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The $A(p, q)$ algebras and singular measures with Fourier transforms in $L(2, q)$, $q>2$

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0. Introduction.

The Banach algebras $A^{p}(G)$ of functions with Fourier transforms in $L^{p}(\hat{G})$ have been thoroughly studied since they were introduced about ten years ago by R. Larsen *et al.*. An excellent survey can be found in [7]. Recently L. Y. H. Yap [11] has proposed a generalization of the $A^{p}(G)$ algebras, namely, the algebras $A(p, q)(G)$ of integrable functions such that the Fourier transforms belong to the Lorentz space $L(p, q)(\hat{G})$. The $A(p, q)$ algebras do not seem to behave markedly different from the $A^{p}(G)$ algebras, but we shall provide proofs for the following, certainly not surprising, facts: The $A(p, q)(G)$ algebras are distinct for any admissible pair (p, q), and there exist singular measures with Fourier transforms in $L(2, q)$ for all $q>2$. This will generalize the corresponding results for the $A^{p}(G)$ algebras, see [8] and [10], and will answer a question by H. C. Lai [5].

1. Preliminaries.

Let G be a locally compact Abelian group with dual group $\hat{G}$. The spaces of p-integrable functions on G are denoted by $L^{p}(G)$, $1\leq p\leq\infty$, and $M(G)$ denotes the Banach space of bounded regular Borel measures. The spaces of continuous functions with compact support and continuous functions vanishing at infinity are denoted by $C^{c}(G)$ and $C^{0}(G)$, respectively. The pseudomeasures are denoted $P(G)$ $(P(G)\cong L^{\infty}(\hat{G}))$. For a measurable function f on a measure space (X, m), let λ_{f}: $[0, \infty)\rightarrow[0, \infty]$ be the *distribution function* of f, that is

$$\lambda_{f}(x)=m(t\,;\ |f(t)|>x)\,.$$

The nonincreasing rearrangement of f is defined by

$$f^{*}(t)=\inf\left\{x>0\,;\ \lambda_{f}(x)\leq t\right\}.$$

(with the convention $\inf\emptyset=\infty$).

The *Lorentz space* $L(p, q)(X)$ consists of all functions f such that $\|f\|_{(p,q)}^{*}<\infty$ where

$$\|f\|^*_{(p,q)}=\begin{cases}\left(\dfrac{q}{p}\displaystyle\int_0^\infty\left(t^{\frac{1}{p}}f^*(t)\right)^q\frac{dt}{t}\right)^{\frac{1}{q}}, & 0<p,\ q<\infty\\ \sup\limits_{t>0} t^{\frac{1}{p}}f^*(t), & 0<p\leq\infty,\ q=\infty\,.\end{cases} \tag{1.1}$$

We note that if $1\leq p\leq\infty$, $\|f\|^*_{(p,p)}=\|f\|_{L^p(X)}$. Moreover, if $1<p<\infty$ and $1\leq q\leq\infty$, then $L(p, q)$ can be equipped with a norm $\|\ \|_{(p,q)}$ equivalent to $\|\ \|^*_{(p,q)}$ which makes $L(p, q)$ into a Banach space ($\|\ \|^*_{(p,q)}$ is not generally a norm, but is easier to handle). We refer to Hunt [2] for further properties of $L(p, q)$ spaces. It will be convenient to introduce an ordering for the pairs (p, q): $(p_1, q_1)<(p_2, q_2)$ if i) $p_1<p_2$ or ii) $p_1=p_2$ and $q_1<q_2$. Following Yap [11], we now define

$$A(p, q)=\left\{f\in L^1(G)\,;\ \hat{f}\in L(p, q)(\hat{G})\right\}$$

and

$$B(p, q)=L^1(G)\cap L(p, q)(G)\,.$$

A *Segal algebra* S on a nondiscrete locally compact Abelian group is a dense subalgebra of $L^1(G)$ which is a Banach algebra under some norm $\|\ \|_S$ such that

i) *if* $f\in S$, *then* $f_x\in S$ *for all* $x\in G$ *where* $f_x(t)=f(t-x)$.
ii) $\|f\|_{L^1(G)}\leq\|f\|_S$ *for all* $f\in S$.
iii) $\|f_x\|_S=\|f\|_S$ *for all* $f\in S$, $x\in G$.
iv) $\|f_x-f_y\|_S\xrightarrow[x\to y]{}0$ *for all* $f\in S$.

PROPOSITION 1.1 *For* $1<p<\infty$ *and* $1\leq q<\infty$

i) $A(p, q)$ *is a Segal algebra with respect to the norm*

$$\|f\|_{A(p,q)}=\|f\|_{L^1(G)}+\|\hat{f}\|_{L(p,q)(\hat{G})}\,.$$

ii) $B(p, q)$ *is a Segal algebra with respect to some norm which is equivalent to*

$$\|f\|_{B(p,q)}=\|f\|_{L^1(G)}+\|f\|_{L(p,q)(G)}\,.$$

A complete proof can be found in [11].

2. Proper Inclusions of $A(p, q)$ and $B(p, q)$ Algebras.

Inclusions of $A(p, q)$ and $B(p, q)$ algebras are easily established. However, as in the case of the A^p-algebras [10], proper inclusions require some arguments. The following technical lemma establishes a sequence of functions analogous to the well known de la Vallee-Poussin kernel on a general nondiscrete locally compact Abelian group. Tewari and Gupta's result $A^p \subsetneqq A^q$ if $q>p$ will be an immediate consequence of Lemma 2.1 and this simplifies to some extent their proof [10].

LEMMA 2.1: *Let G be a nondiscrete locally campact Abelian group. Then there exists a sequence $\{f_i\}_1^\infty \subset L^1(G)$ such that* $\operatorname{supp} \hat{f}_i \subset K_i'$ *and* $\hat{f}_i(\xi)=1$ *for $\xi \in K_i$ where K_i and K_i' are compact sets in $\hat{G}$ and*

i) $\lim_{i\to\infty} m(K_i)=\infty$

ii) $m(K_i') \leq C_1 m(K_i) \qquad i=1, 2, \cdots$

iii) $0 \leq \hat{f}_i \leq 1, \qquad i=1, 2, \cdots$

iv) $\sup_i \|f_i\|_{L^1(G)} \leq C_2$.

PROOF: The general structure theorem asserts that G is topologically isomorphic to $\boldsymbol{R}^n \times G_o$ where G_o contains an open compact subgroup J. Since G is nondiscrete, either $n>0$ or J is infinite. On $\boldsymbol{R}^1$ we prove the lemma by means of functions from the de la Vallee-Poussin kernel $\{V_\alpha\}_{\alpha>0}$. Indeed, $\|V_\alpha\|_{L^1(\boldsymbol{R})} \leq 3$ for all α and

$$\hat{V}_\alpha(\xi)=\begin{cases} 1 & \text{if } |\xi| \leq \alpha \\ 2\alpha-|\xi| & \text{if } \alpha<|\xi|<2\alpha \\ 0 & \text{if } |\xi| \geq 2\alpha \end{cases}$$

On $\boldsymbol{R}^n, n \geq 1$, the result follows with $C_1=2^n$ and $C_2=3$ if we choose

$$f_i(x_1, \cdots, x_n)=\prod_{k=1}^n V_i(x_k), \tag{2.1}$$

$$K_i=\prod_{k=1}^n [-i, i], \qquad K_i'=\prod_{k=1}^n [-2i, 2i]. \tag{2.2}$$

Let $J^\perp$ be the annihilator group of J in $\hat{G}_o$ and let χ_J be the characteristic function of J. Since $J^\perp \simeq \widehat{G_o/J}$ and J is open, $J^\perp$ is compact. Furthermore, $\hat{\chi}_J=m(J)\chi_{J^\perp}$. So, if $G \simeq \boldsymbol{R}^n \times G_o$ with $n \geq 1$, we let

$$f_i(x, y)=\prod_{k=1}^n V_i(x_k) \cdot \frac{\chi_J(y)}{m(J)}, \qquad x \in \boldsymbol{R}^n, \qquad y \in G_o.$$

$$K_i=\prod_{k=1}^n [-i, i] \times J^\perp$$

$$K_i'=\prod_{k=1}^n [-2i, 2i] \times J^\perp.$$

$i=1, 2, \cdots$.

If $n=0$ we first observe that it is enough to establish the lemma on J. Consider this for the moment as done. Extend the functions $\{f_i\}_1^\infty \subset L^1(J)$

to G by defining $\tilde{f}_i(x)=f_i(x)$ if $x\in J$, $\tilde{f}_i(x)=0$ if $x\notin J$. Then

$$\hat{\tilde{f}}_i(\xi)=\begin{cases} 1 & if \quad \xi+J^\perp\in K_i(\subset\hat{J}) \\ 0 & if \quad \xi+J^\perp\in K_i'(\subset\hat{J}). \end{cases}$$

But K_i and K_i' are finite sets and $m(J^\perp)<\infty$, and the lemma follows.

Let us turn to J and suppose first that J is totally disconnected. Then J has a neighbourhood basis at the identity consisting of open and compact subgroups. As J is not discrete, we are able to find a sequence $\{U_i\}_1^\infty$ of open and compact subgroups such that $m(U_i)\xrightarrow[i]{}0$. From the Parceval formula we infer that $m(U_i^\perp)\xrightarrow[i]{}\infty$ and the lemma follows with $f_i=m(U_i)^{-1}\chi_{U_i}$ and $K_i=K_i'=U_i^\perp$.

Finally, if J is not totally disconnected, $\hat{J}$ contains an element of infinite order, say g ([9], p. 47). Let $K_i=\{jg\}_{j=-i}^{i}$. We now invoke Theorem 2.6.1 in [9] with $V=C=K_i$. It follows that there exists a f_i such that $0\leq f_i\leq 1$, $\hat{f}_i(\xi)=1$ on K_i and 0 outside K_i', where $K_i'=K_i+K_i+K_i$. Moreover, $\|f_i\|_{L^1(G)}\leq\{m(K_i+K_i)/m(K_i)\}^{\frac{1}{2}}=\sqrt{2}$. This finishes the proof.

THEOREM 2.2: *Let G be a nondiscrete locally compact Abelian group and let $1<p_1,\ p_2<\infty$, $1\leq q_1,\ q_2\leq\infty$. Then*

i) $A(p_1, q_1)\subset A(p_2, q_2)$

ii) $B(p_2, q_1)\subset B(p_1, q_2)$

if $(p_1, q_1)\leq(p_2, q_2)$. Furthermore, the inclusions are proper unless $(p_1, q_1)=(p_2, q_2)$.

PROOF: Consider the $B(p, q)$ algebras. If $p_1=p_2=p$, then $B(p, q_1)\subset B(p, q_2)$ if $q_1<q_2$ as $L(p, q_1)\subset L(p, q_2)$ [2]. Fix $p_1<p_2$ and suppose that $f\in B(p_2, q_1)$. Since $f\in B(p_2, \infty)\cap L^1(G)$, there exist constants M_1 and M_2 such that $f^*(t)<M_1t^{-1/p_2}$ and $f^*(t)<M_1t^{-1}$. Hence, assuming $q_2<\infty$,

$$\int_0^\infty\left(t^{\frac{1}{p_1}}f(t)\right)^{q_2}\frac{dt}{t}\leq\int_0^1\left(t^{\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}M_1\right)^{q_2}\frac{dt}{t}+\int_1^\infty\left(t^{\frac{1}{p_1}}\frac{M_2}{t}\right)^{q_2}\frac{dt}{t}$$
$$<\infty,$$

which shows that f belongs to $B(p_1, q_2)$. Also, this proves the inclusion if $q_2=\infty$: $B(p_2, q_1)\subset B(p_1, q_0)\subset B(p_1, \infty)$, $q_0<\infty$. It is evident from the similar result for the $L(p, q)$ spaces that $B(p, q)\neq B(p', q')$ if $(p, q)\neq(p', q')$: All we have to observe is that G was assumed to be nondiscrete, and therefore it is possible to find a countable family of pairwise disjoint measurable subsets with finite measure which is contained in a fixed compact set of G. As every function in $L(p, q)$ with compact support is integrable, we may then proceed as in [12].

We turn to the $A(p, q)$ algebras. Arguing as above, $A(p, q_1)\subset A(p, q_2)$ if $q_1>q_2$. Let $p_1<p_2$ and fix $f\in A(p_1, q_1)$. Since $\hat{f}$ is bounded and belongs to $L(p_1, \infty)(\hat{G})$, there are constants M_1 and M_2 such that $\hat{f}^*(t)<M_1$ and $\hat{f}^*(t)<M_2 t^{-1/p_1}$. Thus, if $q_2<\infty$,

$$\int_0^\infty t^{\frac{q_2}{p_2}-1}\hat{f}^*(t)^{q_2}dt\leq\int_0^1 t^{\frac{q_2}{p_2}-1}M_1^{q_2}dt+\int_1^\infty t^{\left(\frac{1}{p_2}-\frac{1}{p_1}\right)q_2-1}M_2^{q_2}dt$$
$$<\infty,$$

and $f\in L(p_2, q_2)(\hat{G})$. The case $q_2=\infty$ is handled as above.

Fix $1<p, q<\infty$. By means of Lemma 2.1 we shall construct an f such that

$$f\in\bigcap_{r>q}A(p, r)\backslash A(p, q).$$

If $\{E_i\}_1^\infty$ is a pairwise disjoint family of measurable sets with finite measure and $\{c_k\}_1^\infty$ is a sequence of positive real numbers tending monotonically to zero, the nonincreasing rearrangement of $h=\sum_{i=1}^\infty c_i\chi_{E_i}$ is given by $h^*(t)=c_n$ if $a_{k-1}\leq t<a_k$, $k=1, 2, \cdots$ where $a_0=0$ and $a_k=\sum_{i=1}^k m(E_i)$. Consequently,

$$\|h\|_{(p,q)}^{*q}=\sum_{k=1}^\infty c_k^q(a_k^{q/p}-a_{k-1}^{q/p}).$$

See [2].

Start with f_1 from Lemma 2.1 and let $c_1=m(K_1)^{-\frac{1}{p}}$. Going if necessary to a subsequence which we again denote by $\{f_i\}_1^\infty$, we inductively choose c_i and f_i such that

$$\left(\frac{c_i}{c_{i-1}}\right)^b\leq\frac{1}{2}, \tag{2. 3}$$

$$c_i^q(a_i^{q/p}-a_{i-1}^{q/p})=\frac{1}{i}, \qquad i=1, 2, \cdots \tag{2. 4}$$

where $a_0=0$, $a_i=\sum_{k=1}^i m(K_k)$.

(Recall that we may find f_i such that K_i has an arbitrary large measure). Let $\{\gamma_i\}_1^\infty$ be a sequence of characters such that $\{K_i'-\gamma_i\}_1^\infty$ are pairwise disjoint and let

$$f=\sum_{i=1}^\infty c_i\gamma_i f_i.$$

The series converges in $L^1(G)$ since $c_i\leq\frac{1}{2^{i/q}}c_1$ according to (2. 3). We observe that

$$\sum_{i=1}^{\infty} c_i \chi_{K_i}(\xi) \leq \hat{f}(\xi) \leq \sum_{i=1}^{\infty} c_i \chi_{K'_i}(\xi)$$

and hence,

$$\left(\sum_{i=1}^{\infty} c_i \chi_{K_i}\right)^* \leq f^* \leq \left(\sum_{i=1}^{\infty} c_i \chi_{K'_i}\right)^* .$$

For simplicity, let $g = \sum_{i=1}^{\infty} c_i \chi_{K_i}$ and $b_o=0$, $b_k = \sum_{i=1}^{k} m(K'_i)$. From Lemma 2. 1 we conclude that $b_k \leq C_1 a_k$. If $t \in [b_{k-1}, b]$, then

$$g^*\left(\frac{t}{C_1}\right) \geq c_k = \left(\sum_{i=1}^{\infty} c_i \chi_{K'_i}\right)^*(t) .$$

Now,

$$\int_0^{\infty} t^{\frac{r}{p}-1} g^*\left(\frac{t}{C_1}\right)^r dt = C_1^{\frac{r}{p}} \int_0^{\infty} t^{\frac{r}{p}-1} g^*(t)^r dt$$

which shows that $\hat{f}$ belongs to $L(p, r)$ if and only if $g \in L(p, r)$. g does not belong to $L(p, q)$ as

$$\|g\|_{(p,q)}^{*q} = \sum_{i=1}^{\infty} c_i^q (a_i^{q/p} - a_{i-1}^{q/p}) = \sum_{i=1}^{\infty} \frac{1}{i} = \infty .$$

By the Mean Value Theorem,

$$a_i^{r/p} - a_{i-1}^{r/p} \leq \frac{r}{q} (a_i^{q/p} - a_{i-1}^{q/p}) \cdot a_i^{\frac{r}{p}-\frac{q}{p}} ,$$

and thus,

$$\begin{aligned} \|g\|_{(p,r)}^{*r} &= \sum_{i=1}^{\infty} c_i^r (a_i^{r/p} - a_{i-1}^{r/p}) \\ &\leq \frac{r}{q} \sum_{i=1}^{\infty} c_i^{r-q} c_i^q (a_i^{q/p} - a_{i-1}^{q/p})\, a_i^{\frac{r}{p}-\frac{q}{p}} \\ &= \frac{r}{q} \sum_{i=1}^{\infty} \frac{1}{i} (c_i^q a_i^{q/p})^{\frac{r}{q}-1} . \end{aligned} \tag{2. 5}$$

Since $\left(\frac{c_i}{c_s}\right)^q \leq 2^{-(i-s)}$ by (2. 3), we may estimate $c_i^q a_i^{q/p}$ as follows:

$$\begin{aligned} c_i^q a_i^{q/p} &= c_i^q \sum_{s=1}^{i} (a_s^{q/p} - a_{s-1}^{q/p}) \\ &= c_i^q \sum_{s=1}^{i} \frac{1}{s} c_s^{-q} \\ &\leq \sum_{s=1}^{i} \frac{1}{s} \frac{1}{2^{i-s}} \end{aligned}$$

$$\leq \sum_{s=1}^{[\sqrt{i}]} \frac{1}{s} \frac{1}{2^{i-s}} + \sum_{s=[\sqrt{i}+1]}^{i} \frac{1}{s} \frac{1}{2^{i-s}}$$
$$\leq \frac{2}{2^{i-[\sqrt{i}]}} + \frac{2}{\sqrt{i}+1}$$
$$\leq \frac{4}{\sqrt{i}}.$$

This may be inserted in (2. 5), and the theorem follows.

3. Singular Measures with Fourier Transforms in $L(2,q)$, $q>2$.

In [1] Hewitt and Zuckerman construct singular measures with absolutely continuous convolution squares by generalizing to compact Abelian groups the classical Riesz products. In this section we shall utilize their technique to construct a singular measure μ such that $\hat{\mu}\in L(2,q)(\hat{G})$ for all $q>2$. Of course, a measure ν such that $\hat{\nu}\in L(p,q)(\hat{G})$ for $(p,q)\leq(2.2)$ is absolutely continuous (with respect to Haar measure). This will answer a question by H. Ċ. Lai [5]. There is a very simple expression for the L^p-norm of the Fourier transform of a Riesz product, but no simple formula seems to be possible for the $L(p,q)$-norm. However, one can obtain precise estimates for special kinds of products.

Let G be a compact Abelian group. A finite set of characters $\{\gamma_1,\cdots,\gamma_m\}$ is said to be *dissociate* if it does not contain 1 and the equality

$$\gamma_1^{\varepsilon_1}\gamma_2^{\varepsilon_2}\cdots\gamma_m^{\varepsilon_m}=1$$

where $\varepsilon_i\in\{-2,-1,0,1,2\}$ implies

$$\gamma_1^{\varepsilon_1}=\gamma_2^{\varepsilon_2}=\cdots=\gamma_m^{\varepsilon_m}=1\,.$$

The order $o(\gamma)$ of a character γ is the smallest number n such that $\gamma^n=1$, and ∞ if there is no such number. An infinite set of characters is said to be *dissociate* if every finite subset is dissociate. If $\{\gamma_i\}_1^\infty$ is a dissociate set, then the *Riesz product*

$$\prod_{1=i}^{\infty}(1+\beta_i\gamma_i+\bar{\beta}_i\bar{\gamma}_i)$$

where $|\beta_i|\leq\frac{1}{2}$ and real if $o(\gamma_i)=2$, $|\beta_i|\leq\frac{1}{2}$ otherwise, converges in the weak-* topology to a positive measure μ. μ is singular if G is $\mathbf{T}$ or a 0-dimensional metrizable group and $\sum_{i=1}^{\infty}|\beta_i|^2=\infty$ [1]. The $L^2(\hat{G})$-norm of $\hat{\mu}$ is easily calculated

$$\|\hat{\mu}\|^2_{L^2(\hat{G})} = \begin{cases} \prod_{i=1}^{\infty}(1+2|\beta_1|^2) & \text{if} \quad o(\gamma_i)>2\,, \qquad i=1, 2, \cdots \\ \prod_{i=1}^{\infty}(1+|2\beta_i|^2) & \text{if} \quad o(\gamma_i)=2\,, \qquad i=1, 2, \cdots \end{cases}$$

Thus, $\hat{\mu}$ belongs to $L^2(\hat{G})$ if and only if $\sum_{i=1}^{\infty}|\beta_i|^2<\infty$.

It was proved in [1] that if G is a 0-dimensional metrizable group, then $\hat{G}$ contains countable dissociate sets where all characters have orders either strictly greater than 2 or all equal to 2. On $\mathbf{Z}\simeq\hat{\mathbf{T}}$, $\{3^k\}_{k=1}^{\infty}$ is a well known dissociate set.

We start with a couple of lemmas. The first one was also noted when $G=\mathbf{T}$ in [3].

LEMMA 3.1. *Let* $\{\gamma_i\}_1^N$ *be a dissociate set and let* $\beta>0$. *Define*

$$R_N=\prod_{i=1}^{N}\left(1+\beta(\gamma_i+\bar{\gamma}_i)\right).$$

Then the polynomial R_N *has* $2^s\binom{N}{s}$ *distinct terms with coefficient* β^s *if* $o(\gamma_i)>2$ *for all* i, *and* $\binom{N}{s}$ *distinct terms with coefficient* $(2\beta)^s$ *if* $o(\gamma_i)=2$ *for all* i. $s=0, 1, \cdots, N$.

PROOF: If $o(\gamma_i)=2$, $i=1, \cdots, N$, then $\gamma_i=\bar{\gamma}_i$ and $R_N=\prod_{i=1}^{N}(1+(2\beta)\gamma_i)$. Clearly, the number of terms consisting of a product of exactly s characters from $\{\gamma_i\}$ is $\binom{N}{s}$. If $o(\gamma_i)>2$ we proceed by induction. The assertion is obviously true for all N if $s=0$ or N. Suppose the formula holds with $N=N_o$.

$$R_{N_o+1}=\left(1+\beta(\gamma_{N_o+1}+\bar{\gamma}_{N_o+1})\right)R_{N_o}.$$

Now,

$$\begin{aligned} &\# \text{ terms in } R_{N_o+1} \text{ with coef. } \beta^s \\ = \quad &(\# \text{ terms in } R_{N_o} \text{ with coef. } \beta^s) \\ &+2(\# \text{ terms in } R_{N_o} \text{ with coef. } \beta^{s-1}) \\ = \quad &2^s\binom{N_o}{s}+2\cdot 2^{s-1}\binom{N_o}{s-1}=2_s\binom{N_o+1}{s}. \end{aligned}$$

Define $S_{-1}^N=0$, $S_k^N=\sum_{s=0}^{k}2^s\binom{N}{s}$, $k=0, 1, \cdots, N$. Then

$$\begin{aligned} S_k^N &= \left(1+2\binom{N}{1}+\cdots+2^k\binom{N}{k}\right) \\ &< 2(1+N+N^2+\cdots+N^k) \\ &\leq 4N^k \quad \text{if} \quad N\geq 3\,. \end{aligned}$$

LEMMA 3.2: *Let $\{\gamma_i\}_1^\infty$ be an infinite dissociate set such that $o(\gamma_i)>2$, $i=1,2,\ldots$ and let f be a trigonometric polynomial. Define $R=\prod_{i=1}^{N}\left(1+\beta(\gamma_i+\bar{\gamma}_i)\right)$, $\beta>0$. Fix $\alpha>0$ and let $2\beta^2 N=\alpha$. Then*

$$\lim_{N\to\infty} \|\widehat{fR}\|_{(2,q)}^{*q} \le e^{\alpha^{q/2}} \|\hat{f}\|_{(2,q)}^{*q}$$

when $q\ge 2$.

PROOF: Let $N>N_o$ where N_o is large enough to ensure that $\beta_o=(\alpha/2N_o)^{\frac{1}{2}}$ is less than the modulus of the smallest nonzero coefficient in f. Suppose supp $\hat{f}^*=[0,a]$. Since $\beta<\beta_o$ it is easy by means of Lemma 3.1 to write down $\widehat{fR}^*$ explisitly:

$$\widehat{fR}^*(t)=\begin{cases}\beta^k f^*\left(\dfrac{t-aS_{k-1}^N}{S_k^N-S_{k-1}^N}\right) & if \quad aS_{k-1}^N\le t<aS_k^N\\ 0 & if \quad t\ge 3^N a.\end{cases}$$

Thus,

$$\begin{aligned}\|\widehat{fR}\|_{(2,q)}^{*q} &= \sum_{k=0}^{N}\frac{2}{q}\int_{aS_{k-1}^N}^{aS_k^N} t^{\frac{q}{2}-1}\beta^{kq}\hat{f}^*\left(\frac{t-aS_{k-1}^N}{S_k^N-S_{k-1}^N}\right)^q dt\\ &=\sum_{k=0}^{N}\frac{2}{q}\int_0^a\left(\tau(S_k^N-S_{k-1}^N)+aS_{k-1}^N\right)^{\frac{q}{2}-1}\beta^{kq}\hat{f}^*(\tau)^q(S_k^N-S_{k-1}^N)\,d\tau\\ &=\frac{2}{q}\sum_{k=0}^{\infty}\int_0^a h_N(k,t)\,\hat{f}^*(t)^q dt\end{aligned}$$

where

$$h_N(k,t)=\begin{cases}\left(t(S_k^N-S_{k-1}^N)+aS_{k-1}^N\right)^{\frac{q}{2}-1}\beta^{kq}(S_k^N-S_{k-1}^N), & k=1,\cdots,N,\\ 0 \quad if \quad k>N.\end{cases}$$

For fixed k, $S_{k-1}^N\cdot(S_k^N-S_{k-1}^N)^{-1}<k!\dfrac{4N^{k-1}}{(N-k+1)^k}=O\left(\dfrac{1}{N}\right)$,

and

$$\begin{aligned}\lim_{N\to\infty} h_N(k,t) &= t^{\frac{q}{2}-1}\lim_{N\to\infty}\beta^{kq}2^{k\frac{q}{2}}\binom{N}{k}^{\frac{q}{2}}\\ &= t^{\frac{q}{2}-1}(\alpha^{\frac{q}{2}})^k\frac{1}{(k!)^{q/2}}.\end{aligned}$$

Moreover,

$$\begin{aligned}h_N(k,t) &\le a^{\frac{q}{2}-1}(S_k^N)^{\frac{q}{2}-1}\beta^{kq}2^k\binom{N}{k}\\ &\le (4a)^{\frac{q}{2}-1}N^{\left(\frac{q}{2}-1\right)k}\beta^{kq}2^k\frac{N^k}{k!}\\ &= (4a)^{\frac{q}{2}-1}(\alpha^{\frac{q}{2}})^k\frac{1}{k!}\\ &= H(k,\tau).\end{aligned}$$

But $H\hat{f}^{*q}$ is $\mathbf{Z}^+\times[0,a]$-integrable and it follows from dominated convergence that

$$\begin{aligned}\lim_{N\to\infty}\|\widehat{fR}\|_{(2,q)}^{*q} &= \frac{2}{q}\sum_{k=0}^{\infty}\int_0^a t^{\frac{q}{2}-1}(\alpha^{\frac{q}{2}})^k\frac{1}{(k!)^{q/2}}\hat{f}^*(t)^q dt\\ &= \sum_{k=0}^{\infty}(\alpha^{\frac{q}{2}})^k\frac{1}{(k!)^{q/2}}\|\hat{f}\|_{(2,q)}^{*q}\\ &\le e^{\alpha^{q/2}}\|\hat{f}\|_{(2,q)}^{*q}.\end{aligned}$$

If the set $\{\gamma_i\}_1^\infty$ consists entirely of characters of order 2, Lemma 3. 2 needs to be slightly modified: The conclusion then holds with α defined as $\alpha=(2\beta)^2N$. If $q=2$, the inequality turns into an equality as is easily seen.

The proof of the following theorem is based on the development in [1].

THEOREM 3. 3: *Let G be a nondiscrete locally compact Abelian group. Then there exists a positive singular measure $\mu\in M(G)$ such that*

$$\hat{\mu}\in\bigcap_{q>2}L(2,q)(\hat{G}).$$

PROOF: Assume first that G is either $\mathbf{T}$ or a 0-dimensional compact metric group. Let $\{\gamma_i\}_1^\infty$ be a countable dissociate set of characters which we assume have orders greater than 2. The argument when $o(\gamma_i)=2$ for all i is quite analogous and is omitted. Let $\{q_s\}_1^\infty$ be a sequence tending monotonically to 2 from above. We construct μ by a repeated use of Lemma 3. 2, namely,

$$\mu=\prod_{s=1}^{\infty}R_s$$

where

$$R_1=\left(1+\frac{1}{4}(\gamma_1+\bar{\gamma}_1)\right)\quad\text{and}\quad R_s=\prod_{k=1}^{N_s}\left(1+\beta_s(\gamma_k^s+\bar{\gamma}_k^s)\right)\quad s=2,3,\cdots.$$

Here $2N_s\beta_s^2=\dfrac{1}{s}$ and $\{\gamma_n^s\}$ consist of distinct elements from $\{\gamma_i\}_1^\infty$. Moreover, N_s is chosen (inductively) large enough to ensure that

$$\|\widehat{(\prod_{r=1}^{s-1} R_r) R_s}\|^{*q_i}_{(2,q_i)} \leq e^{2^{-s}} e^{\left(\frac{1}{s}\right)^{\frac{q_i}{2}}} \|\widehat{\prod_{r=1}^{s-1} R_r}\|^{*q_i}_{(2,q_i)} \qquad \text{for} \quad i=1, 2, \cdots, s \tag{3.1}$$

and

$$(1+2\beta_s^2)^{N_s} \geq \exp\left(-2^{-s}+\frac{1}{s}\right). \tag{3.2}$$

Now,

$$\begin{aligned}\|\hat{\mu}\|_2^2 &= \prod_{s=1}^{\infty}(1+2\beta_s^2)^{N_s} \\ &\geq \exp\left(\sum_{s=1}^{\infty}\left(-2^{-s}+\frac{1}{s}\right)\right) \\ &= \infty .\end{aligned}$$

Thus μ is a positive singular (and continuous) measure [1, Th. 4. 4].

If we fix a $q>2$, there is a q_{i_o} such that $q_{i_o}<q$ and we conclude from (3. 1) that

$$\begin{aligned}\|\hat{\mu}\|^{*q_{io}}_{(2,q_{io})} &\leq \|\widehat{\prod_{s=1}^{i_o} R_s}\|^{*q_{io}}_{(2,q_{io})} \cdot \exp\left(1+\sum_{s=i_o}^{\infty}\left(\frac{1}{s}\right)^{q_{io}/2}\right) \\ &< \infty .\end{aligned}$$

Consequently, $\hat{\mu}\in L(2, q_{i_o})(\hat{G})\subset L(2, q)(\hat{G})$.

The extension to arbitrary compact Abelian groups goes exactly as the proof of Th. 5. 3 in [1] and is omitted.

We then turn to $\mathbf{R}^n$, $n\geq 1$. The construction in [1] Th. 5. 1 seems to be less convenient, so we propose a somewhat simpler approach which gives a measure with similar properties. Let $\mu\in M(\mathbf{T}^n)$ have the desired properties and let μ_o be the unbounded periodic extension of μ to $\mathbf{R}^n$ divided by $(2\pi)^n$. The distributional Fourier transform of μ_o is $\hat{\mu}_o=\sum_{i\in\mathbf{Z}^n}\hat{\mu}(i)\,\delta_i$ where δ_i is the Dirac mesure at the point i. μ is singular with respect to the Haar measure since $m(\operatorname{supp}\mu_o)=\sum_{i\in Z^n} m(Q+i)\cap\operatorname{supp}\mu_o)=0$ where Q is the cube $\prod_1^n(-\pi,\pi)$. Let ϕ be a function (for example made from the Fejer kernel) such that $\hat{\phi}(0)=1$, $0\leq\hat{\phi}\leq 1$, $\operatorname{supp}\hat{\phi}\subset\prod_1^n\left(-\frac{1}{2},\frac{1}{2}\right)$, $\phi\geq 0$, and $\sum_{i\in\mathbf{Z}^n}\sup_{t\in Q+i}\phi(t)<\infty$. A standard partition of unity argument shows that the measure $\tilde{\mu}$ defined by

$$\int_{\mathbf{R}^n} f d\tilde{\mu}=\int_{\mathbf{R}^n} f(t)\,\phi(t)\,d\mu_o(t)$$

is bounded. Furthermore, $\hat{\tilde{\mu}}(\xi)=\sum_{i\in Z^n}\hat{\mu}(i)\hat{\phi}(\xi-i)$.
Since $\operatorname{supp}\tilde{\mu}\subset\operatorname{supp}\mu_o$, $\tilde{\mu}$ is singular. Also, the shape of $\hat{\phi}$ enables us to conclude that

$$\hat{\tilde{\mu}}^*(t)\leq\Big(\sum_{i\in Z^n}\hat{\mu}(i)\,\chi_E(\xi-i)\Big)^*(t)$$
$$=\hat{\mu}^*(t).$$

where $E=\prod_1^n\left[-\frac{1}{2},\frac{1}{2}\right]$. Thus $\tilde{\mu}$ has the desired properties.

If G contains an infinite compact open subgroup J, we argue as in the proof of Th. 5. 4 in [1]: Start with a measure $\mu_o\in M(J)$ fulfilling the requirements in the theorem, and extend μ_o to a bounded measure μ on G by

$$\int_G fd\mu=\int_J fd\mu_o$$

μ is obviously positive and singular, and $\hat{\mu}(\xi)=\hat{\mu}_o(\xi+J^\perp)$. The distribution function of $\hat{\mu}$ is consequently

$$\lambda_{\hat{\mu}}(t)=m\left\{\xi\,;\ \hat{\mu}(\xi)>t\right\}$$
$$=m(J^\perp)\cdot\lambda_{\hat{\mu}_o}(t),$$

and

$$\hat{\mu}^*(t)=\inf\left(y\,;\ \lambda_{\hat{\mu}}(y)\leqslant t\right)=\hat{\mu}_o^*\left(\frac{t}{m(J^\perp)}\right).$$

We thus conclude that

$$\|\hat{\mu}\|^{*q}_{(2,q)}=m(J^\perp)^{\frac{q}{p}}\|\hat{\mu}_o\|^{*q}_{(2,q)}<\infty.$$

Finally, let G be an arbitrary locally compact group. G is topologically isomorphic to $\mathbf{R}^n\times G_o$. If $n=0$ we are back to the situation above, so assume that $n\geq 1$. Let again $\mu_o\in\mathbf{R}^n$ be a singular measure as above and let λ be the Haar measure on J extended to a bounded measure on J. Set $\mu=\mu_o\times\lambda$. As in the proof of Th. 5. 6 in [1] we see that μ is singular. $\hat{\mu}$ has the form

$$\hat{\mu}(\xi,\eta)=\hat{\mu}_o(\xi)\,\chi_{J^\perp}(\eta),\qquad \xi\in\hat{\boldsymbol{R}}^n,\qquad \eta\in\hat{G}_o.$$

Thus,

$$\lambda_{\hat{\mu}}(t)=m\left\{(\xi,\eta)\,;\ \hat{\mu}_o(\xi)\,\chi_{J^\perp}(\eta)\geq t\right\}$$
$$=m(J^\perp)\,\lambda_{\hat{\mu}_o}(t)$$

and

$$\hat{\mu}^*(t) = \hat{\mu}_o^*\left(\frac{t}{m(J^\perp)}\right).$$

But this proves that $\hat{\mu}\in L(2, q)(\hat{G})$ for all $q>2$ and finishes the proof.

REMARK 3.4: The extension to $\mathbf{R}^n$ in the above proof seems to be simpler than the one in Th. 5.1 of [1]. It is easy to show that if ϕ is constructed from the Fejer kernel, then $\operatorname{supp}(\mu*\mu)=\mathbf{R}^n$ if $\operatorname{supp}\mu_o*\mu_o=\mathbf{T}$ as required in [1].

4. Multipliers of $A(p, q)$ Algebras.

A *multiplier* of a commutative Banach algebra A is a bounded linear operator T on A such that $T(ab)=a(Tb)$ for all $a, b\in A$. The set of multipliers is denoted by $M(A)$ and is normed by the operator norm. If (A, V) and (A, W) are *Banach modules*, then $\mathrm{Hom}_A(V, W)$ denotes the Banach space of all continuous *module homomorphisms*. A Segal algebra S is a $L^1(G)$-Banach module, and it is well known that $\mathrm{Hom}_{L^1(G)}(S, S)\cong M(S)$. Furthermore, $T\in M(S)$ if and only if $T(f_t)=(Tf)_t$ for all $f\in S$, $t\in G$ where $f_t(x)=f(x-t)$ [6, 4]. Also, a multiplier may canonically be represented by a pseudomeasure σ such that $Tf=\sigma*f$ ($\widehat{Tf}(\xi)=\hat{\sigma}(\xi)\hat{f}(\xi)$, $\xi\in\hat{G}$). It is clear that $M(G)$ is contained in $M(S)$ in the sense that convolution with a bounded measure is a multiplier of S [6, 4].

DEFINITION 4.1: *A Segal algebra S on a noncompact Abelian group G satisfies condition A if for every $f\in S$, compact set $K\subset G$, and $\varepsilon>0$, there exists a finite set $\{x_i\}_1^N\subset G$ such that*

i) $$(K+x_i)\cap(K+x_j)=\emptyset \quad \textit{if} \quad i\neq j,$$

ii) $$\left\|\frac{1}{N}\sum_{i=1}^N f_{x_i}\right\|_S \leq \left\|\frac{1}{N}\sum_{i=1}^N f_{x_i}\right\|_{L^1(G)} + \varepsilon.$$

Obviously, S satisfies A if i) and ii) can be obtained for a dense set of elements in S.

LEMMA 4.2: *If a Segal algebra S contains a Segal algebra S_o satisfying A, then*

$$M(S)\simeq \mathrm{Hom}_{L^1(G)}\big(S, L^1(G)\big)\simeq M(G).$$

PROOF: Since $M(G)$ is contained in $M(S)$ which in turn is contained in $\mathrm{Hom}_{L^1(G)}(S, L^1(G))$, it is sufficient to show that $M(G)\simeq \mathrm{Hom}_{(L^1G)}(S, L^1(G))$.

The Closed Graph Theorem shows that there is a constant C such that $\|f\|_S \leq C\|f\|_{S_o}$ for all $f \in S_o$. Fix a $T \in \mathrm{Hom}_{L^1(G)}(S, L^1(G))$. For a given $f \in S_o$, let K and $\{x_i\}_1^N$ be large enough to assure that

$$\left\| \frac{1}{N} \sum_{i=1}^{N} (Tf)_{x_i} \right\|_{L^1(G)} \geq \|Tf\|_{L^1(G)} - \varepsilon$$

and

$$\left\| \frac{1}{N} \sum_{i=1}^{N} f_{x_i} \right\|_{S_o} \leq \left\| \frac{1}{N} \sum_{i=1}^{N} f_{x_i} \right\|_{L^1(G)} + \varepsilon .$$

Thus,

$$\begin{aligned} \|Tf\|_{L^1(G)} - \varepsilon &\leq \left\| \frac{1}{N} \sum_{i=1}^{N} (Tf)_{x_i} \right\|_{L^1(G)} \\ &\leq \|T\| \left\| \frac{1}{N} \sum_{i=1}^{N} f_{x_i} \right\|_S \\ &\leq C\|T\| \left\| \frac{1}{N} \sum_{i=1}^{N} f_{x_i} \right\|_{S_o} \\ &\leq C\|T\| \left(\left\| \frac{1}{N} \sum_{i=1}^{N} f_{x_i} \right\|_{L^1(G)} + \varepsilon \right) \\ &\leq C\|T\| \left(\|f\|_{L^1(G)} + \varepsilon \right) . \end{aligned}$$

Consequently, $\|Tf\|_{L^1(G)} \leq C\|T\| \, \|f\|_{L^1(G)}$ for all $f \in S_o$. A routine argument combined with Wendel's theorem ($M(L^1(G)) \cong M(G)$) now finishes the proof.

THEOREM 4.3: *Let G be a nondiscrete locally compact Abelian group and let* $(1, \infty) < (p, q) < (\infty, 1)$.

i) *If G is not compact, then*

$$M\big(A(p, q)\big) \simeq M(G) .$$

ii) *If G is compact, then*

$$M\big(A(p, q)\big) \simeq P(G) \quad \text{if} \quad (p, q) \leq (2, 2)$$

$$M(G) \subsetneqq M\big(A(p, q)\big) \subsetneqq P(G) \quad \text{if} \quad (p, q) > (2, 2) .$$

PROOF: The Segal algebra $A^1 = A(1, 1) = \{f \in L^1(G) ; \hat{f} \in L^1(\hat{G})\}$ is contained in $A(p, q)$ for all admissible pairs (p, q) (this was also noted in [5]). If G is not compact, then A^1 satisfies condition A and i) follows from Lemma 4.2 [6, p. 204–207].

If G is compact and $(p, q) \leq (2, 2)$, then $M(A(p, q)) \simeq P(G)$ by a similar argument as in the proof of the corresponding result for the A^p-algebras [6, p. 207].

It is easily proved that $M(A(p',q'))\subset M(A(p,q))$ if $(p,q)<(p',q')$. Hence, $M(G)\subsetneqq M(A^{2p})=M(A(2p,2p))\subset M(A(p,q))$ [6, p. 208].

The second inclusion of ii) is proved if we establish that $M(A(2,q))\subsetneqq P(G)$ when $q>2$. When G is $\mathbf{T}$ or a 0-dimensional compact metric group, the dissociate sets introduced in § 3 are Sidon sets. We may then argue as follows: Let E be the set of distinct characters from the dissociate set used in the construction of μ in Theorem 3. 3. The operator T represented by χ_E is not a multiplier on $A(2,q)$ for any $q>2$. Indeed, $\hat{\mu}\chi_E\notin L^2(G)$ since

$$\|\hat{\mu}\chi_E\|_2^2=\sum_{s=1}^{\infty}2N_s\beta_s^2=\infty\,.$$

$\left(=\sum_1^{\infty}N_s(2\beta_s)^2=\infty \text{ if } o(\gamma_i)=2 \text{ for all } i\right)$. Therefore $T(\mu)\notin M(G)$ [6, p. 85]. If $\{h_\alpha\}$ is a bounded approximate identity for $L^1(G)$ such that $\hat{h}_\alpha\in C^c(\hat{G})$, then $\|h_\alpha*\mu\|_{A(p,q)}$ is uniformly bounded in α, but $\|T(h_\alpha*\mu)\|_{M(G)}$ cannot be uniformly bounded due to the weak compactness of the unit ball in $M(G)$. On an arbitrary nondiscrete compact Abelian group we argue in a similar way by using the construction in the proof of Th. 5. 3 in [1]. (We need that a Sidon set in a subgroup $H\subset\hat{G}$ is a Sidon set in $\hat{G}$, but this is immediate by virtue of Th. 5. 7. 3 and 2. 7. 2 in [9].)

Remark 4. 4: It is possible to give a characterization of $M(A(p,q))$ as the dual space of a space of continuous functions using tensor products. The general technique is presented in [4] and will not be considered here.

Following Lai [5], let

$$M(p,q)=\left\{\mu\in M(G)\,;\ \hat{\mu}\in L(p,q)(\hat{G})\right\}.$$

Most of the next theorem was proved in [5]. However, the proof may be considerably simplified if one argues as in [8].

Theorem 4. 5: *Let G, p and q be as in Theorem 4. 3. Then*

i) $\operatorname{Hom}_{L^1(G)}\left(L^1(G),\ A(p,q)\right)\simeq A(p,q)$ *if* $(p,q)\leq(2,2)$

ii) $\operatorname{Hom}_{L^1(G)}\left(L^1(G),\ A(p,q)\right)\simeq M(p,q)\supsetneqq A(p,q)$ *if* $(2,2)<(p,q)$.

Proof: The proof would be a repetition of the similar argument for the A^p algebras given in [8] and is omitted. Finally, Theorem 3. 3 provides a proof of the strict inclusion in ii).

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