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The Bochner curvature tensor on almost Hermitian manifolds

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Abstract. We prove a decomposition theorem for curvature tensors on a Hermitian vector space over $\boldsymbol{R}$ and use this to introduce Bochner curvature tensors. Applications which include the well known Kähler case are given for almost Hermitian manifolds.

0. Introduction

Singer and Thorpe [11] established a natural decomposition of curvature tensors on an n-dimensional real vector space with inner product and Nomizu [10] used this decomposition to study generalized curvature tensor fields. Kowalski [8] considered also a decomposition theory to study conformal differential geometry. In these papers the Weyl conformal curvature tensor is obtained in a very natural way as a projection of the Riemann curvature tensor.

Sitaramayya [12] and Mori [9] gave a similar decomposition to study curvature tensors on Kähler manifolds.

In this paper we extend these results, based on [14]. First we prove a decomposition theorem for a class of curvature tensors L on a Hermitian vector space V and derive the Bochner curvature tensor associated with L. Then we consider a large class of almost Hermitian manifolds and study some properties of the Bochner curvature tensor field associated with the Riemann curvature structure.

1. Curvature tensors

Let V be an n-dimensional real vector space with inner product g. A tensor L of type (1, 3) over V is a bilinear mapping $L: V\times V\rightarrow \mathrm{Hom}(V, V)$: $(x, y)\rightarrow L(x, y)$. L is called a *curvature tensor* on V if it has the following properties:

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i) $L(x, y) = -L(y, x)$;

ii) $L(x, y)$ is a skew-symmetric endomorphism of V, i. e., $L(x, y, z, w) + L(x, y, w, z) = 0$ where $L(x, y, z, w) = g(L(x, y)\, z, w)$;

iii) $\mathfrak{S} L(x, y)\, z = 0$, where $\mathfrak{S}$ denotes the cyclic sum over x, y and z. This is the first Bianchi identity.

This means also that L is a *symmetric double form of type* (2, 2) which satisfies the first Bianchi identity [2].

The *Ricci tensor* L_R of type (0, 2) associated with L is a symmetric bilinear function on $V \times V$ defined by

$$L_R(x, y) = \text{trace}\left(z \in V \mapsto L(z, x)\, y \in V\right).$$

Then, the Ricci tensor $Q = Q(L)$ of type (1, 1) is given by $L_R(x, y) = g(Qx, y)$ and the trace of Q is called the *scalar curvature* $l = l(L)$ of L.

2. K_i-curvature tensors

Now let V be a $2n$-dimensional real vector space with a complex structure J and a Hermitian product g, i. e.

$$J^2 = -I, \quad g(Jx, Jy) = g(x, y)$$

for all $x, y \in V$, I denoting the identity transformation on V.

The *Ricci* $*$ *tensor* L_R^* of type (0, 2) resp. Q^* of type (1, 1) associated with a curvature tensor L is defined by [5]

$$L_R^*(x, y) = g(Q^* x, y) = \frac{1}{2}\, \text{trace}\left(z \in V \mapsto L(x, Jy)\, Jz \in V\right).$$

$l^* = \text{trace } Q^*$ is called the *scalar* $*$ *curvature*.

It follows from the theory of almost Hermitian manifolds [6], [7] that it is interesting to consider the following identities for a curvature tensor L:

1) $L(x, y, z, w) = L(x, y, Jz, Jw)$;

2) $L(x, y, z, w) = L(Jx, Jy, z, w) + L(Jx, y, Jz, w) + L(Jx, y, z, Jw)$;

3) $L(x, y, z, w) = L(Jx, Jy, Jz, Jw)$.

1) is called the *Kähler identity*. Further, let $\mathscr{K}_i(V)$ denote the vector space of all curvature tensors over V satisfying the identity i).

DEFINITION A *K_i-curvature tensor* L on V is a curvature tensor $L \in \mathscr{K}_i(V)$. Then it is easy to check the following.

THEOREM 1.

i) $\mathscr{K}_1(V) \subset \mathscr{K}_2(V) \subset \mathscr{K}_3(V)$;

ii) Q^* *is a complex linear* ($Q^* \circ J = J \circ Q^*$) *symmetric endomorphism of*

V for any $L \in \mathscr{K}_i(V)$, $i=1, 2, 3$. The same holds for Q.
We prove, for example, that $\mathscr{K}_2(V) \subset \mathscr{K}_3(V)$. Note that the second identity implies

$$L(z, w, Jx, Jy) = L(Jz, Jw, Jx, Jy) - L(Jz, w, x, Jy) - L(Jz, w, Jx, y)\,.$$

Substituting in 2) gives

$$L(x, y, z, w) = L(Jx, Jy, Jz, Jw) + L(Jx, y, z, Jw) - L(x, Jy, Jz, w)\,.$$

Hence we have also

$$L(z, w, x, y) = L(Jz, Jw, Jx, Jy) + L(Jz, w, x, Jy) - L(z, Jw, Jx, y)\,.$$

The required result follows now at once by adding the last two identities.

Finally we give an example of a K_i-curvature tensor which will play an important role in what follows. Let $x \wedge y$ be the skew-symmetric endomorphism of V defined by $(x \wedge y)\,z = g(y, z)\,x - g(x, z)\,y$. Further we define $L_{A,B,\alpha}$ for any complex linear symmetric endomorphisms A and B of V and any $\alpha \in R$, by

$$\begin{aligned} L_{A,B,\alpha}(x, y) = {} & Ax \wedge By + Bx \wedge Ay + JAx \wedge JBy + JBx \wedge JAy \\ & + 2g(Ax, Jy)\,JB - 2g(Jx, By)\,JA + \alpha\Big\{3x \wedge y - Jx \wedge Jy - 2g(x, Jy)\,J\Big\}\,. \end{aligned}$$

It follows immediately that $L_{A,B,\alpha} \in \mathscr{K}_2(V)$ and $L_{A,B,\alpha} \in \mathscr{K}_1(V)$ if and only if $\alpha = 0$.

We need also the following formulas which are easily verified:

$$(1) \qquad \begin{aligned} & Q(L_{A,B,\alpha}) = A \operatorname{tr} B + B \operatorname{tr} A + 2(AB + BA) + 6\alpha(n-1)\,I\,; \\ & Q^*(L_{A,B,\alpha}) = Q(L_{A,B,\alpha}) - 8\alpha(n-1)\,I\,; \\ & l(L_{A,B,\alpha}) = 2 \operatorname{tr} A \operatorname{tr} B + 4 \operatorname{tr} AB + 12\alpha n(n-1)\,; \\ & l^*(L_{A,B,\alpha}) = l(L_{A,B,\alpha}) - 16\alpha n(n-1)\,. \end{aligned}$$

3. Decomposition theorem

In what follows we denote by $\mathcal{L}(V)$ the vector space of K_3-curvature tensors. This is a subspace of the tensor space of type $(1, 3)$ over V and has a natural inner product induced from that on V:

$$\langle L, \tilde{L} \rangle = \operatorname{trace} \tilde{L}^T \circ L = \sum_{i,j,k=1}^{2n} g\Big(L(e_i, e_j)\,e_k, \tilde{L}(e_i, e_j)\,e_k\Big)\,,$$

$\{e_i\}$ being an orthonormal basis of V. Further, let $\mathrm{Hom}_{cs}(V, V)$ denote the space of all complex linear symmetric endomorphisms of V and define the

Ricci contraction map $\mathscr{R}$ as follows:

$$\mathscr{R}: \mathcal{L}(V) \to \mathrm{Hom}_{cs}(V, V) \times \boldsymbol{R}: L \mapsto (Q+3Q^*, l-l^*).$$

Clearly $\mathscr{R}$ is linear. This is also the case for the map h defined by

$$h: \mathrm{Hom}_{cs}(V, V) \times \boldsymbol{R} \to \mathcal{L}(V): (A, \alpha) \mapsto L_{A,I,\alpha}.$$

The definition of $\mathscr{R}$ and h is based on geometrical considerations [14].

LEMMA 1. *$\mathscr{R} \circ h$ is an isomorphism.*

PROOF. It is sufficient to show that $\mathscr{R} \circ h$ is injective. Let $(A, \alpha) \in \mathrm{Hom}_{cs}(V, V) \times \boldsymbol{R}$ with $\mathscr{R}h(A, \alpha)=0$. Hence $l(L_{A,I,\alpha})=l^*(L_{A,I,\alpha})$ and this implies with (1) that $\alpha=0$. Now, using $Q(L_{A,I,\alpha})+3Q^*(L_{A,I,\alpha})=0$ we obtain from (1) $2(n+2)A+I\,\mathrm{tr}\,A=0$ which implies $\mathrm{tr}\,A=0$ and hence $A=0$.

LEMMA 2. Im h *is orthogonal to* Ker $\mathscr{R}$ *in* $\mathcal{L}(V)$.

PROOF. Let $A \in \mathrm{Hom}_{cs}(V, V)$, $L=h(A, \alpha)=L_{A,I,\alpha}$ and $\tilde{L} \in \mathrm{Ker}\,\mathscr{R}$. It is always possible to choose an orthonormal J-basis $\{e_i, Je_i, i=1, 2, \cdots, n\}$ such that $Ae_i=\lambda_i e_i$, $AJe_i=\lambda_i Je_i$. Further we have $\tilde{l}=\tilde{l}^*=0$. The orthogonality follows now by a straightforward calculation using the first Bianchi identity for $\tilde{L}$.

LEMMA 3. $\dim \mathrm{Im}\, h+\dim \mathrm{Ker}\,\mathscr{R}=\dim \mathcal{L}(V)$.

PROOF. It follows from Lemma 1 that h is injective and $\mathscr{R}$ is surjective. Hence $\dim \mathrm{Im}\, h=\dim \mathrm{Hom}_{cs}(V, V) \times \boldsymbol{R}=\dim \mathcal{L}(V)-\dim \mathrm{Ker}\,\mathscr{R}$.

Putting $\mathcal{L}_B(V)=\mathrm{Ker}\,\mathscr{R}$ we have

LEMMA 4. $\mathcal{L}(V)=\mathrm{Im}\, h \oplus \mathcal{L}_B(V)$.

Now let $L \in \mathcal{L}(V)$. Using Lemma 4 we have

$$(2) \qquad L=h(A, \alpha)+L_B, \quad A \in \mathrm{Hom}_{cs}(V, V), \quad \alpha \in \boldsymbol{R}, \quad L_B \in \mathcal{L}_B(V).$$

Define the map $\mathscr{D}: \mathcal{L}(V) \to \mathrm{Hom}_{cs}(V, V) \times \boldsymbol{R}$ by $\mathscr{D}L=(A, \alpha)$. We call $\mathscr{D}$ the deviation map (see also [8]). Hence we have if j denotes the canonical inclusion of $\mathcal{L}_B(V)$ in $\mathcal{L}(V)$:

DECOMPOSITION THEOREM. *There is a unique linear map $\mathscr{B}: \mathcal{L}(V) \to \mathcal{L}_B(V)$, called the Bochner map, and a unique linear map $\mathscr{D}: \mathcal{L}(V) \to \mathrm{Hom}_{cs}(V, V) \times \boldsymbol{R}$, called the deviation map, such that the following commutative diagram with two exact sequences holds:*

$$
\begin{array}{ccccccc}
 & & & 0 & & & \\
 & & & \downarrow & & & \\
 & & & \mathcal{L}_B(V) & & & \\
 & & & \downarrow j & \searrow id & & \\
0 \rightarrow \mathrm{Hom}_{cs}(V, V)\times \boldsymbol{R} & \xrightarrow{h} & & \mathcal{L}(V) & \xrightarrow{\mathscr{B}} & \mathcal{L}_B(V) \rightarrow 0\,. & \\
 & id \searrow & & \downarrow \mathscr{D} & & & \\
 & & & \mathrm{Hom}_{cs}(V, V)\times \boldsymbol{R} & & & \\
 & & & \downarrow & & & \\
 & & & 0 & & &
\end{array}
$$

Moreover, the decomposition $\mathcal{L}(V)=\mathrm{Im}\, h\oplus\mathcal{L}_B(V)$ *is orthogonal and hence the Bochner map* $\mathscr{B}$ *is the orthogonal projection of* $\mathcal{L}(V)$ *onto its subspace* $\mathcal{L}_B(V)$.

It is possible to express the maps $\mathscr{B}$ and $\mathscr{D}$ explicitly. Indeed we have

THEOREM 2. *Let* $L\in\mathcal{L}(V)$. *Then*

$$\mathscr{D}L=(A, \alpha)\,,$$
$$\mathscr{B}L=L-h\mathscr{D}L=L-L_{A,I,\alpha}$$

where

$$A=\frac{1}{8(n+2)}\left(Q+3Q^*-\frac{l+3l^*}{4(n+1)}\,I\right),\quad \alpha=\frac{1}{16n(n-1)}\,(l-l^*)\,.$$

PROOF. It follows from (2) that $l-l^*=l(L_{A,I,\alpha})-l^*(L_{A,I,\alpha})$. This gives the required expression for α. Further we have $Q+3Q^*=Q(L_{A,I,\alpha})+3Q^*(L_{A,I,\alpha})$ and this implies $8(n+2)\,A=Q+3Q^*-4I\,\mathrm{tr}\,A$. Taking the trace we get $16(n+1)\,\mathrm{tr}\,A=l+3l^*$ and this gives the formula for A.

DEFINITION [14]. Curvature tensors belonging to $\mathcal{L}_B(V)$ are called *Bochner curvature tensors* and the tensor L_B is called the *Bochner curvature tensor* associated with $L\in\mathcal{L}(V)$.

4. Applications

Let M be a Riemannian manifold with metric tensor g and Riemannian connection ∇. For each point $m\in M$ we may consider curvature tensors L over the tangent space $T_m(M)$ with inner product g_m. A differentiable curvature tensor field L on M is called a *generalized curvature tensor field* [10]. We recall that L is *proper* [10] or a *Riemannian double form of* *type* (2, 2) [2] if it satisfies the second Bianchi identity, that is $\mathfrak{S}(\nabla_X L)(Y, Z)=0$ for all $X, Y, Z \in \mathscr{X}(M)$, where $\mathscr{X}(M)$ denotes the Lie algebra of C^∞ vector fields on M.

Now, let M be an almost Hermitian manifold, that is, the tangent bundle has an almost complex structure J and a Riemannian metric g such that $g(JX, JY)=g(X, Y)$ for all $X, Y \in \mathscr{X}(M)$. In the same way as before we may define *generalized K_i-curvature tensor fields.*

In what follows we take for L the Riemann-Christoffel curvature tensor R. This is a proper tensor field and we suppose that $R_m \in \mathcal{L}(V)$, $V=T_m M$. Then we find that the associated Bochner tensor $R_B=B$ is the well known Bochner tensor if M is a Kähler manifold ($\nabla J=0$). For such a manifold we know that M is a *complex space form* if and only if M is Bochner flat ($B=0$) and Einsteinian.

In order to state a generalization we say that a curvature tensor L is *Einsteinian* resp. **Einsteinian* if $Q=\lambda I$ resp. $Q^*=\lambda^* I$ and we recall that an almost Hermitian manifold with $R \in \mathcal{L}(V)$ is a generalized complex space form [13] if an only if $R=L_{\frac{\mu}{8}I, I, \frac{\alpha}{4}}$, i. e.

$$R(X, Y)=\frac{\mu+3\alpha}{4} X \wedge Y+\frac{\mu-\alpha}{4}\{JX \wedge JY+2g(X, JY)J\};$$

μ is the holomorphic sectional curvature and $\frac{1}{4}(\mu+3\alpha)$ the antiholomorphic sectional curvature. Both curvatures are pointwise constant. Further, a *nearly Kähler manifold* M is an almost Hermitian manifold such that $(\nabla_X J)X=0$ for all $X \in \mathscr{X}(M)$ [3]. It follows then that S^6 is a generalized complex space form with respect to the three nearly Kähler structures on S^6 [1]. We have also that for any nearly Kähler manifold $R \in \mathcal{L}(V)$ [3]. So we obtain

THEOREM 3. *Let M be an almost Hermitian manifold with $R \in \mathcal{L}(V)$. Then M is a generalized complex space form if and only if it is a Bochner flat Einstein and *Einstein manifold.*

Finally, using a classification theorem of A. Gray [4], we have

THEOREM 4. *Let M^n be a nearly Kähler manifold with complex dimension $n>2$. Then M^n is a Bochner flat Einstein and *Einstein manifold if and only if it is locally isometric to a complex space from (C^n, $CP^n(\mu)$ or $CD^n(\mu)$) or $S^6(\mu)$, μ denoting the holomorphic sectional curvature.*

5. Remarks

a. Suppose M^n is almost Hermitian and $R \in \mathcal{L}(V)$. One finds that the properties for B in relation with the theory of submanifolds are analogous with those in the Kähler case (see for example [14]).

b. The given decomposition method applies also to derive the well known concircular and projective curvature tensors as projections of the Riemann tensor. This is also true for the complex analogous tensors on an almost Hermitian manifold. Moreover, one may introduce in the same way a Bochner curvature tensor on a class of almost contact metric manifolds which includes for example the Sasakian, nearly Sasakian and normal cosymplectic manifolds. This will be shown in another paper.

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