Hokkaido Mathematical Journal Vol. 7 (1978) p. 300-307

On integrability conditions on the space of sections of jet-bundles

By Yoshifumi ANDO

(Received October 28, 1977; Revised February 22, 1978)

§ 1. Introduction

Let N be a n dimensional differentiable manifold. We consider a differentiable bundle $E(N)$ over N with projection π and the bundle $E^r(N)$ of r-jets of local sections of $E(N)$. Let Ω be an open set of $E^r(N)$. Then we let $\Gamma_\Omega E$ be the space of C^r sections, $s: N\rightarrow E(N)$ such that $j^r s(N)$ is contained in Ω equipped with C^r topology. We let $\Gamma_\Omega E^r$ be the space of continuous sections: $N\rightarrow E^r(N)$ whose image is contained in Ω, equipped with compact-open topology (An element of $\Gamma_\Omega E$ or $\Gamma_\Omega E^r$ will be called *Ω-regular*). Then there is a natural map $j^r: \Gamma_\Omega E\rightarrow\Gamma_\Omega E^r$. We discuss how the map j^r is close to a weak homotopy equivalence. This is related with the integrability of sections of $E^r(N)$ up to homotopy.

THEOREM. *Let Ω and Ω' be open sets in $E^r(N)$ with $\Omega\supseteqq\Omega'$. Let $\Omega-\Omega'$ is a finite union of regular submanifolds of Ω with codimensions greater than $n+\sigma$.*
(i) *If $j^r: \Gamma_{\Omega'}E\rightarrow\Gamma_{\Omega'}E^r$ is a τ-homotopy equivalence, then $j^r: \Gamma_\Omega E\rightarrow\Gamma_\Omega E^r$ is a* $\min(\tau, \sigma)$*-homotopy equivalence.*
(ii) *If $j^r: \Gamma_\Omega E\rightarrow\Gamma_\Omega E^r$ induces the isomorphisms of i dimensional homotopy groups $(0\leqq i\leqq\tau)$, then $j^r: \Gamma_{\Omega'}E\rightarrow\Gamma_{\Omega'}E^r$ induces the isomorphisms of i dimensional homotopy groups* $(0\leqq\tau\leqq\min(\tau, \sigma)-1)$).

A j-homotopy equivalence means the isomorphisms of i dimensional homotopy groups $(0\leqq i<j)$ and a surjection of j dimensional homotopy groups.

This theorem is a generalization of Transversality lemma due to A. du Plessis in [5] which is the case of differentiable maps of the above theorem. The applications of the theorem are given in § 4 to the cace of Thom-Boardman singularities ([2, 7, 9]). The proof is based on the transversality arguments.

All manifolds should be paracompact and Hausdorff.

§ 2. A variant of Thom's transversality theorem

In this section we will show a variant of Thom's transversality theorem.

This is a generalization to the case of differentiable bundles $E(N)$ of Morlet's transversality theorem ([8]) which says that the case of product bundles of the following theorem is valid. Let ΓE be the space of C^∞ sections of $E(N)$ over N with C^∞ topology.

THEOREM 2.1. *Let Σ be a regular differentiable submanifold of $E^r(N)$. Let $\Sigma(N)$ be the space of C^∞ sections, $s: N\to E(N)$ whose r-jet, $j^r s: N\to E^r(N)$ is transverse on Σ. Then $\Sigma(N)$ is represented as the intersections of countable open dense sets of ΓE.*

PROOF. At first we choose a countable covering of Σ by open subsets $\Sigma_1, \Sigma_2, \cdots$ such that each Σ_i satisfies

(i) the closure $\bar{\Sigma}_i$ of Σ_i in $E^r(N)$ is contained in Σ,

(ii) $\bar{\Sigma}_i$ is compact,

(iii) there exists an open neighbourhood U_i in N where $E(N)|_{U_i}$ is trivial and a trivialixation, $t: E(N)|_{U_i}\to U_i\times P$ as follows. This induces a diffeomorphism $t^r: E^r(N)|_{U_i}\mapsto J^r(U_i, P)$ which is composed with a projection $\pi_P: J^r(U_i, P)\to P$. Then there exists an open chart V_i of P such that $\pi_P\circ t^r(\bar{\Sigma}_i)$ is contained in V_i,

(iv) the closure $\bar{U}_i$ of U_i is compact.

Let $\Sigma_i(N)$ be the space of C^∞ sections of ΓE, whose r-jets are transverse on Σ_i. Then it is clear that $\Sigma(N)$ is the intersection of all $\Sigma_i(N)$. Now we show that $\Sigma_i(N)$ is represented as the intersection of countable open dense sets. We let $r_i: \Gamma E\to\Gamma(E|_{U_i})$ be the restriction map of C^∞ sections of $E(N)$. Let $\Sigma(U_i)$ be the space of C^∞ sections of $E(N)|_{U_i}$ whose r-jects are transverse on Σ_i.

Then $r_i^{-1}(\Sigma(U_i))=\Sigma_i(N)$. By the Morlet's transversality theorem $\Sigma(U_i)$ is represented as the intersection of countable open dense sets. Hence it is enough to show that $\Sigma_i(N)$ is dense in ΓE.

Let s be an element of ΓE. We show that there exists a sequence $\{s_l\}$ in $\Sigma_i(N)$ which converges to s. We choose a chart, $\eta: V_i\to R^p$ and differentiable functions $\rho; N\to[0, 1]$ and $\rho': P\to[0, 1]$ such that

$$\rho=\begin{cases}1 & \text{on a neibourhood of } \pi(\bar{\Sigma}_i) \text{ in } U_i,\\ 0 & \text{off } U_i\end{cases}$$

$$\rho'=\begin{cases}1 & \text{on a neibourhood of } \pi_P\circ t^r(\bar{\Sigma}_i) \text{ in } V_i,\\ 0 & \text{off } V_i.\end{cases}$$

This choice of ρ and ρ' is possible since $\bar{\Sigma}_i$ is compact. By Morlet's transversality theorem $\Sigma(U_i)$ is dense in $\Gamma(E|_{U_i})$. Hence there exists a sequence $\{g_l\}$ of $\Sigma(U_i)$ which converges to $s|_{U_i}$ in the fine C^∞ Whitney topology.

Now we define the sequence $\{s_l\}$ as follows

$$s_l(x)=\begin{cases} s(x) & \text{if } x\notin U_i \text{ or } s(x)\notin V_i, \\ \eta_i^{-1}\Big[\eta_i\big(s(x)\big)+\rho(x)\,\rho'\big(g_l x)\big)\,\rho'\big(s(x)\big)\Big[\eta_i\big(g_l(x)\big)-\eta_i\big(s(x)\big)\Big]\Big]. \\ \quad \text{if otherwise}. \end{cases}$$

This definition is possible if l is sufficiently large. It is clear that the sequence $\{s_l\}$ converges to s. The rest of the proof is to show that s_l's are contained on $\Sigma_i(N)$ for sufficiently large l. In fact there exists a large number a such that if $j^r s_l(x_0)\in\bar{\Sigma}_i$ and $l>a$, then $\rho(x_0)=1$, $\rho'(s(x_0))=1$ and $\rho'(g_l(x_0))=1$. For, let ε be a positive number smaller than a half of the distance of the subset of V_i where ρ' is smaller than 1 and the subset $\pi_P\circ t^r(\bar{\Sigma}_i)$. Then a is defined to be an integer such that $|\eta_i(g_l(x))-\eta_i(s(x))|$ is smaller than ε for $l>a$. Since $j^r s_l(x_0)\in\bar{\Sigma}_i$ means that $s_l(x_0)\,(=\pi_P\circ j^r s_l(x_0))$ is in $\pi_P\circ t^r(\bar{\Sigma}_i)$, by the definition of $s_l(x)$ $|s_l(x_0)-s(x_0)|$ and $|g_l(x_0)-s_l(x_0)|$ are smaller than ε. Hence $\rho'(s(x_0))=\rho'(g_l(x_0))=1$ for $l>a$. Since $j^r s_l(x_0)\in\bar{\Sigma}^i$ means $x_0\in\pi_N(\bar{\Sigma}_i)$, we get $\rho(x_0)=1$. Therefore if $j^r s_l(x_0)\in\bar{\Sigma}_i$ and $l>a$, then $s_l(x)=g_l(x)$ near x_0. Hence $s_l(x)$ is transverse on Σ_i for $l>a$. Q. E. D.

REMARK 2. 2. ΓE is a Baire space: We consider the space $C^\infty(N, E(N))$ of differentiable maps of N into $E(N)$. It is well known that $C^\infty(N, E(N))$ is a complete metric space. Then it is clear that ΓE is a closed set of $C^\infty(N, E(N))$, hence, a complete metric space which is a Baire space.

By the above remark we have the following

COROLLARY 2. 3. *$\Sigma(N)$ is dense in ΓE.*

COROLLARY 2. 4. *Let Ω be an open set of $E^r(N)$ and Σ, the regular submanifold of Ω with* codim $\Sigma>$dim N. *Let W be a closed subset of N. Let s be a C^∞ section of $E(N)$ such that $j^r s(N)\subset\Omega$ and $j^r s(W)\cap\Sigma=\phi$. Then there exists a homotopy of sections, $S\colon I\to\Gamma E$ such that $S(0)=s$, $S(t)|_W=s|_W$ for any t and $j^r S(1)\cap\Sigma=\phi$.*

PROOF. Let U be an open small neibourhood of W in N such that $j^r s(x)\notin\Sigma$ for $x\in U$. Then we only needs the deformation of s off U such that we let s be transverse on Σ. This is possible by the similar arguments as the proof of Theorem 2. 1. Q. E. D.

§ 3. Elimination of the singularity Σ

Let Ω and Ω' be open sets in $E^r(N)$ with $\Omega\supsetneqq\Omega'$. Let $\Sigma=\Omega-\Omega'$ and Σ be a finite union of regular submanifolds of Ω with codimensions greater than $n+\sigma$. Then we have the following.

PROPOSITION 3.1. *Let Ω, Ω' and Σ be as above. Then*
(i) *the natural inclusion: $\Gamma_{\Omega'}E \to \Gamma_{\Omega}E$ is a σ-homotopy equivalence,*
(ii) *the natural inclusion: $\Gamma_{\Omega'}E^r \to \Gamma_{\Omega}E^r$ is a σ-homotopy equivalence.*

We need some notations for the proof. Let X be a differentiable manifold. Let p and f be base points of X and Y. Then $\mathscr{F}_0(X, Y)$ denotes the space of continuous maps preserving base point with compact-open topology. A continuous map $\alpha: X \to \Gamma_{\Omega}E$ is called C^r *differentiable* if its associated section, $\alpha': X \times N \to X \times E(N)$ defined by $\alpha(x, n) = (x, \alpha(x)(n))$ is differentiable of class C^r. Let $\mathscr{F}_0^r(X, \Gamma_{\Omega}E)$ denote the space of C^r differentiable maps of $\mathscr{F}_0(X, \Gamma_{\Omega}E)$. Then we have the following lemma. This follows from the differentiable approximation theorem of continuous maps.

LEMMA 3.2. *The canonical inclusion $\mathscr{F}_0^r(X, \Gamma_{\Omega}E) \hookrightarrow \mathscr{F}_0(X, \Gamma_{\Omega}E)$ induces a bijection of the sets of their connected components.*

Next we define a map $\pi: (E_X)^r(X \times N) \to X \times E^r(N)$ where $E_X = X \times E$. Let α be a r-jet, $j^r s$ of a local section $s: X \times N \to X \times E$ defined near (x, y). Then we put $\pi(\alpha) = (x, (j^r s(x))(y))$. Let Ω_X be the pull back $\pi^{-1}(\Omega)$ of an open set Ω of $E^r(N)$. Then we can consider $\Gamma_{\Omega_X}(E_X)$ and the natural map $\Gamma_{\Omega_X}(E_X) \to \mathscr{F}^r(X, \Gamma_{\Omega}E)$ which is a continuous bijection.

PROOF OF PROPOSITION 3.1. We shall begin with proving that the map $\pi_0(\Gamma_{\Omega'}E) \to \pi_0(\Gamma_{\Omega}E)$ is surjective when $\sigma \geqq 0$. Let s be an element of $\Gamma_{\Omega}E$. Then it follows from Corollary 2.4 that there is a path, $S: I \to \Gamma_{\Omega}E$ such that $S(\text{o}) = s$ and $S(1)$ is transverse on Σ. Since $\operatorname{codim} \Sigma > n + \sigma$, this means $j^r S(1)(N) \cap \dot{\Sigma} = \phi$. Hence $S(1)$ is an element of $\Gamma_{\Omega'}E$.

By the above fact we may fix a base point in $\Gamma_{\Omega'}E$ when we consider a connected component of $\Gamma_{\Omega}E$. Let s be a base point in $\Gamma_{\Omega'}E$. Consider the following commutative diagram

$$\begin{array}{ccc} \pi_i(\Gamma_{\Omega'}E, s) & \longrightarrow & \pi_i(\Gamma_{\Omega}E, s) \\ \downarrow & & \downarrow \\ \pi_0\big(\mathscr{F}_0(S^i, \Gamma_{\Omega'}E)\big) & \longrightarrow & \pi_0\big(\mathscr{F}_0(S^i, \Gamma_{\Omega}E)\big) \\ \downarrow & & \downarrow \\ \pi_0\big(\mathscr{F}_0^r(S^i, \Gamma_{\Omega'}E)\big) & \longrightarrow & \pi_0\big(\mathscr{F}_0^r(S^i, \Gamma_{\Omega}E)\big). \end{array}$$

Since both of vertical maps are bijective, it is enough to show that the bottom horizontal map is bijective for $i < \sigma$ and surjective for $i = \sigma$. Let $\alpha: S^i \to \Gamma_{\Omega}E$ be a element of $\mathscr{F}_0(S^i, \Gamma_{\Omega}E)$. Then it is identified with an element $\alpha': S^i \times N \to S^i \times E(N)$ of $\Gamma_{\Omega_{S^i}}E_{S^i}$. Since $\alpha(p) = s$, α' is transverse on Σ at $p \times N$, that is, $j^r\alpha'(p \times N) \cap \Sigma = \phi$. By Corollary 2.4 there exists an Ω_{S^i}-regular differentiable section, $S: I \times S^i \times N \to I \times S^i \times E(N)$ such that

$S|_{0\times S^i\times N}=\alpha'$, $S|_{1\times S^i\times N}\in\Gamma_{\Omega'_{S^i}}E_{S^i}$ and $S|_{t\times p\times N}=s$ for each $t\in I$. Hence i_* is surjective. Let α_0 and α_1 be elements of $\mathscr{F}_0^\infty(S^i, \Gamma_{\Omega'}E)$ such that $i_*\alpha_0=i_*\alpha_1$. By the differentiable approximation theorem there exists an Ω-regular differentiable map $S: (I\times S^i, I\times p)\to(\Gamma_\Omega E, s)$ such that $S|_{j\times S^i}=\alpha_j (j=0, 1)$. We obtain the associated differentiable section $\alpha'_j: S^i\times N\mapsto S^i\times E(N)$ and $S': I\times S^i\times N\to I\times S^i\times E(N)$. We consider $\Omega_{I\times S^i}$ and $\Omega'_{I\times S^i}$. Then S' is an $\Omega_{I\times S^i}$-regular C^r section with $j^rS'(j\times S^i\times N)\subseteqq\Omega'_{I\times S^i}$ $(j=0, 1)$ and $j^rS'(I\times p\times N)\subseteqq\Omega'_{I\times S^i}$. Since $\Omega_{I\times S^i}-\Omega'_{I\times S^i}=\pi^{-1}_{I\times S^i}(\Omega-\Omega')$, it is a finite union of submanifolds with codimensions $>n+\sigma$. By applying Corollary 2. 4 to the case of $\Omega_{I\times S^i}$, $\Omega'_{I\times S^i}$, $\dot{I}\times S^i\times N\cup I\times p\times N$ and S', we get Ω' regular section $\bar{S}': I\times S^i\times N\mapsto I\times S^i\times E(N)$ such that $\bar{S}'|_{j\times S^i\times N}=S|_{j\times S^i\times N}(j=0, 1)$ and $\bar{S}'|_{t\times p\times N}=s$ for each $t\in I$. This completes the proof.

The proof of (ii) follows from the similar arguments as above by the transversality theorem. In fact we consider the bundle $E^r(N)$ over N instead of $E(N)$ over N in the proof of (i) and apply Corollary 2. 4 to the following diagram

$$\begin{array}{ccc} \pi_i(\Gamma_{\Omega'}E^r, *) & \longrightarrow & \pi_i(\Gamma_\Omega E^r, *) \\ \downarrow & & \downarrow \\ \pi_0\left(\mathscr{F}_0(S^i, \Gamma_{\Omega'}E^r)\right) & \longrightarrow & \pi_0\left(\mathscr{F}_0(S^i, \Gamma_\Omega E^r)\right) \\ \downarrow & & \downarrow \\ \pi_0\left(\mathscr{F}_0^r(S^i, \Gamma_{\Omega'}E^r)\right) & \longrightarrow & \pi_0\left(\mathscr{F}_0^r(S^i, \Gamma_\Omega E^r)\right). \end{array}$$

Q. E. D.

Proof of Theorem. This follows from the following commutative diagram

$$\begin{array}{ccc} \pi_i(\Gamma_{\Omega'}E) & \xrightarrow{(j^r)_*} & \pi_i(\Gamma_{\Omega'}E^r) \\ \downarrow & & \downarrow \\ \pi_i(\Gamma_\Omega E) & \xrightarrow{(j^r)_*} & \pi_i(\Gamma_\Omega E^r). \end{array}$$

Q. E. D.

§ 4. Applications

In this section we slightly extend the notion of Thom-Boardman singularities [2, 7, 9] into the space of r-jet bundles $E^r(N)$. Let $J^r(U, P)$ denote the bundle of r-jets over differentiable manifolds U and P. Then the Thom-Boardman singularity with symbol I, $\Sigma^I(U, P)$ is defined in $J^r(U, P)$. $\Sigma^I(U, P)$ is a regular differentiable submanifold of $J^r(U, P)$ and a differentiable subbundle of $J^r(U, P)$ over $U\times P$. Let V be a differentiable manifold which is diffeomorphic to U by h. Let $\bar{h}: U\times P\to V\times P$ be a differentiable bundle map over the diffeomorphism $h: U\to V$. Then we can define a map $j^r\bar{h}$: $J^r(U, P)\to J^r(V, P)$. Let z be an element of $J^r(U, P)$ which is represented by $f: (U, x) \to (P, f(x))$. Then $j^r \bar{h}$ is defined to be the r-jet at $h^{-1}(x)$ of the composition, $p \circ \bar{h} \circ (id_U \times f) \circ h^{-1}$ where p denotes the projection of $V \times P$ onto P.

REMARK 4.1. *The map $j^r \bar{h}$ maps $\Sigma^I(U, P)$ diffeomorphically onto $\Sigma^I(V, P)$ and makes the following diagram commute.*

$$\begin{array}{ccc} J^r(U, P) & \xrightarrow{j^r \bar{h}} & J^r(V, P) \\ \downarrow & & \downarrow \\ U \times P & \xrightarrow{\bar{h}} & V \times P \end{array}$$

PROOF. Let $z = j^r f$ and $y = f(x)$ where $f: (U, x) \to (P, y)$. Let $C(U)_x$(resp. $C(P)_y$) denote the set of C^∞ map germs, $(U, x) \to \boldsymbol{R}$ (resp. $(P, y) \to \boldsymbol{R}$). Let $\mathfrak{M}_x$(resp. $\mathfrak{M}_y$) denote the ideal in $C(U)_x$(resp. $C(P)_y$) consisting of C^∞ map germs which vanish on x(resp. y). It is shown in [7] that the Boardman symbol I is determined only by the ideal $f^*(\mathfrak{M}_y)$ in $C(U)_x$ modulo $\mathfrak{M}_x^{r+1}$. It follows from [6, Proposition in (2.3)] that $(p \circ \bar{h} \circ (id_U \times f) \circ h^{-1})^* (\mathfrak{M}_y)$ is equal to $(h^{-1})^* f^* (\mathfrak{M}_y)$. Hence we know that the Boardman symbol of $j^r \bar{h}(z)$ coincides with that of z by definition. Other statement immediately follows from the definition of $j^r \bar{h}$. Q. E. D.

Let $\pi: E(N) \to N$ be a differentiable bundle over N with fibre P. If π is trivial over an open set U, then $E^r(N)|_U$ is canonically identified with $J^r(U, P)$. Let $\Sigma^I(E|_U)$ denote the differentiable subbundle of $E^r(N)|_U$ which corresponds to $\Sigma^I(U, P)$ by this identification. Let $\{U_\alpha\}$ denote the covering of N such that the bundle π is trivial over U_α for each α. Then we put $\Sigma^I(E) = \cup \Sigma^I(E|_{U_\alpha})$. Then it follows from Remark 4.1 that $\Sigma^I(E)$ is a differentiable subbundle of $E^r(N)$ over N and does not depend on the choice of the covering $\{U_\alpha\}$. We should note that the codimension of $\Sigma^I(E)$ in $E^r(N)$ coincides with that of $\Sigma^I(U, P)$ in $J^r(U, P)$.

DEFINITION 4.2. We call $\Sigma^I(E)$ the *Thom-Boardman singularity with symbol I* of $E^r(N)$.

We define an open set $\Omega^I(E)$ in $E^r(N)$ to be the union of all Thom-Boardman singularities with symbol K such that $K \leqq I$ where we consider the lexicographic order. Since the union of all Thom-Boardman singularities $\Sigma^K(U, P)$, $K \leqq I$ is open in $J^r(U, P)$, we know that $\Omega^I(E)$ is open in $E^r(N)$. Now we consider the integrability of $j^r: \Gamma_\Omega E \to \Gamma_\Omega E^r$ for $\Omega = \Omega^I(E)$.

In the sequel we provide $\pi: E(N) \to N$ with the certain condition which is called '*natural*'. For any n dimensional manifold N there exists a differentiable bundle $E(N)$ such that if U is open in N, then $E(U)$ is the restriction $E(N)$ to U. Moreover for any diffeomorphim h of an open sets V of N, there exists a diffeomorphism $\bar{h}: E(U) \to E(V)$ covering h such that

$\bar{k}\circ\bar{h}=\overline{k\circ h}$ and $\overline{id}_U=id_{E(U)}$. Also $\bar{h}$ depends continuously on h (see, for example [3]).

Let $E'(N')$ be a natural differentiable bundle over $n+1$ dimensional manifolds N' such that $E'(N\times \boldsymbol{R})$ is isomorphic to $E(N)\times \boldsymbol{R}$ over $N\times \boldsymbol{R}$. Then we have a natural map $\bar{\imath}': E'^r(N\times \boldsymbol{R})\to E^r(N)$ which is induced from the inculusion $i: N=N\times O\subset N\times \boldsymbol{R}$. If we consider $\Omega^I(E')$ in $E'^r(N\times \boldsymbol{R})$, then we obtain that $\bar{\imath}(\Omega^I(E'))$ is contained in $\Omega^I(E)$ by the similar arguments as in [4]. It follows from [3, Theorem B] that $j^r: \Gamma_\Omega(E)\to\Gamma_\Omega(E^r)$ is a weak homotopy equivalence for $\Omega=\bar{\imath}(\Omega^I(E'))$. Now we show that $\Omega^I(E)-\bar{\imath}(\Omega^I(E'))$ is a finite union of regular submnaifolds of $\Omega^I(E)$. At first we note that $\Omega^I(E)$ and $\bar{\imath}(\Omega^I(E'))$ are open subbundles over $E(N)$. Their fibers are described as follows. Let $J^r(n,p)$ (resp. $\Omega^I(n,p)$) denote the fibre over the origin $(\boldsymbol{O},\boldsymbol{O})$ of $J^r(\boldsymbol{R}^n,\boldsymbol{R}^p)$ over $\boldsymbol{R}^n\times\boldsymbol{R}^p$ (resp. $\Omega^I(\boldsymbol{R}^n\times\boldsymbol{R}^p)$ where $N=\boldsymbol{R}^n$ and $E(N)=\boldsymbol{R}^n\times\boldsymbol{R}^p$). There is a restriction map $\bar{\imath}: J^r(n+1,p)\to J^r(n,p)$ forgetting the last coordinate. Then the fibre of $\Omega^I(E)$ (resp. $\bar{\imath}(\Omega^I(E'))$ is $\Omega^I(n,p)$ (resp. $\bar{\imath}(\Omega^I(n+1,p))$. If we identify $J^r(n,p)$ with an eucledian space in the usual way, then we know that $\Omega^I(n,p)$ and $\bar{\imath}(\Omega^I(n+1,p))$ are both Zariski open sets. In fact it follows from [7, The Proof of Proposition 2] that $J^r(n,p)-\Omega^I(n,p)$ is a Zariski closed set. It follows from [10] that $J^r(n,p)$ $-\bar{\imath}\Omega^I(n+1,p))$ is a finite union of locally Zariski closed submanifolds of $J^r(n,p)$. Thus $\Omega^I(n,p)-\bar{\imath}(\Omega^I(n+1,p))$ is a finite union of locally Zariski closed submanifolds. Hence we obtain that $\Omega^I(E)-\bar{\imath}(\Omega^I(E'))$ is a finite union of regular submanifolds of $\Omega^I(E)$. We again note that the minimal codimension of these submanifolds coincides with that of the submanifolds of $\Omega^I(n,p)-\bar{\imath}(\Omega^I(n+1,p))$. Let σ^I denote the interger such that σ^I+n+1 is the above codimension. The following theorem is a slight extension of the result in [4, § 1] and we know that $\Omega^I(n,p)$ is equal to $\bar{\imath}(\Omega^I(\mathrm{n}+1,p))$ in this case.

THEOREM 4.3. *Let $\pi: E(N)\to N$ be a natural differentiable fibre bundle with* $\dim N=n$ *and* $\dim P=p$. *Let $I=(i_1,\cdots,i_r)$ and $d^I=\sum_{s=1}^{r-1}\alpha_s$ where*
$$\alpha_s=\begin{cases}1 & \text{if } i_s-i_{s+1}>1.\\ 0 & \text{otherwise}\end{cases}$$
If $i_r>n-p-d^I$, then for $\Omega=\Omega^I(E)$
$$j^r:\Gamma_\Omega(E)\longrightarrow\Gamma_\Omega(E^r)$$
is a weak homotopy equivalence.

Now we give a few applications of Theorem in § 1.

PROPOSITION 4.4. *Let $\pi: E(N)\to N$ be as in Theorem 4.3. Let $\Omega^I(E)$ and σ^I be as defined above. Then $j^r:\Gamma_\Omega(E)\to\Gamma_\Omega(E^r)$ is a σ^I-homotopy equivalence for $\Omega=\Omega^I(E)$.*

Let $K \leqq I$. Then $\Omega^I(E) - \Omega^K(E)$ is the union of Thom-Boardman singularities with symbol H such that $K < H \leqq I$. The codimension of $\Sigma^H(E)$ is determined in [2]. If we take $\Omega^I(E)$ (or $\Omega^K(E)$) as Ω in Theorem 4.3 or Proposition 4.4, we know by applying Theorem in § 1 or Proposition 4.4 how the map $j^r : \Gamma_\Omega(E) \rightarrow \Gamma_\Omega(E^r)$ is close to a homotopy equivalence. For example, let i_1 be fixed. Let i_2 be the positive minimal integer such that $i_2 > n - p - d^I$ where $I = (i_1, i_2)$. Let $K = (i_1, i_2 - 1)$. Then j^r is a homotopy equivalence for $\Omega^I(E)$. Hence j^r induces the isomorphisms of k dimensional homotopy groups where $k < (p - n + i_1)\{i_1(i_2 + 1) - (1/2)\, i_2(i_2 - 1)\} - i_2(i_1 - i_2)$ for $\Omega^K(E)$ since the codimension of $\Sigma^I(E)$ is as mentioned. The examples of the number σ' in Proposition 4.4 are given in [1] for the product bundle $E(N) = N \times P$, which is also valid in our general bundle case.

References

[1] Y. ANDO: On differentiable maps without some Thom-Boardman singularities, to appear.

[2] J. M. BOARDMAN: Singularities of differentiable maps, Publ. I. H. E. S., 33 (1967), 383–419.

[3] A. DU PLESSIS: Homotopy classification of regular sections, Compositio Math., 32 (1976), 301–333.

[4] A. DU PLESSIS: Maps without certain singularities, Comment. Math. Helv., 50 (1975), 363–382.

[5] A. DU PLESSIS: Contact invariant regularity conditions, Springer Lecture Notes, 535 (1975), 205–236.

[6] J. N. MATHER: Stability of C^∞ mappings III, Publ. I. H. E. S., 35 (1968), 127–156.

[7] J. N. MATHER: On Thom-Boardman singularities, Dynamical Systems, Academic Press (1973), 233–248.

[8] C. MORLET: Le lemme de Thom et les théorémes de plongement de Whitney, Séminaire Henri Cartan, 14 (1961/62).

[9] R. THOM: Les singularités des applications différentiables, Ann. Inst. Fourier, 6 (1955–1956), 43–87.

[10] H. WHITNEY: Elementary structure of real algebraic varieties, Ann. of Math., 66 (1956), 545–556.

Department of Mathematics
Hokkaido University