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On the equilibrium existence in abstract economies

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§ 1. Introduction

The purpose of the present paper is to extend earlier equilibrium existence theorems for economies with finitely many agents and with finite-dimensional commodity spaces (see [1] and [8]) to the case of infinite-dimensional commodity spaces. The utilization of Δ or Δ_0, whose existence in the infinite-dimensional space was shown in [12], as the set of price systems is indispensable when it is intended to guarantee the existence of equilibrium for generalized economies. The extension also concerns the recent results to eliminate unnecessary assumptions on consumers' preferences for the proof of equilibrium existence ([4], [6], [9], [10], [11]).

After summarizing the useful auxiliary theorems concerning semi-continuous set-valued mappings in § 2, three types of economies are dealt with in the last three sections of the present paper, respectively. Individual preferences are given in the following three ways: (1) by the utility functions, (2) by the binary relations and (3) by the preference mappings. The present analysis is limited to the pure exchange model only for the sake of conciseness and clarity.

§ 2. Auxiliary theorem concerning semi-continuous set-valued mappings.

Let F be a set-valued mapping assigning to each $x\in X_1$ a subset $F(x)$ of X_2 where X_1 and X_2 are topological spaces. F is called *lower semi-continuous* (briefly l. s. c.) *at* $x_0\in X_1$, if for each open set G meeting $F(x_0)$ there exists a neighbourhood $V(x_0)$ of x_0 such that $F(x)\cap G\neq\phi$ for all $x\in V(x_0)$. F is called *upper semi-continuous* (briefly u. s. c.) *at* $x_0\in X_1$, if for each open set $G\supset F(x_0)$ there exists a neighbourhood $V(x_0)$ of x_0 such that $F(x)\subset G$ for all $x\in V(x_0)$. F is called *lower semi-continuous in* X_1 (briefiy l. s. c. in X_1), if it is lower semi-continuous at each point of X_1. F is a called *upper semi-continuous in* X_1 (briefly u. s. c. in X_1), if it is upper semi-continuous at each point of X_1 and the set $F(x)$ is compact for each $x\in X_1$. If F is both l. s. c. in X_1 and u. s. c. in X_1, then it is called *continuous in* X_1. F

is called *closed*, if the graph of F is closed in the product space $X_1 \times X_2$ endowed with the product topology.

The following lemmas 2.1–2.6 are useful for the later proof of the semi-continuity of the total demand functions.

LEMMA 2.1. *Let F_1 be an u.s.c. mapping of X_1 into X_2 and F_2 be an u.s.c. mapping of X_2 into X_3. Then the composition product $F=F_2F_1$ of F_1 and F_2 is u.s.c. in X_1.*

LEMMA 2.2. *Let F_i be an u.s.c. mapping of a topological space X_0 into a topological space X_i $(i=1, 2, \cdots, n)$. Then the Cartesian product $F=\prod_{i=1}^{n} F_i$ of $F_1, F_2, \cdots, F_n$ is an u.s.c. mapping of X_0 into $\prod_{i=1}^{n} X_i$.*

LEMMA 2.3. *Let F_i be an u.s.c. mapping of a topological space X_i into a topological space Y_i $(i=1, 2)$. Then $F=F_1\times F_2$ is u.s.c. in $X_1\times X_2$, where F is a mapping of $X_1\times X_2$ into $Y_1\times Y_2$ such that $F(x_1, x_2)=F(x_1)\times F(x_2)$ for all $(x_1, x_2)\in X_1\times X_2$.*

LEMMA 2.4. *Let F_1 be an u.s.c. mapping of X_1 into X_2 and F_2 be a closed mapping of X_1 into X_2. Then the mapping $F=F_1\cap F_2$ is u.s.c. in X_1, where F is a mapping of X_1 into X_2 such that $F(x)=F_1(x)\cap F_2(x)$ for each $x\in X_1$.*

LEMMA 2.5. *Let X_2 be compact. Then a mapping $F: X_1\to X_2$ is u.s.c., if and only if it is closed.*

LEMMA 2.6. *A mapping $F: X_1\to X_2$ is l.s.c. if the graph of F is open in $X_1\times X_2$.*

LEMMA 2.7. (*Ky Fan*) *Let E be a non-empty, compact and convex subset of a locally convex Hausdorff linear topological space X and Φ: $E\to E$ be an u.s.c. set-valued mapping such that the set $\Phi(x)$ is a non-empty, closed and convex subset of E for all $x\in E$. Then there exists a point $\bar{x}\in E$ such that $\bar{x}\in\Phi(\bar{x})$.* [2]

LEMMA 2.8. (*An extension of Knaster, Kuratowski, Mazurkiewicz Theorem* [5]). *Let E be a non-empty subset of a Hausdorff linear topological space X. To each $x\in E$, let a closed set $\phi(x)$ in X satisfy the following conditions:*

(1) *the convex hull of any finite subset $\{x_1, x_2, \cdots, x_n\}$ of E is contained in $\bigcup_{i=1}^{n} \phi(x_i)$,*

(2) *$\phi(x)$ is compact for at least one $x\in E$.*

Then $\bigcap_{x\in E}\phi(x)\neq\phi$. [3]

LEMMA 2. 9. *Given* $E=\prod_{i=1}^{n} E_i$, *where* E_i *is a non-empty, compact and convex subset of a separable Banach space* X_i $(i=1, 2, \cdots, n)$, *let* $\psi_i : E \to E_i$ *be a set-valued mapping satisfying*:

(1) *the graph of* ψ_i *is open in* $E \times E_i$,

(2) *the set* $\psi_i(x)$ *is convex or empty for each* $x \in E$.

Then there exists $\bar{x}=(\bar{x}^1, \bar{x}^2, \cdots, \bar{x}^n) \in E$ *such that either* $\bar{x}^i \in \psi_i(\bar{x})$ *or* $\psi_i(\bar{x}) = \phi$ *for each* $i=1, 2, \cdots, n$.

REMARK. Let ψ_{iD_i} denote the restriction of ψ_i to D_i, where $D_i=\{x: x\in E,\ \psi_i(x)\neq\phi\}$. A continuous function $f: D_i \to E_i$ is called a continuous selection for ψ_{iD_i}, if $f(x) \in \psi_{iD_i}(x)$ for every $x \in D_i$. The following Michael's theorem is used in the proof of lemma 2. 9.

THEOREM. *Let* X_1 *be a perfectly normal topological space and* X_2 *be a separable Banach space. Let* $\psi : X_1 \to X_2$ *be a lower semi-continuous set-valued mapping such that the set* $\psi(x_1)$ *belongs to the class* $\mathfrak{D}(X_2)$ *for all* $x_1 \in X_1$. *Then* ψ *admits a continuous selection.*

For the definition of the class $\mathfrak{D}(X_2)$ and the proof of this theorem see [7] p. p. 372—373.

PROOF of LEMMA 2. 9.

Since every metric space is perfectly normal, E is perfectly normal. It follows from (1) that D_i is open in E and the graph of ψ_{iD_i} is also open in $D_i \times E_i$. By making use of lemma 2. 6, ψ_{iD_i} is l. s. c. in D_i. The set $\psi_{iD_i}(x_1)$ is open in E_i for all $x_1 \in D_i$. Hence ψ_{iD_i} satisfies all the assumptions of the above theorem. Therefore ψ_{iD_i} admits a continuous selection $f_i : D_i \to E_i$.

Next, define the mapping $\Phi_i : E \to E_i$ by $\Phi_i(x)=f_i(x)$ if $x \in D_i$, and $\Phi_i(x) = E_i$ otherwise. Then the set $\Phi_i(x)$ is a non-empty, closed and convex subset of E_i for each $x \in E$. Since D_i is open, Φ_i is u. s. c. in E. Define $\Phi : E \to E$ by $\Phi=\prod_{i=1}^{n} \Phi_i$. By making use of lemma 2. 7, there exists $\bar{x}=(\bar{x}^1, \bar{x}^2, \cdots, \bar{x}^n) \in E$ with $\bar{x} \in \Phi(\bar{x})$. $\bar{x} \in \Phi(\bar{x})$ means that $\bar{x}^i \in \Phi_i(\bar{x})$ for all i. If $\bar{x} \in D_i$, then $\bar{x}^i = f_i(\bar{x}) \in \psi_i(\bar{x})$. Otherwise $\psi_i(\bar{x})=\phi$.

§3. Preliminaries on the description of abstract economies.

In the sequel X denotes a locally convex real Hausdorff linear topological space, and its topological dual X^* is assumed to be equipped with the strong topology. Let K be a closed proper cone with vertex θ and K^* be its dual cone with vertex θ^*, where θ and θ^* denote the zero element of X and X^*, respectively. It is assumed that at least one of the following three cases occur:

(α) there exists a non-empty, strongly compact and convex subset $\varDelta$ of K^* such that $\theta^* \notin \varDelta$ and $x^*(x) \geqq 0$ for all $x^* \in \varDelta$ doesn't always imply $x \in K$;

(β) there exists a non-empty strongly compact and convex subset $\varDelta$ of K^* such that $\theta^* \notin \varDelta$ and $x^*(x) \geqq O$ for all $x^* \in \varDelta$ implies $x \in K$;

(γ) there exists a non-empty, strongly compact and convex subset $\varDelta_0$ of a proper subcone $K_0^* \subset K^*$ such that $\theta^* \notin \varDelta_0$ and $x^*(x) \geqq 0$ for all $x^* \in \varDelta_0$ implies $x \in K_0$.

In [12] it is shown that this assumption does not always violate the hypothesis that the space X and the linear subspace $K_0^* - K_0^*$ (or $K^* - K^*$) are infinite-dimensional.

C denotes K and K_0 in the case (α) or (β) and in the case (γ), respectively. The partial order $\underset{c}{\leqq}$ in X is introduced by C, i. e., $x \underset{c}{\leqq} y$ in case $x - y \in C$.

Similarly P is reserved for $\varDelta$ and $\varDelta_0$ in the case (α) or (β) and in the case (γ), respectively, and its typical element is writien as p instead of x^*.

C_i ($i = 1, 2, \cdots, m$) denotes a non-empty, compact and convex subset of C which contains θ and ξ^i satisfying $p(\xi^i) > 0$ for all $p \in P$.

§ 4. Economies with utility functions.

The following definitions are intended to describe a pure exchang economy, in which individual preference relations are given by utility functions:

DEFINITION 4.1. *A set of m ordered triples $\{(C_1, \xi^1, u_1), (C_2, \xi^2, u_2), \cdots, (C_m, \xi^n, u_m)\}$, denoted by $\{(C_i, \xi^i, u_i)\}_{i=1}^m$, is called an economy with utility functions, where $u_i (i = 1, 2, \cdots, m)$ is a continous real-valued function on C. C_i, ξ^i and u_i are called a commodity set, an initial holding and utility function of the economic agent i, respectively.*

DEFINITION 4.2. *A set-valued mapping $F_i : P \to C_i$, defined by $F_i(p) = \{x^i : x^i \in C_i,\ p(x^i) \leqq p(\xi^i)\}$ for all $p \in P$, is called a budget constraint mapping.*

DEFINITION 4.3. *A set-valued mapping $D_i : P \to C_i$, defined by*

$$D_i(p) = \left\{x^i : x^i \in C_i,\ u_i(x^i) = \max_{y^i \in F_i(p)} u_i(y^i)\right\}$$

for all $p \in P$, is called a demand function of the agent i.

A set-valued mapping $D : P \to C_0 = \sum_{i=1}^m C_i$, defined by

$$D(p) = \sum_{i=1}^m D_i(p) = \left\{x : x \in C_0,\ x = \sum_{i=1}^m x^i,\ x^i \in D_i(p)\right\}$$

for all $p \in P$, is called a total demand function of the economy $\{(C_i, \xi^i, u_i)\}_{i=1}^m$.

DEFINITION 4.4. *$(\bar{x}^1, \bar{x}^2, \cdots, \bar{x}^m, \bar{p}) \in C_1 \times C_2 \times \cdots \times C_m \times P$ is called an equi-*

librium (a quasi-equilibrium) for the economy $\{(C_i, \xi^i, u_i)\}_{i=1}^m$ *with respect to* C *and* P, *if* $\bar{x}^i \in D_i(\bar{p})$ *for each* $i=1, 2, \cdots, m$ *and* $\sum_{i=1}^m \bar{x}^i \underset{c}{\leqq} \xi = \sum_{i=1}^m \xi^i \left(p\left(\sum_{i=1}^m \bar{x}^i\right) \leqq p(\xi)\right.$ *for all* $\left. p \in P \right)$.

The notion of quasi-equilibrium is necessary only when C is taken as K in the case (α) of § 3. The expression "with respect to C and P" may be omitted without confusions in the sequel. When there exists an equilibrium (quasi-equilibrium) for the economy, the economy is called to have an equilibrium (quasi-equilibrium). It is noteworthy that the existence of an equilibrium is equivalent to $\xi \in D(\bar{p}) + C$ for some $\bar{p} \in P$.

It plays an important role for the proof of the equilibrium existence that F_i is continuous in P. In other words, it guarantees the equilibrium existence that the budget constraint set satisfying the Walras law in the weak sense varies continuously as p varies.

THEOREM 4. 1. *F_i is continuous in P.*

PROOF. i) Let $F_i(V)$ denote in general $\bigcup_{p \in V} F_i(p)$ for $V \subset P$ and $\mathfrak{V}$ the fundamental (strong) neighbourhood system of $p_0 \in P$. Then it can be shown that $F_i(p_0) = \bigcap_{V \in \mathfrak{V}} (F_i(V))^-$. Here $(F_i(V))^-$ denotes the closure of the set $F_i(V)$.

First $F_i(p_0) \subset \bigcap_{V \in \mathfrak{V}} (F_i(V))^-$. Next let $x_1 \notin F_i(p_0)$ and $x_1 \in C_i$. Then $p_0(x_1) > p_0(\xi^i)$. Put $p_0(x_1) - p_0(\xi^i) = \varepsilon$. Since p_0 is continuous, there exists a neighbourhood $U(x_1)$ of x_1 such that $|p_0(x_2) - p_0(x_1)| < \frac{\varepsilon}{3}$ for all $x_2 \in U(x_1)$. On the other hand, there exists a neighbourhood $V_1(p_0)$ of p_0 such that $|p(\xi^i) - p_0(\xi^i)| < \frac{\varepsilon}{3}$ and $|p(x_2) - p_0(x_2)| < \frac{\varepsilon}{3}$ for all $p \in V_1(p_0)$ and all $x_2 \in U(x_1)$. Then $x_2 \in U(x_1)$ and $p \in V_1(p_0)$ together imply $p(\xi^i) < p_0(\xi^i) + \frac{\varepsilon}{3} < p_0(x_2) - \frac{\varepsilon}{3} < p(x_2)$, which shows $x_2 \notin F_i(V_1)$. Hence $U(x_1) \cap F_i(V_1) = \phi$, and so $x_1 \notin \bigcap_{V \in \mathfrak{V}} (F_i(V))^-$. Thus it has been shown $F_i(p_0) \supset \bigcap_{V \in \mathfrak{V}} (F_i(V))^-$.

ii) Let $p_0 \in P$ and G be an open set in C_i such that $G \supset F_i(p_0)$. By making use of the result of i), $F_i(p_0) = \bigcap_{V \in \mathfrak{V}} (F_i(V))^-$. Putting $H = C_i \cap G^c$, $H \cap \left(\bigcap_{V \in \mathfrak{V}} (F_i(V)^-\right) = \phi$. ($G^c$ denotes in general the complement of G). Since C_i is compact, there exist $V_1, V_2, \cdots, V_n \in \mathfrak{V}$ such that $H \cap \left(\bigcap_{j=1}^n \left(F_i(V_j)\right)^-\right) = \phi$. Then $F_i\left(\bigcap_{j=1}^n V_j\right) \subset G$. Thus there exists a neighbourhood $V_0 = \bigcap_{j=1}^n V_j$ of p_0 such that $F_i(V_0) \subset G$. Combining with the fact that the set $F_i(p_0)$ is compact, this shows that F_i is u. s. c. at p_0. Since p_0 is any point of P, F_i is u. s. c. in P.

iii) Let $p_0 \in P$ and G be an open set in C_i such that $G \cap F_i(p_0) \neq \phi$. Then $p_0(x_1) \leqq p_0(\xi^i)$ for some $x_1 \in G$. First consider the case when $p_0(x_1) < p_0(\xi^i)$. Putting $p_0(\xi^i) - p_0(x_1) = \varepsilon$, there exists a neighbourhood $U_0(p_0)$ of p_0 such that $|p(x_1) - p_0(x_1)| < \frac{\varepsilon}{3}$ and $|p(\xi^i) - p_0(\xi^i)| < \frac{\varepsilon}{3}$ for all $p \in U_0(p_0)$. Then $p \in U_0(p_0)$ implies $p(x_1) < p(\xi^i)$. Hence $x_1 \in F_i(p)$. Thus there exists a neighbourhood $U_0(p_0)$ of p_0 such that $F_i(p) \cap G \neq \phi$ for all $p \in U_0(p_0)$, and hence F_i is l. s. c. at p_0.

Next consider the case when $p_0(x_1) = p_0(\xi^i)$. Since G is open in C_i, there exists λ such that $\lambda x_1 \in G \cap C_i$ and $0 < \lambda < 1$. Then $p_0(\lambda x_1) < p_0(\xi^i)$ and $\lambda x_1 \in G \cap F_i(p_0)$. Using λx_1 instead of x_1 in the above proof of the case when $p_0(x_1) < p_0(\xi^i)$, it can be shown that F_i is l. s. c. at p_0. Since p_0 is any point of P, F_i is l. s. c. in P.

REMARK. It plays a crucial roll that the topology of X^* is strong. If $\theta^* \in P$ and $F_i(\theta^*) = C_i$, then F_i is u. s. c. at θ^*. But F_i is l. s. c. at θ^* if and only if F_i is a constant set-valued mapping.

COROLLARY 4. 1. *D_i is u. s. c. in P.*

PROOF. Define the real-valued function $\psi_i : P \rightarrow R$ by $\psi_i(p) = \max_{y^i \in F_i(p)} u_i(y^i)$. Then, since F_i is continuous in P, ψ_i is continuous in P. The set-valued mapping $D_i^* : P \rightarrow C_i$, defined by $D_i^*(p) = \{x : x \in C_i,\ \psi_i(p) - u_i(x) \leqq 0\}$ for all $p \in P$, is closed, since ψ_i is continuous. By making use of lemma 2. 4, $D_i = F_i \cap D_i^*$ is u. s. c. in P.

THEOREM 4. 2. *The economy $\{(C_i, \xi^i, u_i)\}_{i=1}^m$ has a quasi-equilibrium in the case (α), and it has an equilibrium in the case (β) or (γ).*

PROOF. Consider a price manipulating function, i. e., a set-valued mapping $M : C_0 \rightarrow P$ defined by $M(x) = \{p : p \in P,\ p(x-\xi) = \max_{q \in P} q(x-\xi)\}$ for all $x \in C_0$. Since $X^* \supset P$ is assumed to be endowed with the strong topology, $q(x-\xi)$ is a continuous bilinear function on $C_0 \times P$. Hence the function M^*, defined by $M^*(x) = \max_{q \in P} q(x-\xi)$ for all $x \in C_0$, is continuous in C_0. Therefore, the graph of $M\{(x, p) : M^*(x) - p(x-\xi) \leqq 0\}$ is closed in $C_0 \times P$. Thus M is a closed mapping. Since P is compact, M is u. s. c. in C_0 by making use of lemma 2. 5.

Since the mapping D is a composition product LD^0 of D^0 and L, where $D^0 = \prod_{i=1}^m D_i$ and $L : C_1 \times C_2 \times \cdots \times C_m \rightarrow C_0$ is defined by $L(x^1, x^2, \cdots, x^m) = \sum_{i=1}^m x^i$ for $(x^1, x^2, \cdots, x^m) \in C_1 \times C_2 \times \cdots \times C_m$, D is u. s. c. in P by making use of lemma 2. 1 and 2. 2. Finally the mapping $D \times M : C_0 \times P \rightarrow C_0 \times P$ is u. s. c. in $C_0 \times P$ by making use of lemma 2. 3.

$C_0 \times P$ is convex and compact in a locally convex Hanusdorff linear topological space $X \times X^*$, and the set $D(p) \times M(x)$ is non-empty, convex and closed in $C_0 \times P$. By making use of lemma 2.7, there exists $(\bar{x}, \bar{p}) \in C_0 \times P$ such that $(\bar{x}, \bar{p}) \in D(\bar{p}) \times M(\bar{x})$. For this $(\bar{x}, \bar{p})$, there exists $\bar{x}^i \in D_i(\bar{p})$ for each $i=1, 2, \cdots, m$ such that $\bar{x} = \sum_{i=1}^{m} \bar{x}^i$ and $\bar{p}(\bar{x}-\xi) = \max_{q \in P} q(\bar{x}-\xi)$. Since $\bar{p}(\bar{x}^i) \leqq \bar{p}(\xi^i)$ for all i, $\bar{p}(\bar{x}) \leqq \bar{p}(\xi)$. Consequently $p(\xi - \bar{x}) \geqq 0$ for all $p \in P$. In the case (α), this shows that $(\bar{x}^1, \bar{x}^2, \cdots, \bar{x}^m, \bar{p})$ is a quasi-equilibrium. In the case (β) or (γ), this shows $\xi - \bar{x} \in C$ and hence $\xi \in D(\bar{p}) + C$, which is equivalent to the existence of an equilibrium.

§ 5. Economies with preference orders.

DEFINITION 5.1. *A set of m ordered triples $\{(C_i, \xi^i, \underset{i}{\gtrsim})\}_{i=1}^m$ is called an economy with preference orders, where $\underset{i}{\gtrsim}$, called a preference order of the agent i, is a binary relation on C such that $x \underset{i}{\gtrsim} y$ or $y \underset{i}{\gtrsim} x$ for any $x, y \in C$.*

$x \underset{i}{\gtrsim} y$ and $y \underset{i}{\not\gtrsim} x$ is written as $x \underset{i}{>} y$. $A \underset{i}{>} B$ denotes $x \underset{i}{>} y$ for all $x \in A$ and $y \in B$. Particular cases of preference intervals in C are defined as follows: $C_i^-(x) = \{x' : x' \in C_i,\ x' \underset{i}{\gtrsim} x\}$ and $C_i(x) = \{x' : x' \in C_i,\ x' \underset{i}{>} x\}$.

DEFINITION 5.2. *A set-valued mapping $\tilde{D}_i : P \to C_i$, defined by $\tilde{D}_i(p) = \{x : x \in F_i(p),\ x \underset{i}{\gtrsim} x' \text{ for all } x' \in F_i(p)\}$ for all $p \in P$, is called a demand function of the agent i (in terms of the preference order).*

A set-valued mapping $\tilde{D} : P \to C_0$, defined by

$$\tilde{D}(p) = \left\{x : x \in C_0,\ x = \sum_{i=1}^{m} x^i,\ x^i \in \tilde{D}_i(p) \quad (i = 1, 2, \cdots, m)\right\}$$

for all $p \in P$, is called a total demand function of the economy $\{(C_i, \xi^i, \underset{i}{\gtrsim})\}_{i=1}^m$.

DEFINITION 5.3. *$(\bar{x}^1, \bar{x}^2, \cdots, \bar{x}^m, \bar{p}) \in C_1 \times C_2 \times \cdots \times C_m \times P$ is called an equilibrium (a quasi-equilibrium) for the economy $\{(C_i, \xi^i, \underset{i}{\gtrsim})\}_{i=1}^m$, if $\bar{x}^i \in \tilde{D}_i(\bar{p})$ $(i=1, 2, \cdots, m)$ and $\sum_{i=1}^{m} \bar{x}^i \underset{c}{\leqq} \xi \left(p\left(\sum_{i=1}^{m} \bar{x}\right) \leqq p(\xi) \text{ for all } p \in P\right)$.*

THEOREM 5.1. *Let $C_i^-(x)$ be closed in C_i and $C_i(x)$ be convex for all $x \in C_i$. Then $\tilde{D}_i(p) \neq \phi$.*

PROOF. Let $p \in P$, $E = \{x_1, x_2, \cdots, x_n\} \subset F_i(p)$ and $\psi(x_t) = C_i^-(x_t) \cap \operatorname{conv}(E)$, where $\operatorname{conv}(E)$ denotes the convex hull of E. Then the set $\psi(x_t)$ is closed in C_i. It can be shown that $\operatorname{conv}(\{x_{t_1}, x_{t_2}, \cdots, x_{t_k}\}) \subset \bigcup_{j=1}^{k} \psi(x_{t_j})$ for any subset

$\{x_{t_1}, x_{t_2}, \cdots, x_{t_k}\}$ of E. Suppose the contrary. Then there exists y such that $y \in \mathrm{conv}\,(\{x_{t_1}, x_{t_2}, \cdots, x_{t_k}\})$ and $y \notin \psi(x_{t_j})$ for $j=1, 2, \cdots, k$. $y \notin \psi(x_{t_j})$ implies $y \notin C_i^-(x_{t_j})$, i. e., $x_{t_j} \underset{i}{>} y$ for all $j=1, 2, \cdots, k$. Since $C_i(y)$ is convex, conv $(\{x_{t_1}, x_{t_2}, \cdots, x_{t_k}\}) \subset C_i(y)$. Hence $y \in C_i(y)$, which is a contradiction.

By making use of lemma 2. 8, $\bigcap_{t=1}^{n} \psi(x_t) \neq \phi$. On the other hand, since $F_i(p)$ is convex, $\bigcap_{t=1}^{n} (C_i^-(x_t) \cap F_i(p)) \neq \phi$. Because of the compactness of $F_i(p)$, this shows that $\bigcap_{x \in F_i(p)} (C_i^-(x) \cap F_i(p)) \neq \phi$. Let $x' \in \bigcap_{x \in F_i(p)} (C_i^-(x) \cap F_i(p))$, then $x' \in \tilde{D}_i(p)$.

THEOREM 5. 2. *Let $\tilde{D}_i(p) \neq \phi$ for all $p \in P$. Assume that $x \underset{i}{>} y$ for $x, y \in C_i$ implies $U(x) \underset{i}{>} V(y)$ for a neighbourhood $U(x)$ of x and a neighbourhood $V(y)$ of y. Then $\tilde{D}_i$ is u. s. c. in P.*

PROOF. Consider the set-valued mapping $G_i : P \to C_i$ defined by $G_i(p) = \{x : x \in C_i,\ x \underset{i}{\gtrsim} y$ for all $y \in F_i(p)\}$ for all $p \in P$. Let $p_0 \in P$, $x_1 \in C_i$ and $x_1 \notin G_i(p_0)$. Then $x' \underset{i}{>} x_1$ for some $x' \in F_i(p_0)$, and so there exist a neighbourhood $U(x')$ of x' and a neighbourhood $V(x_1)$ of x_1 such that $U(x') \underset{i}{>} V(x_1)$. On the other hand, since $U(x') \cap F_i(p_0) \neq \phi$, because of the lower semi-continuity of F_i, there exists a negihbourhood $W(p_0)$ of p_0 such that $F_i(p) \cap U(x') \neq \phi$ for all $p \in W(p_0)$. Let $x'' \in F_i(p) \cap U(x')$. Then $x'' \underset{i}{>} V(x_1)$ for $x'' \in F_i(p)$. Hence $V(x_1) \cap G_i(p) = \phi$. Thus it can be shown that the graph of G_i is closed. Finally, since $\tilde{D}_i = F_i \cap G_i$, by making use of lemma 2. 4, $\tilde{D}_i$ is u. s. c. in P.

By utilizing the upper semi-continuity of $\tilde{D}_i(p)$, which is the crucial part of the equilibrium existence proof, the proof of the following theorem is similar to that of theorem 4. 2.

THEOREM 5. 3. *Let the binary relation $\underset{i}{\gtrsim}$ on C satisfy the following conditions*: (1) *$x \underset{i}{\gtrsim} y$ or $y \underset{i}{\gtrsim} x$ for any x, $y \in C$,* (2) *$C_i^-(x)$ is closed in C_i for all $x \in C_i$,* (3) *$C_i(x)$ is convex for all $x \in C_i$,* (4) *$x \underset{i}{>} y$ for x, $y \in C_i$ implies $U(x) \underset{i}{>} V(y)$ for a neighbourhood $U(x)$ of x and a neighbourhood $V(y)$ of y.*

Then the economy $\{(C_i, \xi^i, \underset{i}{\gtrsim})\}_{i=1}^{m}$ has an equilibrium in the case (β) *or* (γ), *and it has a quasi-equilibrium in the case* (α).

§ 6. Economies with preference mappings.

Put $n=m+1$, $C_n=P$ and $C^*=\prod_{j=1}^{n} C_j$.

DEFINITION 6.1. *A set of m ordered triples* $\{(C_i, F_i, \pi_i)\}_{i=1}^{m}$ *is called an economy with preference mappings, where* π_i $(i=1, 2, \cdots, m)$ *is a set-valued mapping of* C^* *into* C_i *and is called a preference mapping of the agent i.*

DEFINITION 6.2. $(\bar{x}, \bar{p})=(\bar{x}^1, \bar{x}^2, \cdots, \bar{x}^m, \bar{p})\in C^*$ *is called an equilibrium (quasi-equilibrium) for the economy* $\{(C_i, F_i, \pi_i)\}_{i=1}^{m}$, *if* $\sum_{i=1}^{m}\bar{x}^i \underset{c}{\leqq} \xi \left(p\left(\sum_{i=1}^{m}\bar{x}^i\right)\leqq p(\xi) \text{ for all } p\in P\right)$ *and, for each* $i=1, 2, \cdots, m$, $\bar{x}^i\in F_i(\bar{p})$ *and* $F_i(\bar{p})\cap\pi_i(\bar{x}, \bar{p})=\phi$.

THEOREM 6.1. *Let the underlying space X be a separable Banach space and* π_i *satisfy, for each* $i=1, 2, \cdots, m$, (1) *the mapping* π_i *has an open graph in* $C^*\times C_i$ *and* (2) *for each* $(x, p)\in C^*$, $\pi_i(x, p)$ *is non-empty, convex and* $x^i\notin\pi_i(x, p)$, *where* $x=(x^1, x^2, \cdots, x^m)\in C_1\times C_2\times\cdots\times C_m$.

Then the economy $\{(C_i, F_i, \pi_i)\}_{i=1}^{m}$ *has an equilibrium in the case* (β) *or* (γ), *and has a quasi-equilibrium in the case* (α).

PROOF. Define $F_i^*: P\to C_i$ $(i=1, 2, \cdots, m)$ by $F_i^*(p)=\{x'^i: x'^i\in C_i,\ p(x'^i)<p(\xi^i)\}$ for all $p\in P$. Then $F_i^*(p)$ is non-empty, and F_i^* has an open graph in $P\times C_i$. By utilizing these F_i^*, define the mapping $\psi_i: C^*\to C_i$ as follows: $\psi_i(x, p)=F_i^*(p)$ if $p(x^i)>p(\xi^i)$; $\psi_i(x, p)=F_i^*(p)\cap\pi_i(x, p)$ if $p(x^i)\leqq p(\xi^i)$, where $x=(x^1, x^2, \cdots, x^m)\in\prod_{i=1}^{m} C_i$ and $p\in C_n$. Furthermore define the mapping ψ_n: $C^*\to C_n(=p)$ by $\psi_n(x, p)=\left\{q: q\in P,\ q\left(\sum_{i=1}^{m}x^i-\xi\right)>p\left(\sum_{i=1}^{m}x^i-\xi\right)\right\}$ for $(x, p)=(x^1, x^2, \cdots, x^m, p)\in C^*$. Then ψ_n has an open graph in $C^*\times C_n$ and the set $\psi_n(x, p)$ is convex. The graph of ψ_i $(i=1, 2, \cdots, m)$ coincides with the set $(G_i\cap H_i)\cup(H_i\cap K_i\cap G_i^c)$, where

$$G_i=\left\{(x, p, x'^i): (x, p)\in C^*, x'^i\in C_i \quad\text{and}\quad p(x^i)>p(\xi^i)\right\},$$

$$H_i=\left\{(x, p, x'^i): (x, p)\in C^*, x'^i\in C_i \quad\text{and}\quad p(x'^i)<p(\xi^i)\right\}$$

and

$$K_i=\left\{(x, p, x'^i): (x, p)\in C^*, x'^i\in C_i \quad\text{and}\quad x'^i\in\pi_i(x, p)\right\}.$$

Since $(G_i\cap H_i)\cup(H_i\cap K_i\cap G_i^c)=(G_i\cap H_i)\cup(H_i\cap K_i)$ and G_i, H_i and K_i are open in $C^*\times C_i$, the graph of ψ_i is also open in $C^*\times C_i$. By making use of lemma 2.9, there exists $(\bar{x}, \bar{p})\in C^*$ such that for each $i=1, 2, \cdots, m$ $\bar{x}^i\in$

$\psi_i(\bar{x}, \bar{p})$ or $\psi_i(\bar{x}, \bar{p})=\phi$ and for ψ_n $\bar{p}\in\psi_n(\bar{x}, \bar{p})$ or $\psi_n(\bar{x}, \bar{p})=\phi$. Since $\bar{x}_i \notin \psi_i(\bar{x}, \bar{p})$ for $i=1, 2, \cdots, m$ and $\bar{p}\notin\psi_n(\bar{x}, \bar{p})$, $\psi_i(\bar{x}, \bar{p})=\phi$ for $i=1, 2, \cdots, n$.

Since it is easy to show that $\bar{x}^i\in F_i(\bar{p})$ and $\sum_{i=1}^{n}\bar{x}^i \underset{c}{\leqq} \xi$ $\left(\text{or } p\left(\sum_{i=1}^{n}\bar{x}^i\right)\leqq p(\xi)\right.$ for all $\left.p\in P\right)$, it remains to show that $F_i(\bar{p})\cap\pi_i(\bar{x}, \bar{p})=\phi$ for all $i=1, 2, \cdots, m$. Contrary to this, suppose that there exists an $x^i\in F_i(\bar{p})\cap\pi_i(\bar{x}, \bar{p})$ for some i. It follows from $F_i^*(\bar{p})\neq\phi$ that $\bar{p}(x'^i)<\bar{p}(\xi^i)$ for some $x'^i\in C_i$. Since the graph of π_i is open in $C^*\times C_i$ and $x^i\in\pi_i(\bar{x}, \bar{p})$, $x''^i=\lambda x'^i+(1-\lambda)x^i\in\pi_i(\bar{x}, \bar{p})$ for sufficiently small λ with $0<\lambda<1$. For this x''^i, $\bar{p}(x''^i)<\bar{p}(\xi^i)$. Hence $x''^i\in F_i^*(\bar{p})\cap\pi_i(\bar{x}, \bar{p})=\psi_i(\bar{x}, \bar{p})$, which is a contradiction.

THEOREM 6.2. *Let the underlying space X be a separable Banach space and π_i satisfy, for each $i=1, 2, \cdots, m$,* (1) *the mapping $\pi_i: C^*\to C_i$ has an open graph in $C^*\times C_i$ and* (2') *for each $(x, p)\in C^*$, $\pi_i(x, p)$ is non-empty and $x^i\notin$ conv $(\pi_i(x, p))$. Then the same conclusion as in theorem* 6.1 *holds.*

The theorem in the form of the generalized n-person game is as follows:

THEOREM 6.3. *Let X be a Banach space. For each i, E_i be a non-empty, compact and convex subset of X, $\Phi_i: E=\prod_{i=1}^{n}E_i\to E_i$ be a continuous set-valued mapping such that $\Phi_i(x)$ is non-empty, compact and convex for each $x\in E$ and $\pi_i: E\to E_i$ be a set-valued mapping such that the graph of π_i is open in $E\times E_i$ and $x^i\notin$ $(\mathrm{conv}\,(\pi_i(x)))^-$ for each $x\in E$, where $x=(x^1, x^2, \cdots, x^n)\in E$.*

Then there exists an $\bar{x}=(\bar{x}^1, \bar{x}^2, \cdots, \bar{x}^n)\in E$ satisfying, for each i, $\bar{x}^i\in\Phi_i(\bar{x})$ and $\pi_i(\bar{x})\cap\Phi_i(\bar{x})=\phi$.

PROOF. By using the norm metric in $E\times E_i$, define the numerical function $d_i: E\times E_i\to R$ so that $d_i(x, x'^i)$ is the distance between (x, x'^i) and the complement of the graph of π_i for each $(x, x'^i)\in E\times E_i$. Then d_i is continuous, and $d_i(x, x'^i)>0$ if and only if $x'^i\in\pi_i(x)$. Define the mapping G_i: $E\to E_i$ by $G_i(x)=\{x''^i: x''^i\in\Phi_i(x),\ d_i(x, x''^i)=\max_{x'^i\in\Phi_i(x)} d_i(x, x'^i)\}$ for all $x\in E$. By making use of the similar argument in the proof of corollary 4.1, $G_i(x)$ is shown to be u.s.c. in E. Furthermore define the mapping $H: E\to E$ by $H(x)=\prod_{i=1}^{n}(\mathrm{conv}\,(G_i(x)))^-$. Then H is u.s.c. in E. Hence, by making use of lemma 2.7, there exists $\bar{x}\in E$ such that $\bar{x}\in H(\bar{x})$. Since $\bar{x}^i\in(\mathrm{conv}(G_i(\bar{x}))^-\subset\Phi_i(\bar{x})$, it remains to prove that $\pi_i(\bar{x})\cap\Phi_i(\bar{x})=\phi$. Contrary to this, suppose that $x'^i\in\pi_i(\bar{x})\cap\Phi_i(\bar{x})$ for some x'^i. Then $d_i(\bar{x}, x'^i)>0$ and $x'^i\in\Phi_i(\bar{x})$. Hence $d_i(\bar{x}, x''^i)>0$ for all $x''^i\in G_i(\bar{x})$, i.e., $G_i(\bar{x})\subset\pi_i(\bar{x})$. Thus $\bar{x}^i\in(\mathrm{conv}\,(G_i(\bar{x}))^-\subset(\mathrm{conv}\,(\pi_i(\bar{x})))^-$, which is a contradiction.

REMARK. If X is finite-dimensional, then the convex hull of a compact set is compact. Hence theorem 6.3 holds under the weaker assumption $x^i \notin \mathrm{conv}(\pi_i(x))$ than the assumption $x^i \notin (\mathrm{conv}(\pi_i(x))^-$. Furthermore, if $\pi_i(x)$ is assumed to be convex, then the assumption $x^i \notin (\mathrm{conv}\, \pi_i(x))$ is equivalent to $x^i \notin \pi_i(x)$. Hence theorem 6.3 gives another simpler proof of theorem 6.1 and theorem 6.2 in the finite-dimensional case. In fact, for each $i=1, 2, \cdots, m$, let $E_i = C_i$, $\Phi_i = F_i$ and π_i satisfy the same assumptions as in theorem 6.1 (6.2), then the economy $\{(C_i, F_i, \pi_i)\}_{i=1}^m$ in theorem 6.1 (6.2) is converted into a generalized n-person game in theorem 6.3.

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