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On cut loci and first conjugate loci of the irreducible symmetric R-spaces and the irreducible compact hermitian symmetric spaces

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Let M be a compact riemannian symmetric space. If M is simply connected, R. J. Crittenden has shown that the tangent cut locus of a point coincides with the first tangent conjugate locus of that point. (See. e. g. [4], [5], [9]).

In this paper we study relations between the cut locus and the first conjugate locus in the case when M is an irreducible compact hermitian symmetric space or an irreducible symmetric R-space. The irreducible symmetric R-spaces are compact riemannian symmetric spaces such as $O(n)/O(m)\times O(n-m)$, $U(n)$, $SO(n)$, $U(2n)/Sp(n)$, $SO(n)\times SO(m)/S(O(n)\times O(m))$ and $T^1\times E_6/E_4$, which are not necessarily simply connected.

In section 1, we give basic notation concerning symmetric spaces and prepare three propositions which will play important roles in section 2.

In section 2, we determine the first tangent conjugate loci and the tangent cut loci for the irreducible compact hermitian symmetric spaces and the irreducible symmetric R-spaces.

In section 3, we culculate the diameters and the injectivity radius of the irreducible compact hermitian symmetric spaces and the irreducible symmetric R-spaces. We study also the closed geodesics in these spaces.

1. Preliminaries.

1.1. Let (G, K) be a compact riemannian symetric pair, which is defined by the following: a) a compact Lie group G and a closed subgroup K of G, b) an involutive automorphism ι of G such that $G_\iota^0\subset K\subset G_\iota=\{g\in G;\ \iota(g)=g\}$, where G_ι^0 is the identity component of G_ι, and c) a G-invariant riemannian structure $\langle\ ,\ \rangle$ on $M=G/K$.

Let $\mathfrak{g}$ (resp. $\mathfrak{k}$) be the Lie algebra of G (resp. K) and put $\mathfrak{p}=\{X\in\mathfrak{g};\ (d\iota)X=-X\}$. Then we have the decomposition $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$. Take a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{p}$, and denote by $\mathfrak{r}$ the restricted root system with respect to $\mathfrak{a}$, by $\mathfrak{r}^+$ the set of positive roots in $\mathfrak{r}$ with respect to a linear order in $\mathfrak{a}$. Then we have the following decompositions of $\mathfrak{k}$ and $\mathfrak{p}$.

$$\mathfrak{k} = \mathfrak{k}_0 + \sum_{\lambda \in \mathfrak{r}^+} \mathfrak{k}_\lambda ,$$
$$\mathfrak{p} = \mathfrak{a} + \sum_{\lambda \in \mathfrak{r}^+} \mathfrak{p}_\lambda ,$$

where

$$\mathfrak{k}_0 = \{X \in \mathfrak{k} ;\ ad^2(H)x = 0 \text{ for all } H \in \mathfrak{a}\} ,$$
$$\mathfrak{k}_\lambda = \{X \in \mathfrak{k} ;\ ad^2(H)x = -\lambda^2(H)x \text{ for all } H \in \mathfrak{a}\} ,$$
$$\mathfrak{p}_\lambda = \{X \in \mathfrak{p} ;\ ad^2(H)x = -\lambda^2(H)x \text{ for } H \in \mathfrak{a}\} .$$

1.2. We recall the notions of cut locus and conjugate locus of a point in a compact riemannian manifold M. Let Exp_x denote the exponential mapping from the tangent space $T_x(M)$ at $x \in M$ into M. Let X be a unit vector in $T_x(M)$. Then t_0X is called the *first tangent conjugate point* of x along the geodesic $\mathrm{Exp}_x\, tX$, if there exists a non-zero Jacobi-field $J(t)$ along $\mathrm{Exp}_x\, tX$ such that $J(0)=0$, $J(t_0)=0$ and if for $0<t_1<t_0$ there exists no non-zero Jacobi-field $J(t)$ such that $J(0)=0$, $J(t_1)=0$.

On the other hand, $\bar{t}_0X$, is called the *tangent cut point* of x along $\mathrm{Exp}_x\, tX$, if the geodesic segment $\mathrm{Exp}_x\, tX/[0, \bar{t}_0]$ is a minimal geodesic segement, but $\mathrm{Exp}_x\, tX/[0, s]$ can not be a minimal geodesic segment for any $s>\bar{t}_0$. Then the following is well-known (see. e.g. [3]). Assume that $\bar{t}_0X$ is the tangent cut point of x along $\mathrm{Exp}_x\, tX$ and is not the first tangent conjugate point of x along $\mathrm{Exp}_x\, tX$. Then there exists a unit vector $Y \in T_x(M)$, $Y \neq X$, such that $\mathrm{Exp}_x\, \bar{t}_0X = \mathrm{Exp}_x\, \bar{t}_0Y$.

The set of first tangent conjugate points of x (resp. tangent cut points of x) is called the *first tangent conjugate locus* (resp. the *tangent cut locus*).

1.3. Let $M=G/K$ be the compact symmetric riemannian manifold associated to the symmetric pair (G, K) in 1.1. We identify the tangent space $T_{ek}(M)$ with $\mathfrak{p}$ in a canonical manner, and regard the tangent cut locus C and the first tangent conjugate locus F as subsets of $\mathfrak{p}$. We denote $C \cap \mathfrak{a}$ (resp. $F \cap \mathfrak{a}$) by $C_\mathfrak{a}$ (resp. $F_\mathfrak{a}$).

PROPOSITION 1.1. (cf. [7]) *We have.*

$$C = \{Ad(k)X ;\ X \in C_\mathfrak{a},\ k \in K\} , \text{ and}$$
$$F = \{Ad(k)X ;\ X \in F_\mathfrak{a},\ k \in K\} .$$

Thus to study C and F, it suffices to determine $C_\mathfrak{v}$ and $F_\mathfrak{a}$. The following two propositions, given in [9], describe $C_\mathfrak{a}$ and $F_\mathfrak{a}$.

PROPOSITION 1.2. *Let X be a unit vector in $\mathfrak{a}$. If t_0X is the first tangent conjugate point along the geodesic* $\mathrm{Epx}_0\, tX$ *emanating from* $0=eK$ *with the initial direction X, then* $t_0 = \underset{\lambda \in \mathfrak{r}^+}{\mathrm{Min}}\, \pi/|\lambda(X)| = \pi/\underset{\lambda \in \mathfrak{r}^+}{\mathrm{Max}}|\lambda(X)|$.

REMARK. If we denote by α the conjugate degree of t_0X, then $\alpha=\sum_{i=1}^{d}\dim\mathfrak{p}_{\lambda_i}$, where $\operatorname*{Max}_{\lambda\in\mathfrak{r}^+}|\lambda(X)|=|\lambda_1(X)|=\cdots=|\lambda_d(X)|$. Moreover the variational completeness of the adjoint action of K implies that $\{\mathrm{Exp}_0\, Ad(h_s)t_0X;\ h_s=\exp sY,\ Y\in\sum_{i=1}^{d}\mathfrak{k}_{\lambda_i}=\{\mathrm{Exp}_o t_0X\}$.

PROPOSITION 1.3. *a) Let X be a unit vector in $\mathfrak{a}$, $\bar{t}_0X$ be the tangent cut point along* $\mathrm{Exp}_o tX$. *Then either $\bar{t}_0X$ is the first tangent conjugate point or there exists a unit vector $Y\in\mathfrak{a}$, $Y\neq X$, such that* $\mathrm{Exp}_0\bar{t}_0X=\mathrm{Exp}_0\,\bar{t}_0Y$. *b) Let $\mathfrak{z}$ be the center of $\mathfrak{g}$ and put $\mathfrak{z}_{\mathfrak{p}}=\mathfrak{z}\cap\mathfrak{p}$. If X is a unit vector in $\mathfrak{z}_{\mathfrak{p}}\subset\mathfrak{a}$, there exists no tangent conjugate point along* $\mathrm{Exp}_0\, tX$.

2. Irreducible symmetric *R*-spaces.

In this section we study the first tangent conjugate locus and the tangent cut locus of an irreducible symmetric R-space. Notations and results used in this section are written in [10].

2.1. Let G be a semi simple real connected Lie group with finite center. We assume that there exists a complexification $\tilde{G}$ of G, and denote by σ the conjugation of $\tilde{G}$ with respect to G.

A subgroup U of G is said to be a *parabolic subgroup* of G if $U=\tilde{U}\cap G$, where $\tilde{U}$ is a parabolic subgroup of $\tilde{G}$, i.e. a subgroup containing a maximal solvable subgroup of $\tilde{G}$. The homogeneous space $M=G/U$ is then called a *R-space*.

Let $\mathfrak{g}$ (resp. $\tilde{\mathfrak{g}}$) be the Lie algebra of G (resp. $\tilde{G}$). We denote by σ the conjugation of $\tilde{\mathfrak{g}}$ with respect to $\mathfrak{g}$. Let K be a maximal compact subgroup of G, $\mathfrak{k}$ be the Lie algebra of K, and $\mathfrak{g}=\mathfrak{k}+\mathfrak{g}$ be the Cartan decompositon. Put $\mathfrak{g}_u=\mathfrak{k}+i\mathfrak{p}$ and denote by G_u the connected Lie subgroup of $\tilde{G}$ corresponding to the Lie subalgebra $\mathfrak{g}_u$. Then we have $K=G\cap G_u$. Let τ be the conjugation of $\tilde{G}$ with respect to G_u and denote by $(\ ,\ )$ the Killing from of $\tilde{\mathfrak{g}}$. Take a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{p}$ and a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ containing $\mathfrak{a}$. We have $\mathfrak{h}=\mathfrak{h}^++\mathfrak{h}^-$, where $\mathfrak{h}^+=\mathfrak{k}\cap\mathfrak{h}$, $\mathfrak{h}^-=\mathfrak{p}\cap\mathfrak{h}=\mathfrak{a}$. Moreover, $\mathfrak{h}_0=i\mathfrak{h}^++\mathfrak{h}^-$ is the real part of the Cartan subalgebra $\tilde{\mathfrak{h}}$ of $\tilde{\mathfrak{g}}$, which is spanned by $\mathfrak{h}$. Let $\tilde{\mathfrak{r}}$ be the root system of $\tilde{\mathfrak{g}}$ with respect to the Cartan subalgebra $\tilde{\mathfrak{h}}$. We shall identify $\alpha\in\tilde{\mathfrak{r}}$ with the element $H_\alpha\in\mathfrak{h}_0$ such that $\alpha(H)=(H, H_\alpha)$ for any $H\in\mathfrak{h}_0$. Now take a σ-order on $\mathfrak{h}_0$. This induces an order on $\mathfrak{h}^-$. Let Δ be the σ-fundamental system of the root system $\tilde{\mathfrak{r}}$ with respect to this order, $\tilde{\Delta}$ be the underlying Dynkin diagram. Then there exists a σ-subsystem Δ_1 in Δ and $Z\in\mathfrak{a}$ such that $(Z,\alpha_j)=0$ for $\alpha_j\in\Delta_1$ and $(Z,\alpha_k)=1$ for $\alpha_k\notin\Delta_1$ and that the Lie algebra of U is the sum of non-negative eigenspaces of adZ in $\mathfrak{g}$. Then $(Z,\alpha_j)=0$ for $\alpha_j\in\check{\Delta}_1$ and $(Z,\alpha_k)=1$ for $\alpha_k\notin\check{\Delta}_1$. Moreover the Lie algebra $\tilde{\mathfrak{u}}$ of $\tilde{U}$ is the sum of all eigenspaces of adZ belonging to non-negative eigenvalues in $\tilde{\mathfrak{g}}$.

There exist vectors $X_\alpha \in \tilde{\mathfrak{g}}$ with the following properties. a) $[H, X_\alpha]=\alpha(H)X_\alpha$ for $\alpha\in\tilde{\mathfrak{r}}$, $H\in\mathfrak{h}_0$. b) $[X_\alpha, X_{-\alpha}]=-\alpha^*=-2\alpha/(\alpha,\alpha)$ for $\alpha\in\tilde{\mathfrak{r}}$. c) $\tau X_\alpha=X_{-\alpha}$ for $\alpha\in\tilde{\mathfrak{r}}$. d) Putting $U_\alpha=X_\alpha+X_{-\alpha}$, $V_\alpha=i(X_\alpha-X_{-\alpha})$ for $\alpha\in\tilde{\mathfrak{r}}^+$, $\mathfrak{g}_u$ is the real subspace spanned by ih_0 and $\{U_\alpha, V_\alpha\,;\ \alpha\in\tilde{\mathfrak{r}}^+\}$. e) If $[X_\alpha, X_\beta]=N_{\alpha,\beta}X_{\alpha+\beta}$, then $N_{-\alpha,-\beta}=\varepsilon_{\alpha,\beta}N_{\alpha,\beta}$, where $\varepsilon_{\alpha,\beta}=\pm1$ and $N^2_{\alpha,\beta}=\varepsilon_{\alpha,\beta}(p+1)^2$, p being the largest integer such that $\alpha-p\,\beta\in\tilde{\mathfrak{r}}$. f) $\sigma X_\alpha=X_{\sigma\alpha}$ for $\alpha\in\tilde{\mathfrak{r}}$, hence $\mathfrak{g}$ is the real subspace spanned by $\mathfrak{h}$, $\{X_\alpha\,;\ \sigma\alpha=\alpha\}$, and $\{X_\alpha+X_{\sigma\alpha}\,;\ \sigma\alpha\neq\alpha\}$.

For $\lambda\in\mathfrak{h}_0$ we define $\bar{\lambda}=(\lambda+\sigma\lambda)/2$, and put

$$\tilde{\mathfrak{r}}_0^+ = \{\alpha\in\tilde{\mathfrak{r}}^+\,;\ \sigma\alpha=-\alpha\} = \tilde{\mathfrak{r}}^+\cap i\mathfrak{h}^+,$$
$$\tilde{\mathfrak{r}}^{(1)} = \{\alpha\in\tilde{\mathfrak{r}}^+\,;\ \sigma\alpha=\alpha\} = \tilde{\mathfrak{r}}\cap\mathfrak{h}^-,$$
$$\tilde{\mathfrak{r}}^{(2)} = \{\alpha\in\tilde{\mathfrak{r}}^+\,;\ \sigma\alpha<\alpha,\ \alpha+\sigma\alpha=2\,\bar{\alpha}\notin\tilde{\mathfrak{r}}\},$$
$$\tilde{\mathfrak{r}}^{(3)} = \{\alpha\in\tilde{\mathfrak{r}}^+\,;\ \sigma\alpha<\alpha,\ \alpha+\sigma\alpha=2\,\bar{\alpha}\in\tilde{\mathfrak{r}}\}.$$

Moreover we define Z_α, Z'_α, S_α, S'_α, T_α, and T'_α for $\alpha\in\tilde{\mathfrak{r}}^+$ as follows;

For $\alpha\in\tilde{\mathfrak{r}}_0^+$, let $Z_\alpha=X_\alpha+X_{-\alpha}$, $Z'_\alpha=i(X_\alpha-X_{-\alpha})$ and $S_\alpha=Z_\alpha$, $Z'_\alpha=S'_\alpha$.
For $\alpha\in\tilde{\mathfrak{r}}^{(1)}$, let $Z_\alpha=X_\alpha$, $Z_{-\alpha}=X_{-\alpha}$ and $S_\alpha=Z_\alpha+Z_{-\alpha}$, $T_\alpha=Z_\alpha-Z_{-\alpha}$.
For $\alpha\in\tilde{\mathfrak{r}}^{(2)}$, let $Z_\alpha=X_\alpha+X_{\sigma\alpha}$, $Z_{-\alpha}=X_{-\alpha}+X_{-\sigma\alpha}$, $Z'_\alpha=i(X_\alpha-X_{\sigma\alpha})$, $Z'_{-\alpha}=i(X_{-\alpha}+X_{-\sigma\alpha})$. For $\alpha\in\tilde{\mathfrak{r}}^{(3)}$, let $Z_\alpha=\sqrt{2}(X_\alpha+X_{\sigma\alpha})$, $Z_{-\alpha}=\sqrt{2}(X_{-\alpha}+X_{-\sigma\alpha})$, $Z'_\alpha=\sqrt{2}i(X_\alpha-X_{\sigma\alpha})$, $Z'_{-\alpha}=\sqrt{2}i(-X_{-\alpha}+X_{-\sigma\alpha})$.
For $\alpha\in\tilde{\mathfrak{r}}^{(2)}\cup\tilde{\mathfrak{r}}^{(3)}$, let $S_\alpha=Z_\alpha+Z_{-\alpha}$, $T_\alpha=Z_\alpha-Z_{-\alpha}$, $S'_\alpha=Z'_\alpha+Z'_{-\alpha}$, $T'_\alpha=Z'_\alpha-Z'_{-\alpha}$.

We denote by Z (resp. R) the set of all the integers (resp. the set of all the real numbers), and by $\{\ \}_Z$ (resp. $\{\ \}_R$) the set spanned by Z and $\{\ \}$ (resp. R and $\{\ \}$).

We denote by $\mathfrak{r}$ the restricted root system which is projection $\bar{\tilde{\mathfrak{r}}}$ of $\tilde{\mathfrak{r}}$, by $\mathfrak{r}^+$ the set of all positive restricted roots, and set $\mathfrak{r}_1=\{\bar{\Delta}_1\}_Z\cap\mathfrak{r}$. We put

$$\mathfrak{k}_r = \{S_\alpha,\ S'_\alpha\,;\ \bar{\alpha}=r\}_R \text{ for } r\in\mathfrak{r}^+,$$
$$\mathfrak{p}_r = \{T_\alpha,\ T'_\alpha\,;\ \bar{\alpha}=r\}_R \text{ for } r\in\mathfrak{r}^+,$$
$$\mathfrak{k}_0 = \{S_\alpha,\ S'_\alpha\,;\ \alpha\in\tilde{\mathfrak{r}}_0^+\}_R+\mathfrak{h}^+, \text{ and}$$
$$\mathfrak{p}_0 = \mathfrak{h}^-.$$

Then we know the following results.

a) $\mathfrak{k}=\mathfrak{k}_0+\sum_{r\in\mathfrak{r}^+}\mathfrak{k}_r$, $\mathfrak{p}=\mathfrak{p}_0+\sum_{r\in\mathfrak{r}^+}\mathfrak{p}_r$.

b) $\tilde{M}=\tilde{G}/\tilde{U}$ and $M=G/U$ are canonically diffeomorphic to G_u/K' and K/K^* respectively, where $K'=\{x\in G_u\,;\ \mathrm{Ad}_x\ Z=Z\}$ and $K^*=K\cap K'$.

c) $\mathfrak{k}^*=\mathfrak{k}_0+\sum_{r\in\mathfrak{r}_1}\mathfrak{k}_r$ and $\mathfrak{m}^*=\sum_{r\in\mathfrak{r}^+-\mathfrak{r}_1}\mathfrak{k}_r$, where $\mathfrak{k}^*$ is the Lie algebra of K^* and $\mathfrak{m}^*$ is the orthogonal complement of $\mathfrak{k}^*$ in $\mathfrak{k}$ with respect to $(\ ,\ )$.

d) Let $\mathfrak{k}'$ be the Lie algebra of K', and $\mathfrak{m}'$ be the orthogonal complement of $\mathfrak{k}'$ in $\mathfrak{g}_u$ with respect to $(\ ,)$. Then $\mathfrak{k}'=i\mathfrak{h}_0+\{U_\alpha,\ V_\alpha:\ \alpha\in\tilde{\mathfrak{r}}_1^+\}_R$ and $\mathfrak{m}'=\{U_\alpha,\ V_\alpha;\ \alpha\in\tilde{\mathfrak{r}}^+-\tilde{\mathfrak{r}}_1\}_R$.

2.2. Suppose that Δ is irreducible and let let $\alpha_0=\sum_{i=1}^l n_i\alpha_i$ be the highest root in $\tilde{\Delta}$. If there exists an index k such that $n_k=1$ and $\tilde{\Delta}_1=\tilde{\Delta}-\{\alpha_k\}$, the pair $(\tilde{\Delta},\tilde{\Delta}_1)$ is called *symmetric*. In general, $(\tilde{\Delta},\tilde{\Delta}_1)$ and the corresponding space $\widetilde{M}$ are called *symmetric*, if every non-trivial irreducible factor of $(\tilde{\Delta},\tilde{\Delta}_1)$ is symmetric; M is called *symmetric* if $\widetilde{M}$ is symmetric. If $\widetilde{M}$ is symmetric, putting $\theta'=Ad(\exp\pi iZ)$, $(G_u, K';\ \theta')$ and $(K, K^*;\ \theta')$ are symmetric pairs, and $\widetilde{M}=G_u/K'$ is a compact hermitian symmetric space.

We assume that $\tilde{G}$ is simply connected. Then $(G_u)_{\theta'}=\{x\in G_u;\ \theta' x=x\}$ is connected so that $K'=\{x\in G_u;\ \theta' x=x\}$ and $K^*=\{x\in K;\ \theta' x=x\}$. From now on we always assume that $\tilde{G}$ is simply connected, and that $\widetilde{M}$, M are the riemannian symmetric spaces with the riemannian structure defined by $-(\ ,\ )$, $(\ ,\ )$ being the Killing form restricted to $\mathfrak{p}$. Moreover we assume that $\tilde{\Delta}$ is irreducible.

LEMMA 2.1. ([6], [8], [10]). *There exists a system* $\mathfrak{f}'=\{\beta_1,\cdots,\beta_{\nu'}\}$ $(\nu'=\text{rank }\widetilde{M})$ *of mutually strongly orthogonal roots in* $\tilde{\mathfrak{r}}^+-\tilde{\mathfrak{r}}_1$ *with the same length* d' *satisfying the following properties.*

a) *Let* $\varpi:\mathfrak{h}_0\to\{\beta_1,\cdots,\beta_{\nu'}\}_R$ *be the orthogonal projection, and put* $h_i=\beta_i/2$ *for* $1\leqq i\leqq\nu'$. *Then* $\varpi\tilde{\mathfrak{r}}=\{\pm(h_i\pm h_j);\ 1\leqq i,\ j\leqq\nu'\}$ *or* $\varpi\tilde{\mathfrak{r}}=\{\pm(h_i\pm h_j),\ \pm h_i;\ 1\leqq i,\ j\leqq\nu'\}$.

b) *One of the following conditions holds*: I) $\mathfrak{f}'\subset\tilde{\mathfrak{r}}^{(1)}$, *and* $\mathfrak{f}^*=\mathfrak{f}'$. II) *Putting* $\mathfrak{f}^*=\mathfrak{f}'\cap\tilde{\mathfrak{r}}^{(2)}$, $\mathfrak{f}'=\mathfrak{f}^*\cup\sigma\mathfrak{f}^*$ *(disjoint sum).*

c) $\nu^*=\text{rank }M$, *where* $\nu^*=\nu'$ *or* $\nu'/2$ *according as to the cases* I) *and* II) *in* b).

Now $\mathfrak{a}_u'=\{U_{\beta_1},\cdots,U_{\beta_{\nu'}}\}_R$ is a maximal abelian subspace in $\mathfrak{m}'$. Set $c'=\exp(\pi/4)\sum_{i=1}^{\nu'}V_{\beta_i}\in G_u$. Then we have $Ad_{c'}\ U_{\beta_j}=i\beta_j^*$, $Ad_{c'}\ (i\beta_j^*)=-U_{\beta_j}$ for $1\leqq j\leqq\nu'$ and $Ad_{c'}\ H=H$ whenever $H\in\mathfrak{h}_0$ is orthogonal to $\{\beta_1,\cdots,\beta_{\nu'}\}_R$. Hence $Ad_{c'}\ (i\mathfrak{h}_0)$ is a maximal abelian subalgebra of $\mathfrak{g}_u$ containing $\mathfrak{a}_u'$, and $Ad_{c'}$ induced an isomophism of $\varpi\tilde{\mathfrak{r}}-\{0\}$ onto the restricted root system of $(\mathfrak{g}_u,\mathfrak{k}')$ with respect to $\mathfrak{a}_u'$. Thus it follows from Lemma 2.1 the following

LEMMA 2.2. *The restricted root system of* $(\mathfrak{g}_u,\mathfrak{k}')$ *with respect to* $\mathfrak{a}_u'$ *is given by*

a) $\left\{\pm i\dfrac{d'^2}{4}(U_{\beta_i}\pm U_{\beta_j})\,;\ 1\leqq i,\ j\leqq\nu'\right\}-\{0\}$, *or*

b) $\left\{\pm i\dfrac{d'^2}{4}(U_{\beta_i}\pm U_{\beta_j}),\ \pm i\dfrac{d'^2}{4}U_{\beta_i}\,;\ 1\leqq i,\ j\leqq\nu'\right\}-\{0\}$.

REMARK. We know the following results. 1°. Any hermitian symmetric space of semi-simple type is simply connected. 2°. Let G/K be a simply connected compact riemannian symmetric space with a simply connected Lie group G. Then $C=F$. (see. [4], [5], [9])

We denote by $'\mathrm{Exp}$ the exponential mapping with respect to $\widetilde{M}$.

THEOREM 2.3. *If $\widetilde{M}$ is an irreducible hermitian compact symmetric space of semi-simple type,*

$$C=F=\{Ad(k')(\pi/2\ \mathrm{Max}\ |x^i|)X;\ X=\textstyle\sum x^iU_{\beta_i}:\ \text{a unit vector in } \mathfrak{a}'_u,\ k'\in K'\}.$$

PROOF. We note that $U_{\beta_1},\cdots,U_{\beta_{\nu'}}$ are mutually orthogonal and have the same length $2/d'$. Let $X=\sum x^iU_{\beta_i}$ be a unit vector in $\mathfrak{a}'_u$.

Suppose that we are in the case a) of Lemma 2.2. We have

$$\begin{aligned}\left|\left(\pm\frac{d'^2}{4}(U_{\beta_i}\pm U_{\beta_j}),\ X\right)\right|&=\left|\left(\frac{d'^2}{4}(U_{\beta_i}\pm U_{\beta_j}),\ \textstyle\sum x^kU_{\beta_k}\right)\right|\\&=\left|\frac{d'^2}{4}x^i(U_{\beta_i},U_{\beta_j})\pm\frac{d'^2}{4}x^j(U_{\beta_i},U_{\beta_j})\right|\\&=|x^i\pm x^j|.\end{aligned}$$

Then we have $\underset{i\leqq i,j\leqq\nu'}{\mathrm{Max}}|x^i\pm x^j|=2\underset{1\leqq i\leqq\nu'}{\mathrm{Max}}|x^i|$.

Suppose that we are in the case b) of Lemma 2.2. We have

$$\begin{aligned}\left|\left(\pm\frac{d'^2}{4}U_{\beta_i},X\right)\right|&=\left|\left(\frac{d'^2}{4}U_{\beta_i},\ \textstyle\sum x^kU_{\beta_k}\right)\right|\\&=\left|\frac{d'^2}{4}x^i(\mathrm{U}_{\beta_i},\mathrm{U}_{\beta_i})\right|=|x^i|.\end{aligned}$$

Then we have $\mathrm{Max}\ \{|x^i\pm x^j|,\ |x^i|\,;\ 1\leqq i,\ j\leqq\nu'\}=2\underset{1\leqq i\leqq\nu'}{\mathrm{Max}}|x^i|$. By Propositon 1.2, the first tangent conjugate point along $'\mathrm{Exp}\ tX$ is $(\pi/2\ \mathrm{Max}\,|x^i|)$ X. Therefore, by Proposition 1.1, $F=\{Ad(k')(\pi/2\,\mathrm{Max}\,|x^i|)X;\ X=\sum x^i\,U_{\beta_i}:$ a unit vector in $\mathfrak{a}'_u$, $k'\in K'\}$. Also, since $\widetilde{M}$ is a hermitian symmetric space of compact type, $C=F$ follows from the above remark.

We put $\mathfrak{f}^*=\{\beta_1,\cdots,\beta_{\nu*}\}$, changing the order of the β_i if necessary. By the definitions in 2.1, for $\beta_i\in\mathfrak{f}^*$ we have $S_{\beta_i}=U_{\beta_i}$, $\bar{\beta}_i^*=\beta_i^*$ in the case I) of Lemma 2.1 b) and $S_{\beta_i}=U_{\beta_i}+U_{\sigma\beta_i}$, $\bar{\beta}_i^*=\beta_i^*+(\sigma\beta_i)^*$ in the case II). Let $\mathfrak{a}_u^*=\{S_{\beta_1},\cdots,S_{\beta_{\nu^*}}\}_R\subset\mathfrak{m}^*$. Then $\mathfrak{a}_u^*$ is a maximal abelian subspace in $\mathfrak{m}^*$ and is contained in $\mathfrak{a}_u'$.

LEMMA 2.4. *$S_{\beta_1},\cdots,S_{\beta_{\nu^*}}$ are mutually orthogonal and have the same length* $2/(\bar{\beta}_i,\bar{\beta}_i)^{\frac{1}{2}}=2/d^*$.

PROOF. We remark $U_{\beta_1},\cdots,U_{\beta_{\nu'}}$ are mutually orthogonal and have the same length $2/d'$.

Case I) of Lemma 2.1. By the above remark we have $(S_{\beta_i},S_{\beta_i})=(U_{\beta_i},U_{\beta_i})=-4/(\beta_i,\beta_i)=-4/d'^2$, and $(\bar{\beta}_i,\bar{\beta}_i)=(\beta_i,\beta_i)$. Thus we get $(S_{\beta_i},S_{\beta_i})=-4/(\bar{\beta}_i,\bar{\beta}_i)$. Again by the above remark we see that $S_{\beta_1},\cdots,S_{\beta_{\nu^*}}$ are mutually orthogonal.

Case II) of Lemma 2.1. By the above remark we have $(S_{\beta_i},S_{\beta_i})=(U_{\beta_i}+U_{\sigma\beta_i},U_{\beta_i}+U_{\sigma\beta_i})=(U_{\beta_i},U_{\beta_i})+(U_{\sigma\beta_i},U_{\sigma\beta_i})=-4/(\beta_i,\beta_i)-4/(\sigma\beta_i,\sigma\beta_i)=-2/d'^2$, and $(\bar{\beta}_i,\bar{\beta}_i)=(\beta_i+\sigma\beta_i,\beta_i+\sigma\beta_i)=2(\beta_i,\beta_i)$. Thus we get $(S_{\beta_i},S_{\beta_i})=-4/(\bar{\beta}_i,\bar{\beta}_i)$. Again by the above remark see that $S_{\beta_1},\cdots,S_{\beta_{\nu^*}}$ are mutually orthogonal.

REMARK. We note that in the case I) $d^*=d'$ and in the case II) $d^*=\sqrt{2}\,d'$.

LEMMA 2.5. *Also, the following is described in* [10].

a) *Let* $\Gamma'=\mathrm{Ker}\,\{\exp:\mathfrak{a}_u'\to\exp\mathfrak{a}_u'\}$. *Then* $\Gamma'=\{\sum 2\pi t^i U_{\beta_i};\ t^i\in Z$ *for all* $i\}$.

b) *Let* $\Gamma^*=\mathrm{Ker}\,\{\exp:\mathfrak{a}_u^*\to\exp\mathfrak{a}_u^*\}$. *Then* $\Gamma^*=\{\sum 2\pi t^i S_{\beta_i};\ t^i\in Z$ *for all* $i\}$.

2.3. We now study the tangent cut locus of a symmetric R-space. We denote by *Exp the exponential mapping with respect to M.

PROPOSITION 2.6. *Let $X=\sum x^i S_{\beta_i}$ be a unit vector in $\mathfrak{a}_u^*$, put* $\alpha^*(X)=\max_{1\leq i\leq\nu^*}|x^i|$ *and* $\bar{t}_0^*(X)=\pi/2\,\alpha^*(X)$. *Then, $\bar{t}_0^*(X)$ equals the minimal value of $t>0$ such that there exists a unit vector $Y\in\mathfrak{a}_u^*$ $(X\neq Y)$, for which* *Exp $tX=$*Exp tY *holds.*

Proof. For such a $t>0$, we have *Exp $tX=$*Exp tY and consequently $\exp 2t(X-Y)=e$. Thus by Lemma 2.5. we have $2t(X-Y)=2\pi\sum t^i S_{\beta_i}$ for some $t^i\in Z$ and consequently $tY=tX-\pi(\sum t^i S_{\beta_i})$. Since X and Y are unit vectors, we get

$$\begin{aligned} t^2 &= \langle tY,tY\rangle \\ &= \langle tX-\pi(\textstyle\sum t^i S_{\beta_i}),\ tX-\pi(\textstyle\sum t^i S_{\beta_i})\rangle \\ &= t^2-2t\pi(\textstyle\sum t^i x^i)\,4/d^{*2}+\pi^2(\textstyle\sum (t^i)^2)\,4/d^{*2}. \end{aligned}$$

Therefore, $t=\pi(\sum(t^i)^2)/2(\sum x^i t^i)$, and $t=\pi(\sum|t^i|^2)/2|\sum x^i t^i|$ since $t>0$. Then it follows that

$$t = \pi(\sum |t^i|^2)/2|\sum x^i t^i| \geqq \pi(\sum |t^i|^2)/2\ \alpha^*(X)\sum |t^i|\,.$$

Since $t^i \in Z$, we have $\sum |t^i|^2/\sum |t^i| \geqq 1$, and so we get $t \geqq \pi/2\ \alpha^*(X) = \bar{t}_0^*(X)$. On the otherhand, if we put $Y = \sum_{j \neq k} x^j S_{\beta_j} - x^k S_{\beta_k}$ where $\alpha^*(X) = |x^k|$ and $t = \pi/2\alpha^*(X)$, then we have $*\mathrm{Exp}\ tX = *\mathrm{Exp}\ tY$. The proposition is proved.

THEOREM 2.7. *Let M be an irreducible symmetric R-space. Then*

$$C = \{Ad(k)\,\pi/2\,\mathrm{Max}\,|x^i|X;\ X = \sum x^i S_{\beta_i}.\ \text{a unit vector in } \mathfrak{a}_u^*,\ k \in K^*\}\,.$$

Proof. Let X be a unit vector in $\mathfrak{a}_u^*$, and t_1X be the tangent cut point along $*\mathrm{Exp}\ tX$. Since t_1X is the tangent cut point, $t_1 \leqq \bar{t}_0^*(X)$. Let t_2X be the first tangent conjugate point along $*\mathrm{Exp}\ tX$. Then there exists a Jacobi-field $J(t)$ along $*\mathrm{Exp}\ tX$ such that $J(0)=0$, $J(t_2)=0$. By the variational completeness of the adjoint action of K^* on K/K^*, there exists $H \in \mathfrak{k}^*$ such that $J(t) = \left.\frac{\partial}{\partial s}\right|_{s=0} f(s,t)$, where $f(s,t) = \pi(\exp -sH\ \exp tX\ \exp sH)$, setting π the projection of K onto K/K^*. Then, $f(s+s_0, t) = \mathrm{L}_{\exp - s_0 H} f(s,t)$, and consequently

$$\left.\frac{\partial}{\partial s}\right|_{s=s_0} f(s,t) = \left.\frac{\partial}{\partial s}\right|_{s=0} f(s+s_0, t)$$
$$= (\mathrm{L}_{\exp - s_0 H})_* J(t)\,.$$

Thus $\left.\frac{\partial}{\partial s}\right|_{(s_0,t_2)} f(s,t) = 0$ and $f(s,t_2) = \pi(\exp t_2 X)$ for any $s \in R$. Thus we have $t_2 \geqq \bar{t}_0^*(X)$. Assume that $t_1 < t_2$. From Proposition 1.3. there exists a unit vector $Y \in \mathfrak{a}_u^*(X \neq Y)$ for which $*\mathrm{Exp}\ t_1X = *\mathrm{Exp}\ t_1Y$ holds. By Proposition 2.6., $t_1 = \bar{t}_0^*(X)$. If $t_1 = t_2$, $t_1 \geqq \bar{t}_0^*(X)$. Thus $t_1 = \bar{t}_0^*(X)$. The theorem follows then from proposition 1.1..

2.4. We study the first tangent conjugate locus of a symmetric R-space. All the irreducible symmetric R-spaces are classified in [10]. By examining these R-spaces, we get the following lemma.

LEMMA 2.8. *Let M be an irreducible symmetric R-space. Then $\nu' = \nu^*$ if M is not simply connected, and $\nu' = 2\nu^*$ if M is simply connected.*

REMARK. The following facts are well-known. 1°. Let G/K be a simply connected compact riemannian symmetric space with a simply connected Lie group G. Then $C=F$. (see [4] [5] [9]). 2°. Let M be a compact riemannian manifold. If $C=F$ at a point $x \in M$, M is simply connected. (cf. [5]).

Now, if M is simply connected, we can determine F by this Remark and Theorem 2.7.

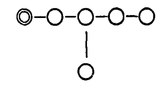
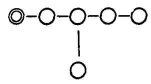
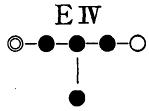
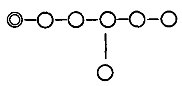
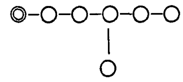
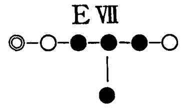
LEMMA 2.9. *If M is not simply connected, then every restricted root* *of K/K^* with respect to $\mathfrak{a}_u^*$ is one of the forms $\pm(x^i \pm x^j)$ $(i<j)$, $\pm x^i$, or $\pm 2x^i$.*

Proof. By Lemma 2. 8., we have $\mathfrak{a}_u^* = \mathfrak{a}'_u$. Therefore all the restricted roots of K/K^* with respect to $\mathfrak{a}_u^*$ are restricted roots of G_u/K' with respect to $\mathfrak{a}'_u$. Our lemma then follows from Lemma 2. 2.

From now on we assume that M is not simply connected. If rank $M \geqq 3$, we can determine F by applying Proposition 1. 2., Lemma 2. 9, and Araki's table [1]. If rank $M=1$, M is the real projective space and the set F is well-known [2]. If rank $M=2$, M is one of the followings: $U(2)$, $U(2)/O(2)$, $U(4)/Sp(2)$, $O(4)/O(2)\times O(2)$, $SO(4)$, $SO(\nu+\nu_0)\times SO(\nu)/S(O(\nu+\nu_0-1)\times O(\nu-1))$, $Sp(4)/<Sp(2)\times Sp(2), \begin{pmatrix} 0 & -1_4 \\ 1_4 & 0 \end{pmatrix}>$, $SO(5)$. If M is $U(2)$, $U(2)/O(2)$, or $U(4)/Sp(2)$, we can determine F directly. If M is $O(4)/O(2)\times O(2)$, $SO(4)$, or $SO(\nu+\nu_0)\times SO(\nu)/S(O(\nu+\nu_0-1)\times O(\nu-1)$, we can determine F by using Lemma 2. 9, Araki's table and the following facts. a) The riemannian universal covering space of $O(4)/O(2)\times O(2)$ (resp. $SO(4)$, $SO(\nu+\varsigma_0)\times SO(\nu)/S(O(\nu+\nu_0-1)\times O(\nu-1))$ is $S^2\times S^2$ (resp. $S^3\times S^3$, $S^{\nu+\nu_0-1}\times S^{\nu-1}$). b) The fundamental group of thses spaces are Z_2. If M is $Sp(4)/<Sp(2)\times Sp(2), \begin{pmatrix} 0 & -1_4 \\ 1_4 & 0 \end{pmatrix}>$ or $SO(5)$, we can determine F by applying Lemma 2. 9, Araki's table, and the above Remark 2°. Thus we get finally the following table.

$(\tilde{\Delta}, \tilde{\Delta}_1)$	$\widetilde{M}$	(Δ, Δ_1)	M	ν^*	$t_0^*(x)$	$\pi_1(M)$
A O-O···◎···O	$Un/U(m)\times U(n-m)$. $n\geqq 2. 1\leqq m\leqq\left[\frac{n}{2}\right]$. (Complex Grassmann and Complex Projective). $\nu'=m$.	A I O-O···◙···O	$On/O(m)\times O(n-m)$. $n\geqq 2. 1<m\leqq[n/2]$. (Real Grassmann).	m	$\dfrac{\pi}{\mathrm{Max}_{i<j}\lvert x^i \pm x^j\rvert}$	$\boldsymbol{Z}_2$
			$O(n)/O(1)\times O(n-1)$. $n\geqq 2$. (Real Projective).	1	$\dfrac{\pi}{\lvert x^1\rvert}$	$\boldsymbol{Z}_2$
		A II ●-O-●-O··· -●-◎-●···◎-●	$Sp(n)/Sp(m)\times Sp(n-m)$. $n\geqq 3. 1\leqq m\leqq[n/2]$. (Quatanion Grassmann).	m	$\dfrac{\pi}{2\,\mathrm{Max}_{i}\lvert x_i\rvert}$	0
			$Sp(n)/Sp(1)\times Sp(n-1)$. $n\geqq 2$. (Quatanion Projective).	1	$\dfrac{\pi}{2\lvert x^1\rvert}$	0
		A III O-O···O-O ╲ ◎ O-O···O-O ╱	$U(m)$ $m\geqq 2$	m	$\dfrac{\pi}{\mathrm{Max}_{i<j}\lvert x_i - x_j\rvert}$	$\boldsymbol{Z}$

$(\tilde{\Delta}, \tilde{\Delta}_1)$	$\tilde{M}$	(Δ, Δ_1)	M	ν^*	$t_0^*(x)$	$\pi_1(M)$
B ◎-○··· ···○-○⇒○	$SO(n)/SO$ $(n-2)\times SO(2)$. $n\geqq 3$; odd. Complex Quadric). $\nu'=\begin{cases}1 (n=3)\\2 (n\geqq 4)\end{cases}$	B I ◎-○···○-●-●··· ···●⇒●	$SO(\nu+\nu_0)\times$ $SO(\nu)/SO(\nu+$ $\nu_0-1)\times$ $O(\nu-1))$. $\nu_0\geqq 1$; odd. $\nu>1$.	2	$\dfrac{\pi}{\underset{i<j}{\mathrm{Max}}\lvert x^i\pm x^j\rvert}$	$\boldsymbol{Z}_2$
			$SO(n+1)/$ $SO(n)$. $n\geqq 2$; odd. (Sphere).	1	$\dfrac{\pi}{2\lvert x^1\rvert}$	0
		B I ○-○···○-●-●··· ···●⇒●	$SO(n+1)/$ $SO(n)$. $n\geqq 2$; odd. (Sphere).	1	$\dfrac{\pi}{2\lvert x^1\rvert}$	0
C ○-○···○⇐◎	$Sp(n)/U(n)$. $n\geqq 1$. $\nu'=n$.	C I ○-○···○⇐◎	$U(n)/O(n)$. $n\geqq 2$.	n	$\dfrac{\pi}{\underset{i<j}{\mathrm{Max}}\lvert x^i-x^j\rvert}$	$\boldsymbol{Z}$
		C II ●-○-●··· ···○-●⇐◎	$Sp(n)$.	n	$\dfrac{\pi}{2\,\underset{i}{\mathrm{Max}}\lvert x^i\rvert}$	0
D ○-○···○<(○, ◎)	$SO(2n)/U(n)$. $n\geqq 3$. $\nu'=\left[\frac{n}{2}\right]$.	D I ○-○···○-●-●···●<(●, ●)	$SO(n+)/$ $SO(n)$. $n\geqq 4$; even. (Sphere).	1	$\dfrac{\pi}{2\lvert x^1\rvert}$	0
		D I ○-○···○<(○, ◎)	$SO(n)$. $n\geqq 4$.	$\left[\frac{n}{2}\right]$	$\dfrac{\pi}{\underset{i<j}{\mathrm{Max}}\lvert x^i\pm x^j\rvert}$	$\boldsymbol{Z}_2$
			$SO(3)$. $(P^3(\boldsymbol{R}))$	1	$\dfrac{\pi}{\lvert x^1\rvert}$	$\boldsymbol{Z}_2$
		D III ●-○-●···●-○<(●, ◎)	$U(2n)/Sp(n)$. $n\geqq 2$.	n	$\dfrac{\pi}{\underset{i<j}{\mathrm{Max}}\lvert x^i-x^j\rvert}$	$\boldsymbol{Z}$
D ◎-○···○<(○, ○)	$SO(n)/SO$ $(n-2)\times SO(2)$. $n\geqq 6$; even. (Complex Quadric). $\nu'=2$.	D I ◎-○···○-●-●···●<(●, ●)	$SO(n+1)/$ $SO(n)$. $n\geqq 4$; even. (Sphere).	1	$\dfrac{\pi}{2\lvert x^1\rvert}$	0
			$SO(\nu+\nu_0)\times$ $SO(\nu)/S(O$ $(\nu_0+\nu_0-1)\times$ $O(\nu-1))$. $\nu_0\geqq 4$; even. $\nu>1$.	2	$\dfrac{\pi}{\underset{i<j}{\mathrm{Max}}\lvert x^i\pm x^j\rvert}$	$\boldsymbol{Z}_2$
		D I ◎-○···○<(○, ○)	$SO(\nu+2)\times$ $SO(\nu)/S(O$ $(\nu+1)\times$ $O(\nu-1))$. $\nu>1$.	2	$\dfrac{\pi}{\underset{i<j}{\mathrm{Max}}\lvert x^i\pm x^j\rvert}$	$\boldsymbol{Z}_2$
		D I ◎-○···○<(○, ○)	$SO(\nu)\times$ $SO(\nu)/S(O$ $(\nu-1)\times$ $O(\nu-1))$. $\nu>1$.	2	$\dfrac{\pi}{\underset{i<j}{\mathrm{Max}}\lvert x^i\pm x^j\rvert}$	$\boldsymbol{Z}_2$

$(\tilde{\Delta}, \tilde{\Delta}_1)$	$\tilde{M}$	(Δ, Δ_1)	M	ν^*	$t_0^*(x)$	$\pi_1(M)$
E_6 	$E_6/T^1\times D_5$. $\nu'=2$.	E I 	$Sp(4)\big/\big\langle Sp(2)\times Sp(2), \begin{pmatrix} 0 & 1_4 \\ -1_4 & 0 \end{pmatrix}\big\rangle$.	2	$\dfrac{\pi}{\underset{i<j}{\mathrm{Max}}\lvert x^i \pm x^j\rvert}$	$\boldsymbol{Z}_2$
		E IV 	$F_4/Spin(9)$	1	$\dfrac{\pi}{2\lvert x^1\rvert}$	0
E_7 	$E_7/T^1\times E_6$ $\nu'=3$	E V 	$SU(8)\big/\big\langle Sp(4), \begin{pmatrix} 1_4 & 0 \\ 0 & -4 \end{pmatrix}\big\rangle$.	3	$\dfrac{\pi}{\underset{i<j}{\mathrm{Max}}\lvert x^i \pm x^j\rvert}$	$\boldsymbol{Z}_2$
		E VII 	$T^1\times E_6/F_4$	3	$\dfrac{\pi}{\underset{i<j}{\mathrm{Max}}\lvert x^i - x^j\rvert}$	$\boldsymbol{Z}$

3. Applications.

THEOREM 3.1. a) *Let $\tilde{M}$ be an irreducible compact hermitian symmetric space of semi-simple type,* $\mathrm{d}(\tilde{M})$ *be the diameter of $\tilde{M}$, and* $\mathrm{inj}(\tilde{M})$ *be the injectivity radius of $\tilde{M}$. Then* $\mathrm{d}(\tilde{M})=\pi\sqrt{\mathrm{rank}\,\tilde{M}}/d'$, $\mathrm{inj}(\tilde{M})=\pi/d'$. b) *Let M be an irreducible symmetric R-space. Then* $\mathrm{d}(M)=\pi\sqrt{\mathrm{rank}\,M}/d^*$, $\mathrm{inj}(M)=\pi/d^*$.

Proof. We prove the theorem for $\tilde{M}$, the second part being proved quite analogously. Let $X=\sum x^i U_{\beta_i}$ be a unit vector in $\mathfrak{a}'_u$. By Theorem 2.3, $\pi/2\,\mathrm{Max}\,|x^i|\;X$ is the tangent cut point along $'\mathrm{Exp}\;tX$. We note that $\sum (x^i)^2=(d')^2/4$. Then the minimal value of $\pi/2\,\mathrm{Max}\,|x^i|$ is $\pi/2d'/2=\pi/\mathrm{d}'$. Thus $\mathrm{inj}(\tilde{M})=\pi/d'$. We also have $\nu'\;(\mathrm{Max}\,|x^i|)^2\geqq\sum (x^i)^2=(d')^2/4$, and consequently $\mathrm{Max}\,|x^i|\geqq d'/2\sqrt{\nu'}$. The maximal value of $\pi/2\,\mathrm{Max}\,|x^i|$ is $\pi/2d'/2\sqrt{\nu'}=\pi\sqrt{\nu'}/d'$. Thus $\mathrm{d}(\tilde{M})=\pi\sqrt{\nu'}/d'$.

COROLLARY 1. *Let N be an irreducible compact hermitian symmetric space of semi-simple type or an irreducible symmetric R-space. If* rank *$N=n$, then* $\mathrm{d}(N)/\mathrm{inj}(N)=\sqrt{n}$.

COROLLARY 2. *Let M be an irreducible symmetric R-space, and $\tilde{M}$ be the irreducible compact hermitian symmetric space corresponding to M. Then M is a totally geodesic submanifold of $\tilde{M}$. If M is simply connected,* $\mathrm{d}(M)=\mathrm{d}(\tilde{M})$ *and* $\mathrm{inj}(M)=\mathrm{inj}(\tilde{M})/\sqrt{2}$. *If M is not simply connected,* $\mathrm{d}(M)=\mathrm{d}(\tilde{M})$ *and* $\mathrm{inj}(M)=\mathrm{inj}(\tilde{M})$.

Proof. As mentioned in 2.1, we know $K\subset G_u$ and $K^*=K\cap K'$. Thus

$(K, K^*; \theta')$ is a symmetric subspace of $(G_u, K'; \theta')$, and consequently M is a totally geodesic submanifold of $\widetilde{M}$. The latter part follows from Theorem 3.1, Remark after Lemma 2.4, and Lemma 2.8.

COROLLARY 3. *Let $\widetilde{M}$ be an irreducible compact compact hermitian symmetric space of semi-simple type. For $m=(m_1, \cdots, m_{\nu'})\in Z^{\nu'}-\{0\}$, put $X(m)=\sum_{i=1}^{\nu'}(m_i/|m|)\,U_{\beta_i}\in\mathfrak{a}'_u$ where $|m|=(\sum m_i^2)^{\frac{1}{2}}$. Then,* i) $'\mathrm{Exp}\ tX(m)$ $(0\leqq t\leqq |m|\pi)$ *is a closed geodesic in $\widetilde{M}$ emanating from eK' and having the length $2\pi|m|/d'$. Moreover the multiplicity is equal to the G. C. M. of $\{m_1, \cdots, m_{\nu'}\}$.* ii) *Let $\gamma(t)$ be a closed geodesic in $\widetilde{M}$ emanating from eK' whose initial vector is of length $2/d'$. Then there exist $m\in Z^{\nu'}-\{0\}$ and $Y\in\mathfrak{k}'$ such that $\gamma(t)={}'\mathrm{Exp}\ t\ Ad(\exp Y)\,X(m)$.*

Let M be an irreducible symmertic R-space. Then the same conclusion as above holds, replacing ν', U_{β_i}, $\mathfrak{a}'_u$, eK', d' $\mathfrak{k}'$, $'\mathrm{Exp}$ *by ν^*, S_{β_i}, $\mathfrak{a}^*_u$, eK^*, d^*, $\mathfrak{k}^*$,* $^*\mathrm{Exp}$ *respectively.*

Proof. We prove the corollary for $\widetilde{M}$, the second part being proved quite analogusly.

i). $'\mathrm{Exp}\,(t+|m|\pi)\,X(m)\approx\exp 2\,(t+|m|\pi)\,X(m)=\exp 2\,tX(m)\cdot\exp 2\,|m|\pi\,X(m)$ $=\exp 2\,tX(m)\cdot\exp\sum 2\pi \mathrm{m}_i\mathrm{U}_{\beta_i}=\exp 2\,tX(m)\approx{}'\mathrm{Exp}\,tX(m)$ by Lemma 2.5. The proof of the other assertions is trivial.

ii). Let $'\mathrm{Exp}\ tX$ $(0\leqq t\leqq t_0)$ be a closed geodeisc with an initial vector $X\in\mathfrak{a}'_u$ such that $|X|=2/d'$. Then $\exp 2\,t_0X=e$. If we put $X=\sum x^iU_{\beta_i}$ $(\sum(x^i)^2=1)$, we have $\exp\sum 2t_0x^iU_{\beta_i}=e$. Thus $2\,t_0x^i=2\pi m_i$ for some $m_i\in Z$. Put $m=(m_1, \cdots, m_{\nu'})$. Then we get $t_0^2\sum(x^i)^2=\pi^2\sum m_i^2=\pi^2|m|^2$. Thus $t_0^2=\pi^2|m|^2$ and $t_0=\pi|m|$. Therefore $x^i=m_i/|m|$ for all i, which proves the assertion.

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