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On homomorphisms of a group algebra into a convolution measure algebra

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Throughout this paper, G denotes a LCA group with the dual group $\hat{G}$. The group operation in G (resp. $\hat{G}$) is expressed under the multiplicative notation. $M(G)$ denotes the convolution algebra of all the bounded regular complex Borel measures on G, and $L^1(G)$ denotes the group algebra of G, the convolution algebra of all the absolutely continuous members of $M(G)$ with respect to the Haar measure of G. $\mathfrak{M}$ denotes a commutative semi-simple convolution measure algebla (cf. J. L. Taylor [6]).

In this paper, we consider the following problem: how can we determine all the homomorphisms of $L^1(G)$ into $\mathfrak{M}$? If $\mathfrak{M}$ is a measure algebra $M(H)$ of some LCA group H, a complete answer to this problem is known (cf. P. J. Cohen [1], [2]). Using the Cohen's results and the Taylor's theory on convolution measure algebras, we will give, in theorem 2 and in theorem 17 below, an anologous answer to this problem in the general setting of $\mathfrak{M}$.

§1. On the range of the homomorphisms.

In this section, we consider the range of a homomorphism of $L^1(G)$ into $\mathfrak{M}$. For this purpose, we can assume without loss of generality that $\mathfrak{M}$ contains an identity of norm 1. By Taylor's representation theorem on convolution measure algebras, there exists a compact commutative topological semi-group S, called the structure semi-group of $\mathfrak{M}$, and we can consider $\mathfrak{M}$ a weak*-dense closed L-subalgebra of the measure algebra $M(S)$ of S. Moreover the maximal ideal space of $\mathfrak{M}$ can be identified with $\hat{S}$, the set of all the non-zero bounded continuous semi-characters on S, and the Gelfand transform of $\mu\in\mathfrak{M}$ is expressed by $\hat{\mu}(f)=\int_S f d\mu (f\in\hat{S})$.

$\hat{S}$ is a compact separately continuous topological semi-group with respect to the Gelfand topology and the pointwise multiplication. $\hat{S}^+=\{f\in\hat{S}\,|\,f\geqq 0\}$ is a closed subsemi-group of $\hat{S}$, and $\hat{S}^+$ becomes a partially ordered set with the natural order: $f\geqq g$ if and only if $f(s)\geqq g(s)$ $(s\in S)$. Every closed subset $(\neq\phi)$ of $\hat{S}^+$ has a minimal element, and this fact will play an important role later.

An element h of $\hat{S}^+$ is called a critical point if and only if h is an isolated point in $h\hat{S}^+=\{hf|f\in\hat{S}^+\}$. If h is a critical point, $\Gamma_h=\{f\in\hat{S}|\,|f|=h\}$ becomes a *LCA* group with respect to the induced topology and the multiplication in Γ_h. The following theorem which we need in this paper is due to J. L. Taylor.

THEOREM 1 (Taylor). *Let h be a critical point of $\hat{S}^+$.*

(a) *There exists a LCA group $G(h)$ such that:*

i) *Γ_h is the dual group of $G(h)$,*

ii) *the kernel $K(h)$ of the compact subsemi-group $S_h=\{s\in S|h(s)=1\}$ is the Bohr compactification of $G(h)$.*

(b) *Let α be the canonical injection of $G(h)$ into $K(h)$, and let i_α be the isomorphic isometry given by $M(G(h))\to M(K(h))$: $\mu\to\mu\circ\alpha^{-1}$. Then, if we identify μ with $i_\alpha(\mu)$ ($\mu\in M(G(h))$), we have*

$$L^1\big(G(h)\big)\subset\mathfrak{M}\cap M\big(K(h)\big)\subset\mathrm{Rad}\,L^1\big(G(h)\big).$$

A commutative semi-simple convolution measure algebra $\mathfrak{N}$ is called an almost group algebra if there exist a *LCA* group G' and a L-subalgebra $\mathfrak{N}'\subset M(G')$, with $L^1(G')\subset\mathfrak{N}'\subset\mathrm{Rad}\,L^1(G')$, such that $\mathfrak{N}'$ is isomorphic to $\mathfrak{N}$ as a measure algebra. By the above theorem, $\mathfrak{M}\cap M(K(h))$ is an almost group algebra for each critical point $h\in\hat{S}^+$. Converse of this result is also true, that is each subalgebra of $\mathfrak{M}$ which is an almost group algebra is a subalgebra of $\mathfrak{M}\cap M(K(h))$ for some critical point $h\in\hat{S}^+$. The closed linear span of $\{L^1(G(h))|h$: critical point of $\hat{S}^+\}$ is a subalgebra of $\mathfrak{M}$, which is called the spine of $\mathfrak{M}$. For the proof of above result, we can refer to [7].

In the rest of this paper, $\boldsymbol{Z}$ and $\boldsymbol{C}$ denote the set of all the rational integers and the complex number field, respectively. If $\boldsymbol{C}\ni\alpha$, we express by $\bar{\alpha}$ the complex conjugate of α. If μ, $\nu\in\mathfrak{M}$, $\mu\perp\nu$ implies that μ and ν are mutually singular.

THEOREM 2. *If Φ is a homomorphism of $L^1(G)$ into $\mathfrak{M}$, there exist a finite number of critical points $h_1,\cdots,h_m$ of $\hat{S}^+$ such that*

$$\Phi\big(L^1(G)\big)\subset\sum_{i=1}^{m}L^1\big(G(h_i)\big).$$

For the proof of theorem 2, we prepare the following lemmas.

DEFINITION 1. *Let 0 be the 0-homomorphism of $L^1(G)$ into $\boldsymbol{C}$. We consider $\hat{G}\cup\{0\}$ the one point compactification of $\hat{G}$.*
If Ψ is a homomorphism of $L^1(G)$ into $\mathfrak{M}$, we call the mapping

$$\hat{S}\longrightarrow\hat{G}\cup\{0\}\ :\ f\longmapsto f\circ\Psi\,,$$

the dual map of Ψ.

We denote by φ *the dual map of the homomorphism* Φ *of theorem* 2. *Obviously* φ *is continuous, and if* $\hat{}$ *express the Gelfand transform, we have*

$$\hat{\mu}\big(\varphi(f)\big)=\widehat{\Phi(\mu)}(f) \qquad \big(f\in\hat{S},\ \mu\in L^1(G)\big).$$

LEMMA 3. *If* $h\in\hat{S}^+$ *is a critical point, the set* $A=\{f\in\hat{S}^+|f\geqq h\}$ *is open and closed in* $\hat{S}^+$.

PROOF. Let μ be a positive normalised measure in $L^1(G(h))$, then the Gelfand transform of μ restricted to $\hat{S}^+$ is the characteristic function of A. This shows that A is open and closed in $\hat{S}^+$.

LEMMA 4. *If* $h\in\hat{S}^+$ *and* $\mu\in\mathfrak{M}$ *satisfies* $\hat{\mu}(f)=0(f\in\hat{S},\ |f|\leqq h)$, *then we have* $h\mu=0$.

PROOF. Since $\mathfrak{M}$ is semi-simple, the relation

$$\widehat{h\mu}(f)=\int_S fhd\mu=\hat{\mu}(hf)=0 \qquad (f\in\hat{S}),$$

implies $h\mu=0$.

LEMMA 5. *For each* $\mu\in L^1(G)$, $\Phi(\mu)$ *is a symmetric measure in* $\mathfrak{M}$.

PROOF. For each $\mu\in L^1(G)$, we define $\tilde{\mu}$ by $\tilde{\mu}(E)=\overline{\mu(E^{-1})}$ (E: Borel set of G). Then the relation

$$\begin{aligned}\widehat{\Phi(\mu)*\Phi(\tilde{\mu})}(f)&=\widehat{\Phi(\mu*\tilde{\mu})}(f)=\widehat{(\mu*\tilde{\mu})}\big(\varphi(f)\big)\\ &=|\hat{\mu}\big(\varphi(f)\big)|^2=|\widehat{\Phi(\mu)}(f)|^2 \qquad (f\in\hat{S}),\end{aligned}$$

shows that $\Phi(\mu)$ is symmetric in $\mathfrak{M}$.

LEMMA 6. *If* $r_1,\cdots,r_N\in\hat{G}$, *and if* V *is a compact neighberhood of the unit of* $\hat{G}$, *there exists* $\mu\in L^1(G)$ *such that*

i) $\|\mu\|\leqq\sqrt{N}$,

ii) $\hat{\mu}(r)=\begin{cases}1: r\in\{r_1,\cdots,r_N\}\\ 0: r\notin\{r_1,\cdots,r_N\}\cdot V.\end{cases}$

PROOF. Let ρ denote the Haar measure of $\hat{G}$. For each open set W in $\hat{G}$ with $W=W^{-1}$ and $WW\subset V$. we can choose $\mu\in L^1(G)$ (cf. [5], 2. 6. 1.) such that

i) $\hat{\mu}(r)=\begin{cases}1: r\in\{r_1,\cdots,r_N\}\\ 0: r\notin\{r_1,\cdots,r_N\}\cdot W\cdot W,\end{cases}$

$$\text{ii)}\quad \|\mu\| \leqq \left\{\rho\left(\{r_1, \cdots, r_N\}\cdot W\right)/\rho(W)\right\}^{\frac{1}{2}} \leqq \sqrt{N},$$

and μ satisfies the required properties i) and ii) of lemma 6.

Let n be a non negative integer, and we express the following set of assumptions by (*).

(*) i) $h_1, \cdots, h_n \in \hat{S}^+$: critical points,

ii) $\varepsilon_1, \cdots, \varepsilon_n \in \boldsymbol{Z}$,

iii) $Y=\left\{f\in \hat{S} \,|\, \widehat{\left(\Phi(\mu)+\sum_{i=1}^{n}\varepsilon_i h_i \Phi(\mu)\right)}(f)\neq 0 \text{ for some } \mu\in L^1(G)\right\}\neq\phi.$

iv) $h_i\hat{S}\cap Y=\phi$ and $h_i\Phi(\mu)\in \sum_{j=1}^{n} \mathrm{Rad}\, L^1\left(G(h_j)\right)$

$\left(\mu\in L^1(G)\right)$ for $i=1, \cdots, n$.

v) $|Y|=\left\{|f| \,\middle|\, f\in Y\right\}$, $|Y|^-$: closure of $|Y|$,

vi) h: one of the minimal points of $|Y|^-$.

Remark 1. Under the assumption (*), $\Phi(\mu)+\sum_{i=1}^{n}\varepsilon_i h_i\Phi(\mu)$ is concentrated in $S\setminus\bigcup_{i=1}^{n}\{s\in S|\ |h_i(s)|=1\}$ by lemma 4 and, $\|\Phi(\mu)\|\geqq\|\Phi(\mu)+\sum_{i=1}^{n}\varepsilon_i h_i \Phi(\mu)\|$ $(\mu\in L^1(G))$.

LEMMA 7. *Under the assumption* (*), $h\in|Y|$ *implies that* h *is a critical point.*

PROOF. Obviously Y is open in $\hat{S}$. Since $h\in|Y|$, there exists $f\in Y$ such that $|f|=h$. Let $\mu\in L^1(G)$ be such that $(\Phi(\mu)+\sum_{i=1}^{n}\varepsilon_i h_i \Phi(\mu))^{\wedge}(f)\neq 0$. Since $\Phi(\mu)$ is symmetric by lemma 3, $\Phi(\mu)$ is concentrated in $S\setminus\{s\in S|\,0<|f(s)|<1\}$ (cf. [6]). Therefore the minimality of h in $|Y|^-$ implies $|f|^2=|f|$, and thus

$$h\hat{S}\longrightarrow h\hat{S}\ :\ g\longmapsto \bar{f}g$$

is a homeomorphism. So $\{h\}=\bar{f}(Y\cap h\hat{S})\cap h\hat{S}^+$ is open in $h\hat{S}^+$, and h is a critical point by definition.

LEMMA 8. *Let the assumption* (*) *be satisfied, and let* $(f_\lambda)_{\lambda\in\Lambda}\subset Y$ *be a net such that* $\lim_{\lambda}|f_\lambda|=h$. *Then for each* i $(1\leqq i\leqq n)$, *we can choose* $\lambda_i\in\Lambda$ *such that* $g\in Y$ *and* $|g|\leqq|f_\lambda|$ *for some* $\lambda\geqq\lambda_i$ *imply either of the following* a) *or* b).

a) $h \ngeqq h_i$ *and* $|g| \ngeqq h_i$,

b) $h \geqq h_i$ *and* $|g| \geqq h_i$.

PROOF. First we suppose that $h \ngeqq h_i$. By lemma 3, $A = \{f \in \hat{S}^+ | f \ngeqq h_i\}$ is open in $\hat{S}^+$, and we can choose $\lambda_i \in \Lambda$ such that $|f_\lambda| \in A$ $(\lambda \geqq \lambda_i)$. Hence if $|g| \leqq |f_\lambda|$ for some $\lambda \geqq \lambda_i$, we get $|g| \ngeqq h_i$.

Next we suppose $h \geqq h_i$. By lemma 3, $A' = \{f \in \hat{S}^+ | f \geqq h_i\}$ is open in $\hat{S}^+$. If b) were not true, $\Lambda_1 = \{\lambda \in \Lambda | \, |g| \leqq |f_\lambda|$ for some $g \in Y$ with $|g| \notin A'\}$ is a cofinal subset of Λ. For each $\lambda \in \Lambda_1$, fix an element $g_\lambda \in Y$ such that $|g_\lambda| \leqq |f_\lambda|$ and $|g_\lambda| \notin A'$. Then $\{g_\lambda\}_{\lambda \in \Lambda_1}$ has a subnet $\{b_\beta\}_{\beta \in B}$ such that $\{|b_\beta|\}_{\beta \in B}$ converges to an element of $\hat{S}^+$, say g^*. Hence we have

$$\int_S g^* d\mu = \lim_\beta \int_S |b_\beta| d\mu \leqq \lim_\lambda \int_S |f_\lambda| d\mu = \int_S h d\mu \qquad (0 \leqq \mu \in \mathfrak{M}) \qquad (1)$$

and (1) implies $g^* \leqq h$. Since h is minimal in $|Y|^-$, we have $g^* = h \in A'$, and there exists $\beta_i \in B$ such that $|b_\beta| \in A'$ $(\beta \geqq \beta_i)$. On the other hand, $|b_\beta| \notin A'$ $(\beta \in B)$ by definition, and this is a contradiction. This completes the proof of lemma 8.

LEMMA 9. *Suppose the assumption* (*) *is satisfied, and let* $(f_\lambda)_{\lambda \in \Lambda}$ *be a net in* Y *such that* $\lim_\lambda |f_\lambda| = h$. *Then there exists* $\lambda_0 \in \Lambda$ *such that*

$$\left(\sum_{i=1}^n \varepsilon_i h_i \Phi(\mu)\right)^\wedge (f) = \left(\sum_{i=1}^n \varepsilon_i h_i \Phi(\mu)\right)^\wedge (hf) \qquad (2)$$
$$\left(\mu \in L_1(G), \quad f \in Y \text{ with } |f_\lambda| \geqq |f| \text{ for some } \lambda \geqq \lambda_0\right)$$

PROOF. By lemma 8, there exists $\lambda_i \in \Lambda$ such that $g \in Y$ and $|f_\lambda| \geqq |g|$ for some $\lambda \geqq \lambda_i$ imply either of the following a) or b).

a) $h \ngeqq h_i$ and $|g| \ngeqq h_i$,

b) $h \geqq h_i$ and $|g| \geqq h_i$.

It is easy to see from (3) that (2) holds for $\lambda_0 = \sup\{\lambda_1, \cdots, \lambda_n\}$, and lemma 9 is proved.

LEMMA 10. *Under the assumption* (*), *h is a critical point.*

PROOF. To prove lemma 10, it is enough to show $h \in |Y|$ by lemma 7. So by assuming $h \notin |Y|$, we will reduce to a contradiction.

Let $(f_\lambda)_{\lambda \in \Lambda}$ be a net in Y such that $\lim_\lambda |f_\lambda| = h$. Here we can assume $|f_\lambda|^2 = |f_\lambda|$ $(\lambda \in \Lambda)$. To prove this, we put $|f_\lambda|^\infty = \lim_{r \to \infty} |f|^r$ and $g_\lambda = |f_\lambda|^\infty f_\lambda$ $(\lambda \in \Lambda)$. Then $|g_\lambda|^2 = |g_\lambda|$ is obvious, and $g_\lambda \in Y$ follows from lemma 5. Choose a subnet $\{b_\beta\}_{\beta\in B}$ of $\{g_\lambda\}_{\lambda\in\Lambda}$ and $g^*\in|Y|^-$ such that $\lim_\beta|b_\beta|=g^*$. Then the relation

$$\int_S g^*d\mu=\lim_\beta\int_S|b_\beta|d\mu\leqq\lim_\lambda\int_S|f_\lambda|d\mu=\int_S hd\mu \qquad (0\leqq\mu\in\mathfrak{M}),$$

implies $h\geqq g^*$. Since h is minimal in $|Y|^-$, we have $h=g^*$ and we can take $\{b_\beta\}_{\beta\in B}$ for $\{f_\lambda\}_{\lambda\in\Lambda}$.

We choose $\lambda_0\in\Lambda$ so that (2) of lemma 9 holds, and put $g_1=f_{\lambda_0}$. Suppose that $g_1,\cdots,g_k$ are already chosen in Y, which satisfy

i) $|g_i|=|g_i|^2 \qquad (i=1,\cdots,k)$,

ii) $\left(\sum_{j=1}^n\varepsilon_jh_j\Phi(\mu)\right)^{\widehat{}}(f)=\left(\sum_{j=1}^n\varepsilon_jh_j\Phi(\mu)\right)^{\widehat{}}(hf)$

$$\left(\mu\in L^1(G), \quad f\in Y \text{ with } |g_i|\geqq|f| \text{ for some } i\ (1\leqq i\leqq k)\right). \tag{4}$$

To choose g_{k+1} in Y, we show that either of the following a) or b) holds.

a) There exists i $(1\leqq i\leqq k)$ and $g\in Y$ such that

i) $|g_i|$ is minimal in $\{|g_1|,\cdots,|g_k|\}$,

ii) $|g|^2=|g|\leqq|g_i|$,

iii) $\varphi(g_i)\neq\varphi(g_i|g|)$.

b) There exists $\lambda_k\in\Lambda$ such that

i) $\lambda_k\geqq\lambda_0$

ii) if $|g_i|$ is minimal in $\{|g_1|,\cdots,|g_k|\}$, we have $(f_{\lambda_k}g_i\hat{S})\cap Y=\phi$.

To prove this, we suppose that both a) and b) are not true. Since b) is not true, there exists, for each $(\Lambda\ni)\lambda\geqq\lambda_0$, $i=i(\lambda)\in\{1,\cdots,k\}$ such that $|g_i|$ is minimal in $\{|g_1|,\cdots,|g_k|\}$ and $(g_if_\lambda\hat{S})\cap Y\neq\phi$. Choose an element of $(g_if_\lambda\hat{S})\cap Y\ni g$ such that $|g|^2=|g|$, then we have $\varphi(g_i)=\varphi(g_i|g|)$ since a) is not true. Putting $g_\lambda=g_i|g|$ $(\lambda\in\Lambda, \lambda\geqq\lambda_0)$, we get a net $\{g_\lambda\}_{\lambda\geqq\lambda_0}$. If we choose a subnet $\{b_\beta\}_{\beta\in B}$ of $\{g_\lambda\}_{\lambda\geqq\lambda_0}$ and $g^*\in|Y|^-$ such that $\{|b_\beta|\}_{\beta\in B}$ converges to g^*, then we have

$$\int_S g^*d\mu=\lim_\beta\int_S|b_\beta|d\mu\leqq\lim_\lambda\int_S|f_\lambda|d\mu=\int_S hd\mu \qquad (0\leqq\mu\in\mathfrak{M}) \tag{5}$$

(5) shows $g^*\leqq h$, and we have $g^*=h$ by the minimality of h. Further, by taking to a subnet of $\{b_\beta\}$ again if necessary, we can find i_0 $(1\leqq i_0\leqq k)$ such that

$$|g_{i_0}| \text{ is minimal in } \left\{|g_1|, \cdots, |g_k|\right\},$$
$$\varphi(g_{i_0}) = \varphi(b_\beta), \qquad b_\beta = g_{i_0}|b_\beta| \qquad (\beta \in B),$$

and thus we have, for each $\mu \in L^1(G)$,

$$\widehat{\Phi(\mu)}(g_{i_0}) = \lim_\beta \widehat{\Phi(\mu)}(b_\beta) = \lim_\beta \widehat{\Phi(\mu)}(g_{i_0}|b_\beta|) = \widehat{\Phi(\mu)}(g_{i_0}h) \tag{6}$$

On the other hand, since $g_{i_0} \in Y$ and $g_{i_0}h \notin Y$, we have from (4)

$$0 \neq \left(\Phi(\mu) + \sum_{i=1}^n \varepsilon_i h_i \Phi(\mu)\right)^{\wedge}(g_{i_0}) - \left(\Phi(\mu) + \sum_{i=1}^n \varepsilon_i h_i \Phi(\mu)\right)^{\wedge}(g_{i_0}h)$$
$$= \widehat{\Phi(\mu)}(g_{i_0}) - \widehat{\Phi(\mu)}(g_{i_0}h) \qquad \text{for some } \mu \in L^1(G),$$

and this contradict to (6). Therefore either of a) or b) must hold.

If a) holds for $g \in Y$, we put $g_{k+1} = g$. If a) dose not hold, then b) must hold for some $\lambda_k \in \Lambda$, and we put $g_{k+1} = f_{\lambda_k}$. It is easy to see that $g_1, \cdots, g_{k+1}$ satisfy (4) and thus we can construct a sequence $\{g_i\}_{i=1}^\infty$ inductively.

Now let N be an integer which satisfies $\sqrt{N} > \|\Phi\| = \sup_{0 \neq \mu \in L^1(G)} \|\Phi(\mu)\| / \|\mu\|$. By taking to a subsequence if necessary, we can suppose that $\{g_i\}_{i=1}^\infty$ satisfies either I or II below.

I. $|g_1| > |g_2| > \cdots ;\ \varphi(g_k) \neq \varphi(|g_{k+1}|g_k) \qquad (k = 1, 2, \cdots)$,

II. $(g_i g_j \hat{S}) \cap Y = \phi \qquad (i \neq j;\ i, j = 1, 2, \cdots)$.

Choose elements $g_{n_1}, \cdots, g_{n_{N+1}}$ from $\{g_i\}_{i=1}^\infty$ which satisfy I′ or II′ below according as $\{g_i\}_{i=1}^\infty$ satisfies I or II.

I′. i) $|g_{n_1}| > |g_{n_2}| > \cdots > |g_{n_{N+1}}|$,

ii) $\left\{\varphi(g_{n_i}),\ \varphi(g_{n_i}|g_{n_i+1}|) \,\middle|\, i = 1, \cdots, N+1\right\} = A \cup B,\ A \cap B = \phi$.

iii) A dose not contain both $\varphi(g_{n_i})$ and $\varphi(g_{n_i}|g_{n_i+1}|)$ for each $i (1 \leqq i \leqq N+1)$, and the same is true for B.

II′. i) $(g_{n_i} g_{n_j} \hat{S}) \cap Y = \phi \qquad (i \neq j,\ i, j = 1, \cdots, N+1)$,

ii) $\left\{\varphi(g_{n_i}),\ \varphi(g_{n_i}h) \,\middle|\, i = 1, \cdots, N+1\right\} = A \cup B,\ A \cap B = \phi$,

iii) A dose not contain both $\varphi(g_{n_i})$ and $\varphi(g_{n_i}h)$ for each $i (1 \leqq i \leqq N+1)$, and the same is true for B.

Such a choice of $g_{n_1}, \cdots, g_{n_{N+1}}$ in the case II is possible by (4) and the fact $g_{n_i}h \notin Y$.

In either cases of I′ and II′ above, there exists by lemma 6 an element $\mu \in L^1(G)$ such that $\|\mu\| \leqq \sqrt{N}$ and $\hat{\mu}|A$ (the restriction of $\hat{\mu}$ to A) $= 1$, $\hat{\mu}|B = 0$.

In the case I′, $\|\Phi(\mu)\| \geqq N$ is obvious. In the case II′, $\|\Phi(\mu)\| \geqq \|\Phi(\mu)+\sum_{i=1}^{n}\varepsilon_i h_i \Phi(\mu)\| \geqq N+1$ follows from remark 1, (4) and lemma 4. Thus, in both cases, we have $\|\Phi\| \geqq \|\Phi(\mu)\|/\|\mu\| \geqq \sqrt{N}$, which contradict to the choice of N. The proof of lemma 10 is now complete.

Proof of theorem 2. Let N be an integer such that $\sqrt{N} > \|\Phi\|$. In the first, we show that there exists a finite number of critical points $h_1, \cdots, h_m \in \hat{S}^+$ such that

$$\Phi\big(L^1(G)\big) \subset \sum_{i=1}^{m} \operatorname{Rad} L^1\big(G(h_i)\big) \tag{7}$$

Let $Y_1 = \{f \in \hat{S} \mid \Phi(\mu)(f) \neq 0 \text{ for some } \mu \in L^1(G)\}$. If $Y_1 = \phi$, we have $\Phi = 0$ and theorem 2 is trivial. If $Y_1 \neq \phi$, there exists a minimal element h of $|Y_1|^-$, and since the assumption (*) is trivially satisfied with $n=0$ for $Y = Y_1$ and h, h is a critical point by lemma 10. We put $h_1 = h$ and $Y_2 = \{f \in \hat{S} \mid (\widehat{\Phi(\mu) - h_1\Phi(\mu)})(f) \neq 0$ for some $\mu \in L^1(G)\}$. If $Y_2 = \phi$, we have $\Phi(\mu) \in \operatorname{Rad} L^1(G(h_1))$ $(\mu \in L^1(G))$, and we get (7). If $Y_2 \neq \phi$, and if h is a minimal point of $|Y_2|^-$, (*) is satisfied with $n=1$ for h_1, $\varepsilon_1^{(2)} = -1$, $Y = Y_2$ and h, and h is a critical point by lemma 10 again. We put $h_2 = h$ and $Y_3 = \{f \in \hat{S} \mid (\widehat{\Phi(\mu) - h_1\Phi(\mu) - h_2\Phi(\mu) + h_1h_2\Phi(\mu)})(f) \neq 0$ for some $\mu \in L^1(G)\}$. If $Y_3 = \phi$, we have $\Phi(\mu) \in \operatorname{Rad} L^1(G(h_1)) + \operatorname{Rad} L^1(G(h_2))$ $(\mu \in L^1(G))$ and we have (7) again. If $Y_3 \neq \phi$ and h is a minimal point of $|Y|^-$, (*) is satisfied with $n=2$ for h_1, h_2, $\varepsilon_1^{(3)} = a$, $\varepsilon_2^{(3)} = -1$, $Y = Y_3$ and h, where $a = 0$ or -1 according as $h_1h_2 = h_1$ or $h_1h_2 < h_1$. Suppose this process continues to the k-th step and that (*) is satisfied with $n = k-1$ for $h_1, \cdots, h_{k-1}$, $\varepsilon_i = \varepsilon_i^{(k)}$ $(i = 1, \cdots, k-1)$, $Y = Y_k$ and h, where $Y_k = \{f \in \hat{S} \mid (\widehat{\Phi(\mu) + \sum_{i=1}^{k-1}\varepsilon_i^{(k)} h_i \Phi(\mu)})(f) \neq 0$ for some $\mu \in L^1(G)\}$, then h is a critical point by lemma 10.
We put $h_k = h$, and choose integers $\varepsilon_1^{(k+1)}, \cdots, \varepsilon_k^{(k+1)}$ such that $Y_{k+1} \cap h_i \hat{S} = \phi$ $(i = 1, \cdots, k)$, where $Y_{k+1} = \{f \in \hat{S} \mid (\widehat{\Phi(\mu) + \sum_{i=1}^{k}\varepsilon_i^{(k+1)} h_i \Phi(\mu)})(f) \neq 0$ for some $\mu \in L^1(G)\}$. Such a choice of $\varepsilon_1^{(k+1)}, \cdots, \varepsilon_k^{(k+1)}$ is possible by theorem 1. If $Y_{k+1} = \phi$, we have $\Phi(\mu) = -\sum_{i=1}^{k}\varepsilon_i^{(k+1)} h_i \Phi(\mu) \in \sum_{i=1}^{k} \operatorname{Rad} L^1(G(h_i))$ $(\mu \in L^1(G))$, and the process ends here. If $Y_{k+1} \neq \phi$ and h is a minimal point of $|Y_{k+1}|^-$, it is easy to see that (*) is satisfied with $n = k$ for $h_1, \cdots, h_k$, $\varepsilon_1^{(k+1)}, \cdots, \varepsilon_k^{(k+1)}$, $Y = Y_{k+1}$ and h, and we go on the same way as before.

Suppose that this process continues infinitely. Then we have the infinite sequences $\{h_k\}_{k=1}^{\infty}$, $\{\varepsilon_1^{(k)}, \cdots, \varepsilon_{k-1}^{(k)}\}_{k=2}^{\infty}$ and $\{Y_k\}_{k=1}^{\infty}$, where h_{k-1}, $\varepsilon_1^{(k)}, \cdots, \varepsilon_{k-1}^{(k)}$ and Y_k are given at the k-th step of the above process. Let $f_i \in \hat{S}$ be such that $|f_i| = h_i$ $(i = 1, 2, \cdots)$, and put

$$A_n = \left\{ r \in \hat{G} \mid \sum_{\substack{1 \leq k \leq n \\ \varphi(h_k f_n) = r}} \varepsilon_k^{(n)} \neq 0 \right\} \qquad (n = 1, 2, \cdots),$$

where $\varepsilon_n^{(n)} = 1$ for each n. Since $Y_n \neq \phi$, we have $A_n \neq \phi$ $(n = 1, 2, \cdots)$.

CASE I. Assume that $\bigcup_{n=1}^{\infty} A_n$ is an infinite set. Then there exists an increasing sequence of positive integers $n_1, \cdots, n_N$ such that

$$A_{n_1} \subsetneqq A_{n_1} \cup A_{n_2} \subsetneqq \qquad \subsetneqq A_{n_1} \cup \cdots \cup A_{n_N}.$$

Choose $r_i \in (A_{n_1} \cup \cdots \cup A_{n_i}) \setminus (A_{n_1} \cup \cdots \cup A_{n_{i-1}})$ $(i = 1, 2, \cdots, N)$, and define a function F of $\bigcup_{i=1}^{N} A_{n_i} \cup \{0\}$ into $\{0, 1\}$ such that

$$\text{i)} \quad F(r) = 0 \qquad \left(r \notin \{r_1, \cdots, r_N\} \right),$$

$$\text{ii)} \quad \left| \sum_{i=1}^{n_k} \varepsilon_i^{(n_k)} F\left(\varphi(h_i f_{n_k})\right) \right| \geqq 1 \qquad (k = 1, \cdots, N).$$

From the choice of r_i, such function F can be defined inductively on $\bigcup_{i=1}^{k} A_{n_i}$. from $k = 1$ to N. By lemma 4, we can find $\mu_1 \in L^1(G)$ such that $\|\mu_1\| \leqq \sqrt{N}$ and $\hat{\mu}_1 | \bigcup_{i=1}^{N} A_{n_i} = F$. Therefore we get

$$\begin{aligned} \left| \widehat{\left(\Phi(\mu_1) + \sum_{i=1}^{n_k - 1} \varepsilon_i^{(n_k)} h_i \Phi(\mu) \right)} (f_{n_k}) \right| &= \left| \sum_{i=1}^{n_k} \varepsilon_i^{(n_k)} \hat{\mu}_1 \left(\varphi(f_{n_k} h_i \right) \right| \\ &= \left| \sum_{i=1}^{n_k} \varepsilon_i^{(n_k)} F\left(\varphi(f_{n_k} h_i) \right) \right| \geqq 1 \qquad (k = 1, \cdots, N) \end{aligned} \tag{8}$$

For each $\nu \in \mathfrak{M}$ and $i (1 \leqq i < \infty)$, we decompose ν as

$$\nu = (\nu)_i + (\nu)_i' \ : \ (\nu)_i \in M\left(K(h_i)\right), \qquad (\nu)_i' \in \left(\mathfrak{M} \cap M\left(K(h_i)\right) \right)^{\perp} \tag{9}$$

Then (8) with the notation of (9) becomes

$$\left\| \left(\Phi(\mu_1) \right)_{n_k} \right\| = \left\| \left(\Phi(\mu_1) + \sum_{i=1}^{n_k - 1} \varepsilon_i^{(n_k)} h_i \Phi(\mu_1) \right)_{n_k} \right\| \geqq 1 \quad (k = 1, \cdots, N),$$

and thus $\|\Phi(\mu_1)\| \geqq \sum_{k=1}^{N} \|(\Phi(\mu_1))_{n_k}\| \geqq N$. From this we have $\|\Phi\| \geqq \|\Phi(\mu_1)\| / \|\mu_1\| \geqq \sqrt{N}$, whic contradicts to the choice of N.

CASE II. Assume next $\bigcup_{k=1}^{\infty} A_k$ is a finite set. Then there exist a strictly increasing sequence $n_1, \cdots, n_N$ of positive integers and $r_1, \cdots, r_l \in \hat{G}$ such that $A_{n_1} = \cdots = A_{n_N} = \{r_1, \cdots, r_l\}$.

Let $\mu_2 \in L^1(G)$ be such that $\|\mu_2\| \leqq 1$, $\hat{\mu}_2(r_1) = 1$ and $\hat{\mu}_2(r_2) = \cdots = \hat{\mu}_2(r_l) = 0$. In the same way as the Case I, we have for this μ_2,

$$\left\|\left(\Phi(\mu_2)\right)_{n_k}\right\|=\left\|\left(\Phi(\mu_2)+\sum_{i=1}^{n_k-1}\varepsilon_i^{(n_k)}h_i\Phi(\mu_2)\right)_{n_k}\right\|\geqq 1 \qquad (k=1,\cdots,N),$$

and we have $\|\Phi(\mu_2)\|\geqq N$. Hence we have $\|\Phi\|\geqq\|\Phi(\mu_2)\|/\|\mu_2\|\geqq N$, which again contradicts to the choice of N. This proves (7).

To complete the proof of theorem 2, suppose that theorem 2 is false. Then there exist $k(1\leqq k\leqq m)$ and $\nu_1\in L^1(G)$ such that

$$\begin{aligned}&\left(\Phi(\nu_1)\right)_k\in \operatorname{Rad} L^1\left(G(h_k)\right)\backslash L^1\left(G(h_k)\right),\\ &\left(\Phi(\nu_1)\right)_i\in L^1\left(G(h_i)\right) \qquad (h_i<h_k).\end{aligned} \tag{10}$$

If $h_i<h_k$ and if δ_{e_k} denotes the unit mass at the unit e_k of the group $K(h_k)$, $L^1(G(h_i))\to M(K(h_k)):\mu\mapsto\mu*\delta_{e_k}$ and $L^1(G)\to M(K(h_k):\nu\to(h_k(\Phi(\nu)))*\delta_{e_k}$ are homomorphisms, and hence both $\mu*\delta_{e_k}$ and $(h_k(\Phi(\nu)))*\delta_{e_k}$ belong to the spine of $M(K(h_k))$ (cf. [4]). Therefore

$$\left(\Phi(\nu_1)\right)_k=\left(\Phi(\nu_1)\right)_k*\delta_{e_k}=\left(h_k\Phi(\nu_1)\right)*\delta_{e_k}-\sum_{h_i<h_k}\left(\Phi(\nu_1)\right)_i*\delta_{e_k}$$

belongs to the spine of $M(G(h_k))$, which contradict to (10). This completes the proof of theorem 2.

REMARK 2. In the proof of theorem 2, the only properties which we required to $L^1(G)$ were that $L^1(G)$ is a commutative symmetric Banach algebra which satisfies lemma 6. In other words, theorem 2 remains true if we replace $L^1(G)$ with a commutative symmetric Banach algebra A which satisfies the following propery (**).

(**). There exists $c>0$ such that if $r_1,\cdots,r_N$ are a finite mimber of elements of the maximal ideal space Δ_A of A, and if W is a compact subset of Δ_A which contains $\{r_1,\cdots,r_N\}$ in the interior, then there exists $a\in A$ which satisfies

$$\|a\|\leqq c\sqrt{N}, \qquad \hat{a}(r)=\begin{cases}1 : r\in\{r_1,\cdots,r_N\}\\ 0 : r\notin W.\end{cases}$$

COROLLARY 11. *Let A be a commutative symmetric Banach algebra which satisfies (**). If Φ is a homomorphism of A into $\mathfrak{M}$, there exist a finite number of critical points $h_1,\cdots,h_m\in\hat{S}^+$ such that $\Phi(A)\subset\sum_{i=1}^m L^1(G(h_i))$.*

COROLLARY 12. *Let $\mathfrak{M}$ be a commutative semi-simple symmetric convolution measure algebra which satisfies (**). Then there exist a finite number of critical points $h_1,\cdots,h_m\in\hat{S}^+$ such that $\mathfrak{M}=\sum_{i=1}^m L^1(G(h_i))$.*

PROOF. Let Φ be an identity map of $\mathfrak{M}$ into $\mathfrak{M}$. Then by corollary

11, there exist a finite number of critical points $h_1, \cdots, h_m \in \hat{S}^+$ such that

$$\mathfrak{M} = \Phi(A) \subset \sum_{i=1}^{m} L^1\big(G(h_i)\big) \subset \mathfrak{M}.$$

This completes the proof.

Corollary 13. *Let A be a commutative symmetric Banach algebra which satisfies* (**), *and let S be a commutative discrete semi-group such that $\hat{S}$, the set of all the bounded semi-characters, separates points of S. Then if Φ is a homomorphism of A into $M(S)$, there exist subgroups $G_1, \cdots, G_m$ of S such that $\Phi(A) \subset M(G_1 \cup \cdots \cup G_m)$.*

Proof. Since $\hat{S}$ separates points of S, $M(S)$ is semi-simple (cf. [3]). The structure semi-group of $M(S)$ is the Bohr compactification $\bar{S}$ of S, and the Taylor's representation of $M(S)$ is given by

$$i_\alpha : M(S) \longrightarrow M(\bar{S}) : \mu \longmapsto \mu \circ \alpha^{-1} \tag{11}$$

where α is the canonical injection of S into $\bar{S}$ (cf. [7] § 4. 1).
By corollary 11, there exist a finite number of critical points $h_1, \cdots, h_m \in \hat{\bar{S}}^+$ such that

$$i_\alpha\big(\Phi(A)\big) \subset \sum_{i=1}^{m} L^1\big(G(h_i)\big) \subset \sum_{i=1}^{m} M\big(K(h_i)\big) \tag{12}$$

By (11) and (12), we have $\Phi(A) \subset M(\alpha^{-1}(K(h_1)) \cup \cdots \cup \alpha^{-1}(K(h_m)))$, and $\alpha^{-1}(K(h_i))$ $(i=1, \cdots, m)$ is a subgroup of S. This completes the proof.

§2. A characterization of the dual maps.

At the point of view of theorem 2, we restrict ourselves to the case:

$$\hat{S}^+ = \{h_1, \cdots, h_m\}, \qquad \mathfrak{M} = \sum_{i=1}^{m} L^1\big(G(h_i)\big) \text{ and } \hat{S} = \bigcup_{i=1}^{m} \Gamma_{h_i}.$$

We can suppose without loss of generality that h_i is maximal in $\{h_1, \cdots, h_i\}$ $(i=1, \cdots, m)$.

Let $\hat{H}$ be a LCA group. A subset E of $\hat{H}$ is called an open coset if E is a coset of some open subgroup of $\hat{H}$. The cost ring of $\hat{H}$ means the ring generated by all the open cosets of $\hat{H}$. A map α of an open coset K of $\hat{H}$ into $\hat{G}$ is called affine if α satisfies

$$\alpha(rr'r''^{-1}) = \alpha(r)\,\alpha(r')\,\alpha(r'')^{-1} \qquad (r,\ r',\ r'' \in K).$$

Definition 2. *Let α be a map of $\hat{H}$ into $\hat{G} \cup \{0\}$. Suppose that:*

(1) *$\alpha^{-1}(\hat{G})$ is a finite disjoint union of elements $E_1, \cdots, E_n$ of the coset ring of $\hat{H}$,*

(2) *for each* $l(1 \leqq l \leqq n)$, *there exist an open coset* K_l *and a map* α_l *of* K_l *into* $\hat{G}$ *such that* $E_l \subset K_l$ *and* α_l *is continuous affine with* $\alpha_l | E_l = \alpha | E_l$.

Then such α *is called a piecewise affine map.*

DEFINITION 3. *Let* X *be a topological space and let* α *be a map of* X *into* $\hat{G} \cup \{0\}$. *If* $\alpha(X) \subset \hat{G}$, *we call* α *a k-map if the inverse image of a compact set is also compact. If* $\alpha(X) = \{0\}$, α *will be called a trivial map.*

DEFINITION 4. *Let* Y *be a subset of a set* X, *and let* α *be a map of* Y *into* $\hat{G}$. *A trivial extension* α^* *of* α *to* X *is the map of* X *into* $\hat{G} \cup \{0\}$ *such that*

$$\alpha^*(x) = \begin{cases} \alpha(x) & : x \in Y, \\ 0 & : x \notin Y, \end{cases}$$

Let H be the dual group of $\hat{H}$. We consider $\hat{H}$ an open subset of the maximal ideal space $\Delta_{M(H)}$ of the measure algebra $M(H)$. If $\mathfrak{M} = M(H)$, the following theorem determines all the homomorphisms of $L^1(G)$ into $\mathfrak{M}$ by characterizing the dual maps restricted to $\hat{H}$.

THEOREM 14 (Cohen). *Let* α *be a map of* $\hat{H}$ *into* $\hat{G} \cup \{0\}$.

(a). α *is the restriction to* $\hat{H}$ *of the dual map of a homomorphism of* $L^1(G)$ *into* $M(H)$ *if and only if* α *is piecewise affine.*

(b). α *is the dual map of a homomorphism of* $L^1(G)$ *into* $L^1(H)$ *if and only if* α *is piecewise affine and* $\alpha | \alpha^{-1}(\hat{G})$ *is a k-map.*

DEFINITION 5. *Let* $J(\Gamma_{h_i})$ *denote the coset ring of* $\Gamma_{h_i} (1 \leqq i \leqq m)$. *Suppose* $J(\Gamma_{h_i}) \ni E$ *has a representation of the form*

$$E = r_0 H_0 \setminus \bigcup_{j=1}^{n} r_j H_j \tag{13}$$

with i) $\{r_0, \cdots, r_n\} \subset \Gamma_{h_i}$, ii) $H_0, \cdots, H_n$ *is a set of open subgroups of* Γ_{h_i}, iii) $H_0 / H_j \cap H_0$ *is an infinite group* $(j = 1, \cdots, n)$.

Such E *will be called a canonical element of* $J(\Gamma_{h_i})$.

LEMMA 15. *Every non-void element of* $J(\Gamma_{h_i})$ $(1 \leqq i \leqq m)$ *can be represented as a finite disjoint union of canonical elements of* $J(\Gamma_{h_i})$.

PROOF, Let E be a non-void element of $J(\Gamma_{h_i})$ and let χ_E be the characteristic function of E. It is easy to see from the definition of the coset ring that χ_E has a representation of the form

$$\chi_E = \sum_{j=1}^{n} a_j \chi_{r_j H_j} \tag{14}$$

with i) $a_j \in \boldsymbol{Z}$ and $r_j \in \Gamma_{h_i}$, ii) H_j is an open subgroup of Γ_{h_i}, iii) if $H_j = H_{j'}$, then we have $r_j r_{j'}^{-1} \notin H_j$.

Further, by dividing $r_jH_j(1\leqq j\leqq n)$ into the cosets of a subgroup of H_j if necessary, we can assume without loss of generality that $\mathfrak{H}=\{r_iH_i|i=1,\cdots,n\}$ satisfies the following: iv) $r_jH_j\cap r_{j'}H_{j'}$ is a finite desjoint union of elements in $\mathfrak{H}$, and if $H_j\not\subset H_{j'}$ we have $^{\#}(H_j/H_j\cap H_{j'})=\infty$, for $1\leqq j,\ j'\leqq n$. By rearranging the sequence $r_1H_1,\cdots,r_nH_n$ if necessary, we can suppose H_n is maximal in $\{H_1,\cdots,H_n\}$. Then $r_nH_n\not\subset\bigcap_{j=1}^{n-1}r_jH_j$ is obvious, and from (14) we have $a_n=0$ or $a_n=1$. If $a_n=0$, the representation

$$\chi_E=\sum_{j=1}^{n-1}a_j\chi_{r_jH_j} \tag{15}$$

satisfies i)$\sim$iv) above. If $a_n=1$, $F=r_nH_n\backslash(\bigcup_{j=1}^{n-1}r_jH_j)$ is a canonical element of $J(\Gamma_{h_i})$ and

$$\chi_{E,F}=\chi_E-\chi_F=\sum_{j=1}^{n-1}a'_j\chi_{r_jH_j}\qquad\text{for some } a'_j\in\boldsymbol{Z} \tag{16}$$

If $n-1\neq 0$, we repeat, using (15) or (16), the same process as before, and n times repetition of this process will give the conclusion of lemma 15.

DEFINITION 6. *For each* $\nu\in M(G(h_i))$ $(1\leqq i\leqq m)$, *we decompose* ν *as*

$$\nu=\nu'+\nu'',\qquad \nu'\in L^1\big(G(h_i)\big),\qquad \nu''\in L^1\big(G(h_i)\big)^{\perp},$$

and define the projection P_{h_i} *by*

$$P_{h_i}:M\big(G(h_i)\big)\longrightarrow L^1\big(G(h_i)\big)^{\perp}:\nu\longmapsto\nu''.$$

LEMMA 16. *Let* $E\in J(\Gamma_{h_i})$ $(1\leqq i\leqq m)$ *be canonical, and let* K *be an open coset of* Γ_{h_i} *which contains* E. *Suppose that* α *is a continuous affine map of* K *into* $\hat{G}$ *and that* Ψ *is a homomorphism of* $L^1(G)$ *into* $M(G(h_i))$ *such that the dual map* ϕ *of* Ψ *satisfies*

$$\phi(r)=\begin{cases}0 & : r\in\Gamma_{h_i},\quad r\notin E,\\ \alpha(r) & : r\in E.\end{cases}$$

Then, if $\phi|E$ *is not a k-map, we have*

$$\widehat{P_{h_i}\big(\Psi(\mu)\big)}(r)=\widehat{\Psi(\mu)}(r)\qquad\big(r\in E,\ \mu\in L^1(G)\big).$$

PROOF. Let ν be the idempotent of $M(G(h_i))$ such that the Fourier-Stieltjes transform of ν is χ_E. Since E is canonical, we can represent ν in the form

$$\nu=r_0\rho_{H_0^{\perp}}+\sum_{j=1}^{n}a_jr_j\rho_{H_j^{\perp}},$$

with i) $\{r_0, \cdots, r_n\} \in \Gamma_{h_i}$, ii) $a_j \in \boldsymbol{Z}$ and H_j is an open subgroup of Γ_{h_i}, iii) $\rho_{H_j^\perp}$ is the normalized Haar measure of the annihilator $H_j^\perp$ of H_j in $G(h_i)$, iv) $r_0 H_0 \supset r_j H_j$ and H_0/H_j is an infinite group, v) $r_j H_j \cap E = \phi$ $(j=1, \cdots, n)$.

Let Ψ' be a homomorphism of $L^1(G)$ into $M(G(h_i))$ such that the dual map ψ' of Ψ' satisfies

$$\psi'(r) = \begin{cases} 0 & : r \in \Gamma_{h_i}, \quad r \notin r_0 H_0, \\ \alpha(r) & : r \in r_0 H_0, \end{cases}$$

then we have $\Psi'(\mu) * \nu = \Psi(\mu)$ $(\mu \in L^1(G))$. Since $\psi'|E$ is not a k-map, we have $P_{h_i}(\Psi'(\mu) * r_0 \rho_{H_0^\perp}) = \Psi'(\mu)$ and $P_{h_i}(\Psi'(\mu) * r_j \rho_{H_j^\perp})$ is either $\Psi'(\mu) * r_j \rho_{H_j^\perp}$ or $0 (1 \leqq j \leqq n)$ (cf. [4]). Therefore we get from v) above

$$\begin{aligned} \widehat{P_{h_i}\big(\Psi(\mu)\big)}(r) &= \widehat{P_{h_i}\big(\Psi'(\mu)*\nu\big)}(r) \\ &= \widehat{P_{h_i}\big(\Psi'(\mu)*r_0\rho_{H_j^\perp}\big)}(r) + \sum_{j=1}^{n} a_j \widehat{P_{h_i}\big(r_j \rho_{H_j^\perp} * \Psi'(\mu)\big)}(r) \\ &= \widehat{\Psi'(\mu)}(r) = \widehat{\Psi(\mu)}(r) \qquad \big(r \in E,\ \mu \in L^1(G)\big) \end{aligned} \tag{17}$$

This completes the proof.

DEFINITION 7. *Let α be a map of $\hat{S} = \bigcup_{i=1}^{m} \Gamma_{h_i}$ into $\hat{G} \cup \{0\}$. We call a non-void subset E of $\hat{S}$ a α-admissible set (in abbr. α-set) if E is a canonical element of the coset ring of some Γ_{h_i} and that either $\alpha|E$ is trivial or there exist an open coset K of Γ_{h_i} and a continuous affine map α' of K into $\hat{G}$ such that $\alpha|E = \alpha' E$. E is called a (k, α)-set if E is a α-set and that $\alpha|E$ is a non-trivial k-map.*

DEFINITION 8. *If $h_j < h_i$, we denote by η_j^i the continuous homomorphism*

$$\Gamma_{h_i} \longrightarrow \Gamma_{h_j} : f \longmapsto f h_j .$$

DEFINITION 9. *Let α be a map of $\hat{S}$ into $\hat{G} \cup \{0\}$, and let*

$$\mathfrak{A} = \left\{ E_{i;l} \mid E_{i;l} \subset \Gamma_{h_i},\ l = 1, \cdots, n_i,\ i = 1, \cdots, m \right\}$$

be a collection of subsets of $\hat{S}$. $\mathfrak{A}$ will be called a finite disjoint system of α-sets if $\mathfrak{A}$ satisfies the following conditions.

i) $E_{i;l}$ *is a α-set* $\quad (l = 1, \cdots, n_i,\ i = 1, \cdots, m)$,

ii) $E_{i;l} \cap E_{i';l'} = \phi \quad$ *($i \neq i'$ ar $l \neq l'$)*,

iii) $\hat{S} = \bigcup_{i=1}^{m} \bigcup_{l=1}^{n_i} E_{i;l}$

iv) *For each* i, j *and* l *with* $h_j < h_i$ *and* $1 \leqq l \leqq n_i$, *we have*

$\eta_j^i(E_{i;l}) \subset E_{j;l(j,i)}$ *for some* $l(j, i) \in \{1, \cdots, n_j\}$.

DEFINITION 10. *Let* α *be a map of* $\hat{S}$ *into* $\hat{G} \cup \{0\}$, *and let*

$$\mathfrak{A} = \left\{E_{i;l} | E_{i;l} \subset \Gamma_{h_i},\ l = 1, \cdots, n_i,\ i = 1, \cdots, m\right\}$$

be a finite desjoint system of α*-sets. For each* i, j *and* l *with* $h_j < h_i$ *and* $1 \leqq l \leqq n_i$, *we put*

$$\mathfrak{T}(j, i\,; l) = \left\{h_k | h_j \leqq h_k \leqq h_i \text{ and } \alpha \circ \eta_j^k | E_{k;l(k,i)} \text{ is a non-trivial } k\text{-map}\right\}$$

DEFINITION 11. *Let* α *be a map of* $\hat{S}$ *into* $\hat{G} \cup \{0\}$, *and suppose that*

$$\mathfrak{A} = \left\{E_{i;l} | E_{i;l} \subset \Gamma_{h_i},\ l = 1, \cdots, n_i,\ i = 1, \cdots, m\right\}$$

be a finite disjoint system of α*-sets. Suppose moreover* $\mathfrak{A}$ *satisfies the following conditions.*

(a) *If* $E_{i;l}$ *is a* (k, α)*-set or* $\alpha | E_{i;l}$ *is trivial, and if* $h_j < h_i$ *such that* $E_{j;l(j,i)}$ *is a* (k, α)*-set we have* $^{\#}\mathfrak{T}(j, i\,; l) \geqq 2$ ($^{\#}A$ *denotes the cardinal number of* A).

(b) *If* $E_{i;l}$ *is not a* (k, α)*-set and that* $\alpha | E_{i;l}$ *is non-trivial, then there exists one and only one* j *such that* $h_j < h_i$ *and* $E_{j;l(j,i)}$ *is a* (k, α)*-set with* $^{\#}\mathfrak{T}(j, i\,; l) = 1$. *Moreover in this case, we have* $\alpha | E_{i;l} = \alpha \circ \eta_j^i | E_{i;l}$.

We call such a system $\mathfrak{A}$ *a compatible system of* α*-sets.*

DEFINITION 12. *Let* α *be a map of* $\hat{S}$ *into* $\hat{G} \cup \{0\}$, *and suppose that*

$$\mathfrak{A} = \left\{E_{i;l} | E_{i;l} \subset \Gamma_{h_i},\ l = 1, \cdots, n_i,\ i = 1, \cdots, m\right\}$$

is a finite disjoint system of α*-sets. If* $E_{i;l} \in \mathfrak{A}$ *and* $h_j < h_i$, *we denote by* $\Phi_{i;l}$ *(resp.* $\Phi_{j,i;l}$*) the homomorphism of* $L^1(G)$ *into* $M(G(h_i))$ *such that the trivial extension of* $\alpha | E_{i,l}$ *(resp.* $\alpha \circ \eta_j^i | E_{i,l}$*) to* Γ_{h_i} *is the restriction to* Γ_{h_i} *of the dual map of* $\Phi_{i;l}$ *(resp.* $\Phi_{j,i;l}$*).* $\nu_{i,l}$ *is the idempoint of* $M(G(h_i))$ *such that the Fourier-Stieltjes transform of* $\nu_{i;l}$ *is the characteristic function of* $E_{i;l}$. *With these notations we have*

$$\Phi_{j;l(j,i)}(\mu) * \nu_{i;l} = \Phi_{j,i;l}(\mu) \qquad \left(\mu \in L^1(G)\right).$$

THEOREM 17. *A map* α *of* $\hat{S}$ *into* $\hat{G} \cup \{0\}$ *is the dual map of a homomorphism* Φ *of* $L^1(G)$ *into* $\mathfrak{M}$ *if and only if there exists a compatible system of* α*-sets.*

LEMMA 18. *If* α *is the dual map of a homomorphism* Φ *of* $L^1(G)$ *into* $\mathfrak{M}$, *there exists a finite disjoint system of* α*-sets.*

PROOF. Let $\Phi_i (1 \leqq i \leqq m)$ be the homomorphism of $L^1(G)$ into $M(G(h_i))$ defined by $L^1(G) \to M(G(h_i)) : \mu \mapsto (h_i \Phi(\mu)) * \delta_{e_i}$, where δ_{e_i} is the unit mass at the unit e_i of $K(h_i)$. Then the restriction to Γ_{h_i} of the dual map of Φ_i is equal to $\alpha | \Gamma_{h_i}$, and by theorem 14 a) and lemma 15, there exist a finite disjoint system of $\alpha | \Gamma_{h_i}$-sets $\mathfrak{A}^{(i)} = \{E_{i;l} | E_{i;l} \subset \Gamma_{h_i},\ l = 1, \cdots, n_i\}$ $(i = 1, \cdots, m)$. If $h_j < h_i$ and $E \in J(\Gamma_{h_j})$, $(\eta_j^i)^{-1}(E)$ is an element of $J(\Gamma_{h_i})$. So, dividing elements of $\mathfrak{A}^{(i)}$ $(i = 1, \cdots, m)$ if necessary, we can suppose without loss of genefality that $\mathfrak{A} = \bigcup_{i=1}^{m} \mathfrak{A}^{(i)}$ satisfies the condition iv) of definition 9. This completes the proof.

LEMMA 19. *Let α be the dual map of a homomorphism Φ of $L^1(G)$ into $\mathfrak{M}$, and suppose*

$$\mathfrak{A} = \left\{E_{i;l} | E_{i;l} \subset \Gamma_{h_i},\ l = 1, \cdots, n_i,\ i = 1, \cdots, m\right\}$$

is a finite disjoint system of α-sets. If $h_j < h_i$ and $1 \leqq l \leqq n_i$ such that $E_{j;l(j,i)}$ is a (k, α)-set, then $\mathfrak{T}(j, i; l)$ has an unique maximal element.

PROOF. If $h_u, h_v \in \mathfrak{T}(j, i; l)$, there exists $h_w \in \hat{S}^+$ such that (cf. [7])

$$L^1\big(G(h_u)\big) * L^1\big(G(h_v)\big) \subset L^1\big(G(h_w)\big) \tag{18}$$

By definition 12, we have for each $\mu \in L^1(G)$,

$$\begin{aligned} \Phi_{j,u;l(u,i)}(\mu) * \nu_{w;l(w,i)} &= \Phi_{j,w;l(w,i)}(\mu) \\ \Phi_{j,v;l(v,i)}(\mu) * \nu_{w;l(w,i)} &= \Phi_{j,w;l(w,i)}(\mu) \end{aligned} \tag{19}$$

By (18) and (19), we have

$$\begin{aligned} &\Phi_{j,w;l(w,i)}(\mu) * \Phi_{j,w;l(w,i)}(\mu) \\ &= \Phi_{j,u;l(i,u)}(\mu) * \Phi_{j,v;l(v,i)}(\mu) * \nu_{w;l(w,i)} \in L^1\big(G(h_w)\big) \end{aligned} \tag{20}$$

By theorem 14 b), (20) shows that $\alpha \circ \eta_j^w | E_{w;l(w,i)}$ is a k-map. Since $\mathfrak{T}(j, i; l)$ is a finite set, this completes the proof of lemma 19.

PROOF OF THEOREM 17. Suppose that α is the dual map of a homomorphism Φ of $L^1(G)$ into $\mathfrak{M}$. By lemma 18, there exists a finite disjoint system of α-sets

$$\mathfrak{A} = \left\{E_{i;l} | E_{i;l} \subset \Gamma_{h_i},\ l = 1, \cdots, n_i,\ i = 1, \cdots, m\right\}$$

If $m = 1$, $\mathfrak{A}$ is a compatible system of α-sets by theorem 14 b). If $m > 1$, $\{E_{1;1}, \cdots, E_{1;n_1}\}$ is a compatible system of $\alpha | \Gamma_{h_1}$-sets by theorem 14 b) again. If we put

$$\varepsilon(1, 1 ; l) = 1 \qquad (l = 1, \cdots, n_1),$$
$$\mathfrak{A}_i = \{E_{j;l} | h_j \leqq h_i,\ 1 \leqq l \leqq n_j\} \qquad (i = 1, \cdots, m),$$

it is clear that the following (21) holds with $k=1$.

$$\left.\begin{array}{l}
\text{a) } \mathfrak{A}_i \text{ is a compatible system of } \alpha|h_i\hat{\mathfrak{S}}\text{-sets } (i=1, \cdots, k). \\
\text{b) } \{\varepsilon(j, i ; l) | h_j \leqq h_i,\ i \leqq k,\ l=1, \cdots, n_i\} \text{ is a set of integers which satisfies the following conditions.} \\
\quad \text{i) If } i \leqq k \text{ and } 1 \leqq l \leqq n_i, \text{ then we have} \\
\qquad \varepsilon(i, i ; l) = \begin{cases} 1 : \alpha|E_{i;l} \text{ is either trivial or } k\text{-map} \\ 0 : \text{otherwise}. \end{cases} \\
\quad \text{ii) If } h_j < h_i \text{ and } 1 \leqq l \leqq n_i, \text{ and if } E_{j;l(j,i)} \text{ is not a } (k, \alpha)\text{-set or } \alpha \circ \eta_j^i | E_{i,l} \text{ is not a } k\text{-map, then } \varepsilon(j, i ; l) = 0. \\
\quad \text{iii) If } h_j < h_i \text{ and } 1 \leqq l \leqq n_i, \text{ and if } E_{j;l(j,i)} \text{ is a } (k, \alpha)\text{-set and that } \alpha \circ \eta_j^i | E_{i;l} \text{ is a } k\text{-map, then we have} \\
\qquad \sum_{h_u \in \mathfrak{T}(j,i;l)} \varepsilon\big(j, u ; l(u, i)\big) = 0. \\
\text{c) If } i \leqq k, \text{ we have} \\
h_i \Phi(\mu) = \sum_{h_j \leqq h_i} \sum_{h_j \leqq h_u \leqq h_i} \sum_{l=1}^{n_u} \varepsilon(j, u ; l)\, \Phi_{j,u;l}(\mu) \qquad \big(\mu \in L^1(G)\big).
\end{array}\right\} \quad (21)$$

We suppose that (21) holds with $k=p(<m)$, and we will show that (21) also holds with $k=p+1$ by defining an appropriate set of integers

$$\big\{\varepsilon(j, p+1 ; l) \big| h_j \leqq h_{p+1},\ 1 \leqq l \leqq n_{p+1}\big\}.$$

To prove (21) a) with $k=p+1$, fix an integer $1 \leqq l \leqq n_{p+1}$ arbitrary, and put

$A(p+1 ; l) = \{j | h_j < h_{p+1},\ E_{j;l(j,p+1)}$ is a (k, α)-set and ${}^{\#}\mathfrak{T}(j, p+1 ; l) = 1\}$.
If $h_j < h_{p+1}$ and $E_{j;l(j,p+1)}$ is a (k, α)-set, and if we put

$$\Psi_{j;l}(\mu) = \sum_{h_j \leqq h_u < h_{p+1}} \varepsilon\big(j, u ; l(u, p+1)\big)\, \Phi_{j,u;l(u,p+1)}(\mu) * \nu_{p+1;l} \qquad \big(\mu \in L^1(G)\big), \tag{22}$$

we have from lemma 16, lemma 19 and (21) with $k=p$ that the following i), ii) and iii) hold for each $\mu \in L^1(G)$ and $r \in E_{p+1;l}$.

i) If ${}^{\#}\mathfrak{T}(j, p+1 ; l) \geqq 2$ and $h_{p+1} \notin \mathfrak{T}(j, p+1 ; l)$ then $\Psi_{j;l}(\mu) = 0$.

ii) If ${}^{\#}\mathfrak{T}(j, p+1 ; l) \geqq 2$ and $h_{p+1} \in \mathfrak{T}(j, p+1 ; l)$, then

$$\Psi_{j;l}(\mu) \in L^1\big(G(h_{p+1})\big) \tag{23}$$

iii) If ${}^{\#}\mathfrak{T}(j,p+1\,;l)=1$, then $\widehat{P_{h_{p+1}}\big(\Psi_{j;l}(\mu)\big)}(r)=\widehat{\Phi_{j,p+1;l}(\mu)}(r)$
$=\hat{\mu}\Big(\alpha\big(\eta_j^{p+1}(r)\big)\Big)$.

From (21) with $k=p$, (22) and (23) we get

$$0=P_{h_{p+1}}\Big(\Big(h_{p+1}\Phi(\mu)-\sum_{h_j<h_{p+1}}\sum_{h_j\leqq h_u<h_{p+1}}\varepsilon\big(j,u\,;l(u,p+1)\big)$$
$$\Phi_{j,u;l(u,p+1)}(\mu)\Big)*\nu_{p+1;l}\Big)=P_{h_{p+1}}\big(\Phi_{p+1;l}(\mu)\big)-\sum_{j\in A(p+1;l)}P_{h_{p+1}}\big(\Psi_{j;l}(\mu)\big)$$
$$\big(\mu\in L^1(G)\big) \tag{24}$$

Suppose first, $E_{p+1;l}$ is a (k,α)-set or $\alpha|E_{p+1;l}$ is trivial that is $\Phi_{p+1;l}(\mu)\in L^1(G(h_{p+1}))$ for each $\mu\in L^1(G)$. If we choose $r_0\in E_{p+1;l}$ and $\mu_0\in L^1(G)$ such that $\hat{\mu}_0(\alpha(\eta_j^{p+1}(r_0)))=1$ ($j\in A(p+1\,;l)$), then we have from (24) that $A(p+1;l)=\phi$, and definition 11 (a) holds. Next, suppose $E_{p+1;l}$ is a non (k,α)-set and that $\alpha|E_{p+1;l}$ is non-trivial, then from lemma 16 and (21) (b) iii) with $k=p$, (24) becomes

$$0=\widehat{P_{h_{p+1}}\big(\Phi_{p+1;l}(\mu)\big)}(f)-\sum_{j\in A(p+1;l)}\widehat{P_{h_{p+1}}\big(\Psi_{j;l}(\mu)\big)}(f)$$
$$=\hat{\mu}\big(\alpha(f)\big)-\sum_{j\in A(p+1;l)}\hat{\mu}\Big(\alpha\big(\eta_j^{p+1}(f)\big)\Big)\qquad\big(f\in E_{p+1;l},\ \mu\in L^1(G)\big). \tag{25}$$

From (25), it follows easily that ${}^{\#}A(p+1\,;l)=1$ and that (b) of definition 11 holds. Thus we have proved that $\mathfrak{A}_{p+1}$ is a compatible system of $\alpha|h_{p+1}\hat{S}$-sets. It is easy to define integers $\{\varepsilon(j,p+1\,;l)|h_j\leqq h_{p+1},\ l=1,\cdots,n_{p+1}\}$ so that b) and c) of (21) hold with $k=p+1$.

From above, we can conclude by induction that $\mathfrak{A}=\mathfrak{A}_m$ is a compatible system of α-sets.

Conversely, let $\mathfrak{A}=\{E_{i;l}|E_{i;l}\subset\Gamma_{h_i},\ l=1,\cdots,n_i,\ i=1,\cdots,m\}$ be a compatible system of α-sets. If we put $\varepsilon(1,1\,;l)=1$ ($l=1,\cdots,n_1$), it is clear that (21) b) holds with $k=1$. Suppose that $p<m$ and we have already defined $\{\varepsilon(j,i\,;l)|h_j\leqq h_i,\ i\leqq p,\ l=1,\cdots,n_i\}$ so that b) of (21) holds with $k=p$. Since $\mathfrak{A}$ is a compatible system of α-sets, we can define a set of integers $\{\varepsilon(j,p+1\,;l)|h_j\leqq h_{p+1},\ l=1,\cdots,n_{p+1}\}$, by definition 11, so that b) of (21) holds with $k=p+1$. Thus we can define by induction a set integers $\{\varepsilon(j,i\,;l)|h_j\leqq h_i,\ l=1,\cdots,n_i,\ i=1,\cdots,m\}$ so that b) of (21) holds with $k=1,\cdots,m$. Therefore, if we put

$$\Phi(\mu)=\sum_{i=1}^{m}\sum_{h_j\leqq h_i}\sum_{l=1}^{n_i}\varepsilon(j,i\,;l)\,\Phi_{j,i;l}(\mu)\qquad\big(\mu\in L^1(G)\big),$$

we get

$$\widehat{\Phi(\mu)}|E_{i;l}=\hat{\mu}\circ\alpha|E_{i;l} \qquad (1\leqq i\leqq m,\ l=1,\cdots,n_i) \tag{26}$$

It is easy to see from (26) that Φ is a homomorphism of $L^1(G)$ into $\mathfrak{M}$ with the dual map α, and this completes the proof of theorem 17.

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