D_t^l D_x^\alpha D_\xi^\beta p_{0\sigma}(t, x, \xi)| \leqq C(1+|\xi|)^{m(\sigma)+\varepsilon-|\beta|}(|\xi|^{-1}+t^2)^{m(\sigma)+\varepsilon-l/2}$$

and

(3.10) $$|D_t^l D_x^\alpha D_\xi^\beta q_{0\sigma}(t, x, \xi) \leqq C(1+|\xi|)^{m(\sigma)+\varepsilon-1/2-|\beta|}(|\xi|^{-1}+t^2)^{m(\sigma)+\varepsilon-l/2}$$

for $t \geqq 0$, $x \in K$, $|\xi| \geqq 1$. *Here* ε *is an arbitrary positive number and* C *is a positive constant depending on* K, ε, l, α, β; $m(\sigma)$ *is defined by* (0.4).

PROOF. In view of semi-homogeneity, we can apply Proposition 2.6 with $\Omega=\boldsymbol{R}^n \times \boldsymbol{S}^{n-1}$ and $B=B_0(x, \xi/|\xi|)$. Then (3.9) and (3.10) are immediate from (3.4) and (3.8).

REMARK 3.4. If $n=1$ and $B_0(x, \xi)$ is independent of x, then ε in (3.9) and (3.10) can be dropped. The choice of $m(\sigma)$ can be localized (cf. (0.4) and the discussions after Theorem in § 0).

To determine $p_{j\sigma}(t, x, \xi)$ and $q_{j\sigma}(t, x, \xi)$ for $j \geqq 1$ we proceed by induction on j.

Assume for $k<j$

(3.11) $$p_k(t, x, y, \xi) = \sum_{\sigma=\pm 1} e^{i\sigma t^2|\xi|/2} p_{k\sigma}(t, x, \xi)$$

and

(3.12) $$q_k(t, x, y, \xi) = \sum_{\sigma=\pm 1} e^{i\sigma t^2|\xi|/2} q_{k\sigma}(t, x, \xi).$$

Here, $p_{k\sigma}(t, x, \xi)$ and $q_{k\sigma}(t, x, \xi)$ are semi-homogeneous of degree $-k/2$ and $-k/2-1/2$, respectively.

Set

(3.13) $$r_{j-k,\sigma}^k(t, x, \xi) = e^{-i\sigma t^2|\xi|/2} P_k^\frown \left\{e^{i\sigma t^2|\xi|/2} p_{j-k,\sigma}(t, x, \xi)\right\}$$

and

$$(3.14)\qquad s^{k}_{j-k,\sigma}(t, x, \xi) = e^{-i\sigma t^2 \xi|/2} P^{\frown}_{k} \{ e^{i\sigma t^2|\xi|/2} q_{j-k,\sigma}(t, x, \xi) \}$$

for $k=1, \cdots, j$. Then since $P^{\frown}$ is semi-homogeneous of degree $-k/2+1$, $r^{k}_{j-k,\sigma}(t, x, \xi)$ and $s^{k}_{j-k,\sigma}(t, x, \xi)$ are semi-homogeneous of degree $-j/2+1$ and $-j/2+1/2$, respectively. By (1.11) and (1.16), $p_j(t, x, y, \xi)$ and $q_j(t, x, y, \xi)$ satisfy

$$(3.15)\qquad (D_t^2 + B_0(x, \xi) - t^2|\xi|^2) p_j(t, x, y, \xi) = \sum_{\sigma=\pm 1} e^{i\sigma t^2|\xi|/2} r_{j\sigma}(t, x, \xi),$$

$$(3.16)\qquad (D_t^2 + B_0(x, \xi) - t^2|\xi|^2) q_j(t, x, y, \xi) = \sum_{\sigma=\pm 1} e^{i\sigma t^2|\xi|/2} s_{j\sigma}(t, x, \xi)$$

with the null initial data. Here,

$$(3.17)\qquad r_{j\sigma}(t, x, \xi) = -\sum_{k=1}^{j} r^{k}_{j-k,\sigma}(t, x, \xi)$$

and

$$(3.18)\qquad s_{j\sigma}(t, x, \xi) = -\sum_{k=1}^{j} s^{k}_{j-k,\sigma}(t, x, \xi)$$

are semi-homogeneous of degree $-j/2+1$ and $-j/2+1/2$, respectively.

Then

$$(3.19)\qquad p_{j\sigma}(t, x, \xi) = p^{1}_{j\sigma}(t, x, \xi) - \sum_{\sigma'=\pm 1} p^{1}_{j\sigma'}(0, x, \xi)\, p_{0\sigma}(t, x, \xi) - \sum_{\sigma'=\pm 1} (D_t p^{1}_{j\sigma'})(0, x, \xi)\, q_{0\sigma}(t, x, \xi)$$

and

$$(3.20)\qquad q_{j\sigma}(t, x, \xi) = q^{1}_{j\sigma}(t, x, \xi) - \sum_{\sigma'=\pm 1} q^{1}_{j\sigma'}(0, x, \xi)\, p_{0\sigma}(t, x, \xi) - \sum_{\sigma'=\pm 1} (D_t q^{1}_{j\sigma'})(0, x, \xi)\, q_{0\sigma}(t, x, \xi).$$

Here $p^{1}_{j\sigma}(t, x, \xi)$ and $q^{1}_{j\sigma}(t, x, \xi)$ are solutions of

$$(3.21)\qquad (D_t^2 + 2\sigma t|\xi| D_t + B_0(x, \xi) - i\sigma|\xi|)\, p^{1}_{j\sigma}(t, x, \xi) = r_{j\sigma}(t, x, \xi)$$

and

$$(3.22)\qquad (D_t^2 + 2\sigma t|\xi| D_t + B_0(x, \xi) - i\sigma|\xi|)\, q^{1}_{j\sigma}(t, x, \xi) = s_{j\sigma}(t, x, \xi),$$

respectively. In particular, $p_{j\sigma}(t, x, \xi)$ and $q_{j\sigma}(t, x, \xi)$ are semi-homogeneous of degree $-j/2$ and $-j/2-1/2$, respectively.

PROPOSITION 3.5. *Let K be any compact subset of $\boldsymbol{R}^n$. For any non-negative integer l, and non-negative integral multi-indices α, β, we have*

$$(3.23)\qquad |D_t^l D_x^\alpha D_\xi^\beta p_{j\sigma}(t, x, \xi)| \leqq C(1+|\xi|)^{m(\sigma)+\epsilon-|\beta|} (|\xi|^{-1}+t^2)^{m(\sigma)+\epsilon+(j-l)/2},$$

$$(3.24)\qquad |D_t^l D_x^\alpha D_\xi^\beta q_{j\sigma}(t, x, \xi)| \leqq C(1+|\xi|)^{m(\sigma)+\epsilon-1/2-|\beta|} (|\xi|^{-1}+t^2)^{m(\sigma)-\epsilon+(j-l)/2}$$

and

(3.25) $$|D_t^l D_x^\alpha D_\xi^\beta r_{j-k,\sigma}^k(t, x, \xi)| \leqq C(1+|\xi|)^{m(\sigma)+\varepsilon+1-|\beta|}(|\xi|^{-1}+t^2)^{m(\sigma)+\varepsilon+(j-l)/2},$$

(3.26) $$|D_t^l D_x^\alpha D_\xi^\beta s_{j-k,\sigma}^k(t, x, \xi)| \leqq C(1+|\xi|)^{m(\sigma)+\varepsilon+1/2-|\beta|}(|\xi|^{-1}+t^2)^{m(\sigma)+\varepsilon+(j-l)/2}$$

for $t\geqq 0$, $x\in K$, $|\xi|\geqq 1$ $(j=0, 1, 2, \cdots;\ k=1, \cdots, j,\ j\geqq 1)$. *Here,* ε *is an arbitrary positive integer and* C *is a positive constant depending on* K, ε, l, α, β.

PROOF. We show the estimates (3.23) and (3.25) for $l=|\alpha|=|\beta|=0$. (3.24) and (3.26) for $l=|\alpha|=|\beta|=0$ follow in the same way. These estimates for general α, β, l follow in a similar manner by differentiations in respective variables.

Since $p_{j\sigma}(t, x, \xi)$ and $r_{j-k,\sigma}^k(t, x, \xi)$ are respectively semi-homogeneous of degree $-j/2$ and $-j/2+1$, we may restrict ξ on the unit sphere and write ω instead of ξ, so $|\omega|=1$. We claim

(3.27) $$p_{j\sigma}(t, x, \omega)=C_j t^{-2\tilde{\mu}_\sigma+j}\big(1+o(1)\big)$$

and, for $k=1, \cdots, j$,

(3.28) $$r_{j-k,\sigma}^k(t, x, \omega)=C'_{j,k} t^{-2\tilde{\mu}_\sigma+j}\big(1+o(1)\big)$$

in the sector S defined by (2.48). Here $o(1)\in O^{\varepsilon-1}(S, \boldsymbol{R}^n\times \boldsymbol{S}^{n-1})$, ε being arbitrary positive number, and

(3.29) $$\tilde{\mu}_\sigma=\frac{1}{4}+\frac{i\sigma}{4}B_0(x, \omega).$$

The estimates (3.27) for $j=0$ follows from Propositions 3.1 and 2.6. (3.28) for $j=k=1$ follows from (3.13) and (1.8). Now assume (3.27) for $j<J$. Then from (1.8), (1.9) and (3.13), we have for $k=1, \cdots, J$,

$$\begin{aligned} r_{J-k,\sigma}^k(t, x, \omega) &= C'_k t^k\big(1+o(1)\big)C_{J-K} t^{-2\tilde{\mu}_\sigma+J-k}\big(1+o(1)\big) \\ &= C'_{J,k} t^{-2\tilde{\mu}_\sigma+J}\big(1+o(1)\big), \end{aligned}$$

that is, (3.28) for $j=J$ and $k=1, \cdots, J$. Then by (1.11), Corollary 2.11 and Remarks 2.12 and 2.13, we have (3.27) for $j=J$. By the semi-homogeneity,

(3.30) $$p_{j\sigma}(t, x, \xi)=|\xi|^{-j/2}p_{j\sigma}(|\xi|^{1/2}t, x, \omega),$$

(3.31) $$r_{j-k,\sigma}^k(t, x, \xi)=|\xi|^{1-j/2}r_{j-k,\sigma}^k(|\xi|^{1/2}t, x, \omega),$$

$\omega=\xi/|\xi|$. (3.30) and (3.27) now give

$$|p_{j\sigma}(t, x, \xi)|\leqq C|\xi|^{-j/2}(1+|\xi|^{-j/2}(1+|\xi|^{1/2}t)^{-2\mathrm{Re}\tilde{\mu}_\sigma+j},$$

which implies (3.23) for $l=|\alpha|=|\beta|=0$. Similarly, (3.31) and (3.28) yield

(3. 25) for $l=|\alpha|=|\beta|=0$.

REMARK 3. 6. ε can be dropped in (3. 23) (3. 24) (3. 25) (3. 26) if $n=1$ and $B_0(x, \xi)$ is independent of x. Also $m(\sigma)$ can be localized.

By Propositions 3. 3 and 3. 5, we can now apply a modified version of results of Boutet de Movel [1]. For that purpose, we begin by introducing several notions in the forms we need. The first is the following variant of the symbol classes of Hörmander (e. g., [4]).

DEFINITION 3. 7. For real m, we denote by S^m the space of all C^∞ functions $p(t, x, \xi)$ on $\bar{\boldsymbol{R}}_+\times\boldsymbol{R}^n\times\boldsymbol{R}^n$ such that for any non-negative integer l, and non-negative integral multi-indces α, β, we have

$$|D_t^l D_x^\alpha D_\xi^\beta p(t, x, \xi)| \leqq C(1+|\xi|)^{m-|\beta|}$$

for $0\leqq t\leqq T$, $x\in K$, $|\xi|\geqq 1$. Here, T is an arbitrary positive number, K any compact subset of $\boldsymbol{R}^n$, and C is a positive constant depending on T, K, l, α, β. An element of S^m is called a symbol of degree m.

Just as Hörmander's classes, we have

PROPOSITION 3. 8. *Let $p_j\in S^{m-j}$, $j=0, 1, 2, \cdots$. Then there exists a $p\in S^m$ such that for all N*

$$p-\sum_{j<N} p_j\in S^{m-N}.$$

Two such symbols differ by a symbol of degree $-\infty$.

The following is a variant of the classes of Boutet de Monvel [1].

DEFINITION 3. 9. For real m, κ, we denote by $S^{m,\kappa}$ the space of all C^∞ functions $p(t, x, \xi)$ on $\bar{\boldsymbol{R}}_+\times\boldsymbol{R}^n\times\boldsymbol{R}^n$ such that for any non-negative integer l, and non-negative integral multi-indices α, β, we have

$$|D_t^l D_x^\alpha D_\xi^\beta p(t, x, \xi)| \leqq C(1+|\xi|)^{m-|\beta|}(|\xi|^{-1}+t^2)^{\kappa/2-l/2}$$

for $0\leqq t\leqq T$, $x\in K$, $|\xi|\geqq 1$. Here T is any positive number, K any compact subset of $\boldsymbol{R}^n$, and C is a positive constant depending on T, K, l, α, β.

In this terminology, Proposition 3. 5 reads:

COROLLARY 3. 10 *For $j=0, 1, 2$,*

$$p_{j\sigma}(t, x, \xi)\in S^{m(\sigma)+\varepsilon, 2m(\sigma)+2\varepsilon+j}, \tag{3. 32}$$

$$q_{j\sigma}(t, x, \xi)\in S^{m(\sigma)+\varepsilon-1/2, 2m(\sigma)+2\varepsilon+j}, \tag{3. 33}$$

and for $j=1, 2, \cdots$, $k=1, \cdots, j$,

$$r_{j-k,\sigma}^k(t, x, \xi)\in S^{m(\sigma)+1+\varepsilon, 2m(\sigma)+2\varepsilon+j}, \tag{3. 34}$$

$$s_{j-k,\sigma}^k(t, x, \xi)\in S^{m(\sigma)+1/2+\varepsilon, 2m(\sigma)+2\varepsilon+j}. \tag{3. 35}$$

For most properties of $S^{m,\kappa}$, we invite the reader to consult Boutet de Monvel [1]. However, we require the following properties.

PROPOSITION 3.11. (i) $S^{m}\subset S^{m,0}$;

(ii) $S^{m,\kappa}\subset S^{m',\kappa'}$, *if* $m\leqq m'$ *and* $m-\kappa/2\leqq m'-\kappa'/2$.

PROOF. These are consequences of

$$|\xi|^{-1}\leqq|\xi|^{-1}+t^{2}\leqq 1+T^{2}$$

for $|\xi|\geqq 1,\ 0\leqq t\leqq T$.

PROPOSITION 3.12. *Let* $p\in S^{m,\kappa}$ *and* $q\in S^{m',\kappa'}$. *Then* $pq\in S^{m+m',\kappa+\kappa'}$.

PROOF. This is an immediate consequence of the Leibniz formula.

PROPOSITION 3.13. *Let* $p_{j}\in S^{m,\kappa+j}$, $j=0, 1, 2, \cdots$. *Then there exists a symbol* $p\in S^{m,\kappa}$ *such that for all* N

$$p-\sum_{j<N}p_{j}\in S^{m,\kappa+N}.$$

Two such symbols differ by a symbols of degree m, vanishing to the infinite order modulo $S^{-\infty}$ *on* $t=0$.

PROOF. This is just a variant of Proposition 1.11 (ii) of Boutet de Monvel [1]. More explicitly, let $\chi(\theta)$ be a C^{∞} function of $\theta\in\boldsymbol{R}$ such that $\chi(\theta)=0$ for $\theta<1/2$ and $\chi(\theta)=1$ for $\theta>1$.

Then by setting

$$\tilde{\chi}_{\lambda}(t, \xi)=\chi(|\xi|/\lambda)\left(1-\chi(\lambda t^{2})\right)$$

for $\lambda\geqq 1$, we can choose a sequence $\lambda_{j}\nearrow\infty$ so that, for any N, the sequence

$$\{2^{j}\tilde{\chi}_{\lambda_{j}}p_{j}\}_{j\geqq N}$$

is bounded in $S^{m,\kappa+N-2}$. Then

$$p=\sum_{j\geqq 0}\tilde{\chi}_{\lambda_{j}}p_{j}$$

satisfies the requirement.

Thus the construction of the taming terms are completed by the following

COROLLARY 3.14. *There exist symbols*

$$p_{\sigma}(t, x, \xi)\in S^{m(\sigma)+\varepsilon, 2m(\sigma)+2\varepsilon}$$

and

$$q_{\sigma}(t, x, \xi)\in S^{m(\sigma)+\varepsilon-1/2, 2m(\sigma)+2\varepsilon}$$

such that for all N

(3.36) $$p_\sigma(t, x, \xi) - \sum_{j<N} p_{j\sigma}(t, x, \xi) \in S^{m(\sigma)+\varepsilon, 2m(\sigma)+2\varepsilon+N}$$

and

(3.37) $$q_\sigma(t, x, \xi) - \sum_{j<N} q_{j\sigma}(t, x, \xi) \in S^{m(\sigma)+\varepsilon-1/2, 2m(\sigma)+2\varepsilon+N}$$

PROOF. These are immediate consequences of Proposition 3.13 and (3.23), (3.33).

Recalling (1.1) and (1.2), we have

COROLLARY 3.15.

(3.38) $$E_0(f)(0, x) = f(x) + (R_0 f)(x),$$

(3.39) $$D_0 E_0(f)(0, x) = (R_1 f)(x),$$

(3.40) $$E_1(g)(0, x) = (R'_0 g)(x),$$

(3.41) $$D_t E_1(g)(0, x) = g(x) + (R'_1 g)(x).$$

Here R_0, R_1, R'_0, R'_1 *are operators with* C^∞ *kernels.*

PROOF. These follow from Corollary 3.14 and definitions of $p_{j\sigma}(t, x, \xi)$ and $q_{j\sigma}(t, x, \xi)$.

PROPOSITION 3.16. *Let* $\tilde{r}_\sigma(t, x, \xi)$ *and* $\tilde{s}_\sigma(t, x, \xi)$ *be determined by* (1.23) *and* (1.24), *i.e.,*

$$P^\wedge\left(e^{i\sigma t^2|\xi|/2} p_\sigma(t, x, \xi)\right) = e^{i\sigma t^2|\xi|/2} \tilde{r}_\sigma(t, x, \xi)$$

and

$$P^\wedge\left(e^{i\sigma t^2|\xi|/2} q_\sigma(t, x, \xi)\right) = e^{i\sigma t^2|\xi|/2} \tilde{s}_\sigma(t, x, \xi).$$

Then

(3.42) $$\tilde{r}_\sigma(t, x, \xi) \in S^{m(\sigma)+1+\varepsilon, \infty} = \bigcap_{N=0}^\infty S^{m(\sigma)+\varepsilon+1, 2m(\sigma)+2\varepsilon+N}$$

and

(3.43) $$\tilde{s}(t, x, \xi) \in S^{m(\sigma)+1/2+\varepsilon, \infty} = \bigcap_{N=0}^\infty S^{m(\sigma)+1/2+\varepsilon, 2m(\sigma)+2\varepsilon+N}$$

PROOF. Let N be any positive integer. Then by (3.36)

$$p_\sigma^{N+2} = p_\sigma - \sum_{k<N+2} p_{k\sigma} \in S^{m(\sigma)+\varepsilon, 2m(\sigma)+2\varepsilon+N+2}.$$

By (1.5),

$$P^\wedge(e^{i\sigma t^2|\xi|/2} p_\sigma^{N+2}) = e^{i\sigma t^2|\xi|/2} \tilde{r}_\sigma^{N+2}$$

where

$$\begin{aligned}\tilde{r}_\sigma^{N+2} = \Big(&-2t^2 \textstyle\sum_{j=1}^n \xi_j D_j + \sigma a(t, x) t|\xi| + \sum_{j=1}^n b_j(t, x)\xi_j \\ &- i\sigma|\xi| + 2\sigma t|\xi| D_t - t^2 \textstyle\sum_{j=1}^n D_j^2 + \sum_{j=1}^n b_j(t, x) D_j \\ &+ c(t, x) + a(t, x) D_t + D_t^2\Big) p_\sigma^{N+2}.\end{aligned}$$

Hence, by Proposition 3.11 and 3.12,

(3.44) $$\tilde{r}_{\sigma}^{N+2} \in S^{m(\sigma)+\varepsilon+1, 2m(\sigma)+2\varepsilon+N}.$$

By the Taylor expansion, we have

$$P^{\wedge} = \sum_{j=0}^{N+1} P_j^{\wedge} + P^{\wedge N+2},$$
$$P^{\wedge N+2} = t^{N+2} a_{N+2}(t, x) D_t + t^{N+3} B_{N+3}(t, x, \xi) + t^{N+1} B_{N+1}(t, x, D_x) + t^{N+1} C_{N+1}(t, x),$$

$P_j^{\wedge}$ being defined by (1.8) and (1.9). Then

$$\sum_{j=0}^{N+1} P_j^{\wedge}(e^{i\sigma t^2|\xi|/2} \sum_{k<N+2} p_{k\sigma}) = \sum_{j+k\leq N+1} P_j^{\wedge}(e^{i\sigma t^2|\xi|/2} p_{k\sigma}) + \sum_{j+k\geq N+2} P_j^{\wedge}(e^{i\sigma t^2|\xi|/2} p_{k\sigma}).$$

By (3.15) (3.13) (3.17) the first sum vanishes. On the other hand

$$\sum_{j+k\geq N+2} P_j^{\wedge}(e^{i\sigma t^2|\xi|/2} p_{k\sigma}) = e^{i\sigma t^2|\xi|/2} \tilde{r}_{\sigma}^{\prime N+2},$$

where

$$\tilde{r}_{\sigma}^{\prime N+2} = \sum_{j+k\geq N+2} r_{k\sigma}^{j}$$

by (3.13). Hence, by (3.34) and Proposition 3.11,

(3.45) $$\tilde{r}_{\sigma}^{\prime N+2} \in S^{m(\sigma)+\varepsilon+1, 2m(\sigma)+2\varepsilon+N+2}.$$

Finally,

$$P^{\wedge N+2}(e^{i\sigma t^2|\xi|/2} \sum_{k<N+2} p_{k\sigma}) = e^{i\sigma t^2|\xi|/2} \tilde{r}_{\sigma}^{\prime\prime N+2},$$

where

$$\tilde{r}_{\sigma}^{\prime\prime N+2} = \Big(t^{N+2} a_{N+2}(t, x) D_t + \sigma t^{N+3} a_{N+2}(t, x)|\xi| + t^{N+3} B_{N+3}(t, x, \xi) + t^{N+1} B_{N+1}(t, x, D) + t^{N+1} c_{N+1}(t, x)\Big) \sum_{k<N+2} p_{k\sigma}.$$

Hence, by Propositions 3.11 and 3.12,

(3.46) $$\tilde{r}_{\sigma}^{\prime\prime N+2} \in S^{m(\sigma)+1+\varepsilon, 2m(\sigma)+2\varepsilon+N+1}.$$

Since $\tilde{r}_{\sigma} = \tilde{r}_{\sigma}^{N+2} + \tilde{r}_{\sigma}^{\prime N+2} + \tilde{r}_{\sigma}^{\prime\prime N+2}$, we have (3.32) from (3.44), (3.45), (3.46). (3.43) follows in a similar manner.

Let

(3.47) $$F_0(f)(t, x) = \sum_{\sigma=\pm1} (2\pi)^{-n} \iint e^{i\{\langle x-y, \xi\rangle + \sigma t^2|\xi|/2\}} \tilde{r}_{\sigma}(t, x, \xi) f(y)\, dy\, d\xi$$

and

$$(3.48)\qquad F_1(g)(t,x) = \sum_{\sigma=\pm 1}(2\pi)^{-n}\iint e^{i\{<x-y,\xi>+\sigma t^2|\xi|/2\}}\tilde{s}_\sigma(t,x,\xi)\,g(y)\,dy\,d\xi\,.$$

Now from Definition 3. 9, we see $S^{m,\infty}$ is the set of symbols S^m which vanish to the infinite order on $t=0$ modulo $S^{-\infty}$, that is, p is the sum of a symbol of degree $-\infty$ and of a symbol of degree m which is flat at $t=0$. In our discussions, we may discard symbols of degree $-\infty$. In this sense, we may assume in the following section that

$$(3.49)\qquad \tilde{r}_\sigma(t,x,\xi)\in S^{m(\sigma)+1+\epsilon},$$

$$(3.50)\qquad \tilde{s}_\sigma(t,x,\xi)\in S^{m(\sigma)+1/2+\epsilon},$$

and

(3.51) $\tilde{r}_\sigma(t,x,\xi)$ *and* $\tilde{s}_\sigma(t,x,\xi)$ *are flat at* $t=0$.

Summarizing, we have shown

PROPOSITION 3. 17.

$$PE_0(f)(t,x)=F_0(f)(t,x)+K_0(f)(t,x),$$
$$PE_1(g)(t,x)=F_1(g)(t,x)+K_1(g)(t,x).$$

Here F_0 *and* F_1 *are respectively defined by* (3. 47) *and* (3. 48) *with* $\tilde{r}_\sigma$ *and* $\tilde{s}_\sigma$ *satisfying* (3. 49), (3. 50) *and* (3. 51); K_0 *and* K_1 *are operators with smooth kernels on* $\overline{\boldsymbol{R}}_+\times\boldsymbol{R}^n\times\boldsymbol{R}^n$.

§ 4. Completion of the proof of Theorem

We begin by the following

PROPOSITION 4. 1. *Let* $a(t)$ *be a* C^∞ *function of* $t\geqq 0$. *If a* C^∞ *function* $f(t)$ *of* $t\geqq 0$ *is flat* $t=0$, *then the equation*

$$(4.1)\qquad \big(td/dt+a(t)\big)u(t)=f(t),\quad t\geqq 0,$$

has a solution $u(t)$ *which is flat at* $t=0$.

PROOF. Let $b(t)=(a(t)-a(0))/t$. $b(t)$ is a smooth function of $t\geqq 0$. Set $v(t)=t^{a(0)}u(t)$. Then $v(t)$ satisfies the equation

$$(4.2)\qquad \big(d/dt+b(t)\big)v(t)=t^{a(0)-1}f(t).$$

The right hand side of (4. 2) is flat at $t=0$.
Hence,

$$v(t)=\int_0^t \exp\Big(-\int_\tau^t b(s)ds\Big)\tau^{a(0)-1}f(\tau)\,d\tau$$

is a solution of (4. 2). Furthermore, $v(t)$ is flat at $t=0$. Therefore,

$$u(t)=t^{-a(0)}v(t)$$

is flat at $t=0$, and is the solution of (4. 1) as required.

Now we proceed to the construction of $\tilde{p}_\sigma(t, x, \xi)$ and $\tilde{q}_\sigma(t, x, \xi)$. We set

$$\tilde{E}_0(f)(t, x) = \sum_{\sigma=\pm 1}(2\pi)^{-n}\iint e^{i\{<x-y,\xi>+\sigma t^2|\xi|/2\}}\tilde{p}_\sigma(t, x, \xi)f(y)dy\, d\xi\,. \tag{4. 3}$$

and

$$\tilde{E}_1(g)(t, x) = \sum_{\sigma=\pm 1}(2\pi)^{-n}\iint e^{i\{<x-y,\xi>+\sigma t^2|\xi|/2\}}\tilde{q}_\sigma(t, x, \xi)\, g(y)\, dy\, d\xi\,. \tag{4. 4}$$

A formal application of P to $\tilde{E}_0(f)$ and to $\tilde{E}_1(g)$ gives

$$P\tilde{E}_0(f)(t, x) = \sum_{\sigma=\pm 1}(2\pi)^{-n}\iint e^{i\{<x-y,\xi>+\sigma t^2|\xi|/2\}}\tilde{P}_\sigma\tilde{p}_\sigma(t, x, \xi)f(y)dy\, d\xi \tag{4. 5}$$

and

$$P\tilde{E}_1(g)(t, x) = \sum_{\sigma=\pm 1}(2\pi)^{-n}\iint e^{i\{<x-y,\xi>+\sigma t^2|\xi|/2\}}\tilde{P}_\sigma q_\sigma(t, x, \xi)\, g(y)\, dy\, d\xi\,, \tag{4. 6}$$

where

$$\begin{aligned} \tilde{P}_\sigma &= \tilde{P}_{1\sigma}+\tilde{P}_0, \\ \tilde{P}_{1\sigma} &= 2\sigma t|\xi|D_t-2t^2\textstyle\sum_{j=1}^n\xi_jD_j+\sigma ta(t, x)|\xi| \\ &\quad +\textstyle\sum_{j=1}^n b_j(t, x)\,\xi_j-\sigma i|\xi|\,, \\ \tilde{P}_0 &= P=D_t^2-t^2\textstyle\sum_{j=1}^n D_j^2+a(t, x)D_t+\sum_{j=1}^n b_j(t, x)D_j+c(t, x). \end{aligned} \tag{4. 7}$$

Therefore, in view of Proposition 3. 17 and (4. 5), (4. 6),

$$E(f, g)=E_0(f)+\tilde{E}_0(f)+E_1(g)+\tilde{E}_1(g) \tag{4. 8}$$

is a parametrix of (0. 1) if

$$\tilde{P}_\sigma\tilde{p}_\sigma(t, x, \xi)+\tilde{r}_\sigma(t, x, \xi)\in S^{-\infty} \tag{4. 9}$$

and

$$\tilde{P}_\sigma\tilde{q}_\sigma(t, x, \xi)+\tilde{s}_\sigma(t, x, \xi)\in S^{-\infty}. \tag{4. 10}$$

To solve the equations (4. 9) and (4. 10) we set formally

$$\tilde{p}_\sigma(t, x, \xi) = \textstyle\sum_{j=0}^{\infty} \tilde{p}_{j\sigma}(t, x, \xi) \tag{4.11}$$

and

$$\tilde{q}_\sigma(t, x, \xi) = \textstyle\sum_{j=0}^{\infty} \tilde{q}_{j\sigma}(t, x, \xi). \tag{4.12}$$

Then (4. 10) and (4. 11) are decomposed to the following systems of equations:

$$\tilde{P}_{1\sigma}\tilde{p}_{0\sigma}(t, x, \xi) + \tilde{r}_\sigma(t, x, \xi) = 0, \tag{4.13}$$

$$\tilde{P}_{1\sigma}\tilde{p}_{j\sigma}(t, x, \xi) + P\tilde{p}_{j-1,\sigma}(t, x, \xi) = 0, \quad j \geqq 1, \tag{4.14}$$

and

$$\tilde{P}_{1\sigma}\tilde{q}_{0\sigma}(t, x, \xi) + \tilde{s}_\sigma(t, x, \xi) = 0, \tag{4.15}$$

$$\tilde{P}_{1\sigma}\tilde{q}_{j\sigma}(t, x, \xi) + P\tilde{q}_{j-1,\sigma}(t, x, \xi) = 0, \quad j \geqq 1. \tag{4.16}$$

Then we have

PROPOSITION 4. 2. *For* $j = 0, 1, 2, \cdots$.

$$\tilde{p}_{j\sigma}(t, x, \xi) \in S^{m(\sigma)+\epsilon-j} \tag{4.17}$$

$$\tilde{q}_{j\sigma}(t, x, \xi) \in S^{m(\sigma)+\epsilon-1/2-j} \tag{4.18}$$

and

$$\tilde{p}_{j\sigma}(t, x, \xi) \textit{ and } \tilde{q}_{j\sigma}(t, x, \xi) \textit{ are flat at } t = 0. \tag{4.19}$$

PROOF. We note that the operator $\tilde{P}_{1\sigma}$ is of the form $|\xi|(tX_\sigma + \tilde{a}_\sigma(t, x, \xi/|\xi|))$, where

$$X_\sigma = 2\sigma D_t - 2t \textstyle\sum_{j=1}^{n} |\xi|^{-1}\xi_j D_j$$

and

$$\tilde{a}_\sigma(t, x, \xi/|\xi|) = \sigma t a(t, x) + \textstyle\sum_{j=1}^{n} b_j(t, x)|\xi|^{-1}\xi_j - \sigma i.$$

By (3. 49), (3. 51) and Proposition 4. 1, we see $\tilde{p}_{0\sigma}(t, x, \xi) \in S^{m(\sigma)+\epsilon}$ and is flat at $t=0$, solving (4. 13). Hence, from (4. 14) and Proposition 4. 1 we see $\tilde{p}_{1\sigma}(t, x, \xi) \in S^{m(\sigma)+\epsilon-1}$ and is flat at $t=0$. In this way, we obtain (4. 17) (4. 18) and (4. 19).

Therefore, by Proposition 3. 8 we can now interprete (4. 11) and (4. 12) in the following sense.

PROPOSITION 4. 3. *There exist symbols* $\tilde{p}_\sigma(t, x, \xi) \in S^{m(\sigma)+\epsilon}$ *and* $\tilde{q}_\sigma(t, x, \xi) \in S^{m(\sigma)+\epsilon+1/2}$ *such that for all* N

$$\tilde{p}_\sigma(t, x, \xi) - \textstyle\sum_{j<N} \tilde{p}_{j\sigma}(t, x, \xi) \in S^{m(\sigma)+\epsilon-N}$$

and

$$\tilde{q}_\sigma(t, x, \xi) - \sum_{j<N} \tilde{q}_{j\sigma}(t, x, \xi) \in S^{m(\sigma)+\varepsilon-1/2-N}.$$

COROLLARY 4.4.

$$\tilde{E}_0(f)(0, x) = (\tilde{R}_0 f)(x), \tag{4.20}$$

$$D_t \tilde{E}_0)f)(0, x) = (\tilde{R}_1 f)(x), \tag{4.21}$$

$$\tilde{E}_1(g)(0, x) = (\tilde{R}_0' g)(x), \tag{4.22}$$

$$D_t \tilde{E}_1(g)(0, x) = (\tilde{R}_1' g)(t, x), \tag{4.23}$$

$$P\tilde{E}_0(f) + F_0(f) = \tilde{K}_0(f)(t, x), \tag{4.24}$$

$$P\tilde{E}_1(g) + F_1(g) = \tilde{K}_1(g)(t, x). \tag{4.25}$$

Here $\tilde{R}_0$, $\tilde{R}_0'$, $\tilde{R}_1$, $\tilde{R}_1'$ *are smoothing operators on* $\boldsymbol{R}^n$ *and* $\tilde{K}_0$, $\tilde{K}_1$ *are operators with smooth kernels on* $\bar{\boldsymbol{R}}_+ \times \boldsymbol{R}^n \times \boldsymbol{R}^n$.

PROOF. These follow from our constructions. In fact, we can find $\tilde{p}_\sigma$ and $\tilde{q}_\sigma$ so that $\tilde{R}_0 = \tilde{R}_0' = \tilde{R}_1 = \tilde{R}_1' = 0$.

The proof of Theorem is now complete by (4.8) and Corollaries 4.4, 3.15 and Proposition 3.17.

References

[1] BOUTET DE MONVEL, L.: Hypoelliptic operators with double characteristics and related pseudo-differential operators, Comm. Pure Appl. Math., 27 (1974) 585–639.

[2] CHI, MIN-YOU: On the Cauchy problem for a class of hyperbolic equations with initial data on the parabolic degenerating line, Acta Math. Sinica, 8 (1958) 521–529 (Chinese Math. (1967) 246–254).

[3] ERDÉLYI, A. et al.: Higher Transcendental Functions, Vol. 1, McGraw Hill, New York (1953).

[4] HÖRMANDER, L.: Fourier integral operators I, Acta Math., 127 (1971) 79–183.

[5] HÖRMANDER, L.: On the Cauchy problem for differential equations with double characteristics (preprint).

[6] IVRII, V. YA.: The Cauchy problem for nonstrictly hyperbolic equations of second order, Soviet Math. Dokl., 12 (1971) 1712–1716 (Dokl. Akad. Nauk SSSR, 201 (1971) 778–781).

[7] IVRII, V. YA. and PETKOV, V. M.: Necessary conditions for the Cauchy problem for non-strictly hyperbolic equations to be well-posed, Russian Math. Surv., 29 (1974) 1–70) (Uspekhi Mat. Nauk 29 (1974) 3–70).

[8] MENIKOFF, A.: The Cauchy problem for weakly hyperbolic equations, Amer. J. Math., 97 (1975) 548–558.

[9] OLEINIK, O. A.: On the Cauchy problem for weakly hyperbolic equations, Comm. Pure Appl. Math., 23 (1970) 569-589.

Added in the proof: Similar results to our present part have recently been obtained by

S. ALINHAC: Paramétrix pour un système hyperbolique à multiplicité variable (to appear).

Department of Mathematics
Hokkaido University