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Characteristic mixed problems for hermitian systems in three unknowns

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§ 1. Introduction and results

The purpose of this paper is to prove that the results of Strang [10] for 2×2 systems are also valid for hermitian 3×3 systems with characteristic boundary including the linearized shallow water equations. This has been conjectured by Majda and Osher [5].

We consider the mixed problems for hermitian systems of first order in the quarter space $t\geqq 0$, $x\geqq 0$, $y=(y_1, \cdots, y_n)\in \boldsymbol{R}^n$:

$$(1.1)\qquad \begin{aligned} &\frac{\partial u}{\partial t}+A\frac{\partial u}{\partial x}+\sum_{j=1}^{n}A_j\frac{\partial u}{\partial y_j}=f && \text{in } t>0,\ x>0,\ y\in \boldsymbol{R}^n, \\ &u(0, x, y)=u_0(x, y) && \text{in } x>0,\ y\in \boldsymbol{R}^n, \\ &Bu(t, 0, y)=0 && \text{in } t>0,\ y\in \boldsymbol{R}^n. \end{aligned}$$

Here we assume A and A_j to be constant, hermitian 3×3 matrices, the boundary $x=0$ to be characteristic; that is, $\det A=0$. Furthermore, we assume B to be a constant $l\times 3$ matrix whose rank l is equal to the number of positive eigenvalues of A. In the treatment of characteristic mixed problems it is natural to assume that the problem (1.1) is reflexive, that is, $\ker A\subset \ker B$ (see Kubota and Ohkubo [4] and Rauch [9]).

Our problem is whether there exists a solution u of (1.1) satisfying the following energy inequality: There is a constant $C_T>0$ for each $T>0$ such that

$$(1.2)\qquad \|u(t)\|\leqq C_T\Big(\|u_0\|+\int_0^t\|f(s)\|\,ds\Big)$$

for any t with $0\leqq t\leqq T$. Here $\|\cdot\|$ stands for the usual L^2-norm in the half space $x>0$, $y\in \boldsymbol{R}^n$.

A sufficient condition for the existence of a solution of (1.1) satisfying (1.2) has been already established by Friedrichs [2] and Lax and Phillips [7]. This condition is called "maximally non-positive"; that is, after a non-singular transformation $v=Tu$ of unknowns such that $A'=T^{-1}AT$ and $A'_j=T^{-1}A_jT$ are hermitian, it holds that

(1. 3) $\quad (A'v, v) \leqq 0 \quad$ for any $v \in \ker BT$

and $\ker BT$ is a maximal subspace of $\boldsymbol{C}^3$ satisfying the above property.

For the mixed problems with zero initial data, the necessary and sufficient conditions for "L^2-well posedness" have been developed in [1] and Ohkubo and Shirota [8] for the case of non-characteristic boundary and in [4] for mainly Maxwell's equations. The problem (1. 1) with zero initial data is said to be L^2-well posed if there exist positive constants C and T which satisfy the following condition: For every f whose first derivatives belong to L^2_T and that $f=0$ in $t<0$, the mixed problem has a unique solution u such that $\partial u/\partial t$, $\partial u/\partial y_j$ and $A\partial u/\partial x$ belong to L^2_T and it holds

(1. 4) $$\int_0^T ||u(t)||\,dt \leqq C\int_0^T ||f(t)||\,dt\,.$$

Here L^2_T denotes the space of all square intergrable functions in $0<t<T$, $x>0$, $y \in \boldsymbol{R}^n$.

In [1], [4] and [8] L^2-well posed mixed problems are characterized by compensating functions or reflection coefficients. Using the methods of these characterizations we prove the following

THEOREM. *If the mixed problem* (1. 1) *is L^2-well posed, then it is maximally non-positive.*

As compared with the case of non-characteristic boundary one of significant features of characteristic mixed problems is the behavior of reflection coefficient $C(\tau, \sigma)$ in a neighbourhood of $(0, \sigma)$, where (τ, σ) being a covector of (t, y) and (1. 1) will be transformed a simple form stated in § 2. In particular, from the proof of Proposition 3. 6 in § 3 we present the following example such that Lopatinski determinant $R(\tau, \sigma)$ does not vanish for $\tau=0$ but there exist microlocally no symmetrizers:

$$A=\begin{pmatrix} 1 & & 0 \\ & -1 & \\ 0 & & 0 \end{pmatrix}, \quad A_1=\begin{pmatrix} 3 & 2 & 1 \\ 2 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix},$$
$$B=(1, 2, 0)\,.$$

The 2×2 systems considered in [10] are not necessarily hermitian, because they are symmetrizable in virtue of strong hyperbolicity (see Strang [11]). However, Lax [6] gave already an example of 3×3 system which is strongly hyperbolic but not symetrizable. Thus our problem is left open for non-symmetrizable systems.

The author thanks K. Kubota for useful conversations related to this work.

§ 2. Proof of Theorem (special cases) and necessary conditions for L^2-well posedness.

Since A and A_j are hermitian and $\det A=0$, there exists a unitary transformation $u=Tv$ of unknowns such that the problem (1. 1) with zero initial data is equivalent to the following one:

$$
\begin{aligned}
&\frac{\partial v}{\partial t}+A'\frac{\partial v}{\partial x}+\sum_{j=1}^{n}A'_j\frac{\partial v}{\partial y_j}=g\,,\\
&v(0,x,y)=0\,,\\
&B'v(t,0,y)=0\,.
\end{aligned}
\tag{2. 1}
$$

Here $B'=BT$, $A'_j=T^{-1}A_jT$ are hermitian and

$$
A'=\begin{pmatrix} a_1 & & 0\\ & a_2 & \\ 0 & & 0\end{pmatrix}.
$$

If $a_1=a_2=0$, then (2. 1) is the Cauchy problem. Thus we may assume that $a_1\neq 0$. Furthermore, if $a_1<0$ and $a_2\leqq 0$, then there exist no non-trivial L^2-solutions of (2. 1). This contradicts L^2-well posedness of (2. 1).

There are two special cases in the proof of Theorem. First if $a_1>0$ and $a_2=0$, then $\operatorname{rank} B'=1$ and the reflexivity of the problem (2. 1) implies that only the first component of B' is non-zero. Hence we see easily that the inequality (1. 3) holds. If $a_1>0$ and $a_2>0$, then $\operatorname{rank} B'=2$ and the third column of B' is zero. Hence (1. 3) holds. *Therefore we may assume hereafter that*

$$
a_1>0,\ a_2<0 \ \textit{and}\ B'=(1,b,0)\,, \tag{2. 2}
$$

where b is a complex number. The form of B' follows from the reflexivity and L^2-well posedness of (2. 1) (see Lemma 2. 10 in [4], I).

Second special cases are as follows. If either

$$
a_{12}^{(j)}=a_{13}^{(j)}=0 \text{ for all } j \text{ or } a_{12}^{(j)}=a_{23}^{(j)}=0 \text{ for all } j\,, \tag{2. 3}
$$

then we verify similar to [10] that the inequality (1. 3) holds after some change $v=Sw$ of unknowns. Here $a_{kl}^{(j)}$ stands for (k,l)-element of A'_j. In fact, let S be

$$
\begin{pmatrix} \alpha & & 0\\ & 1 & \\ 0 & & 1\end{pmatrix} \text{ or } \begin{pmatrix} 1 & & 0\\ & \alpha & \\ 0 & & 1\end{pmatrix},
$$

respectively, where $\alpha \neq 0$. Then we see that $S^{-1}A'_j S = A'_j$, $S^{-1}A'S = A'$ and

$$B'S = (\alpha, b, 0) \quad \text{or} \quad (1, \alpha b, 0)$$

respectively. Therefore we obtain

$$(A'w, w) = (a_1|b|^2/\alpha^2 + a_2)|w_2|^2 \quad \text{or} \quad (a_1|b|^2\alpha^2 + a_2)|w_2|^2$$

for $w = {}^t(w_1, w_2, w_3) \in \ker B'S$, respectively. Thus the assertion is valid by taking α as a sufficiently large or small number, respectively. We remark that the above argument is applicable to the case when y variables are absent.

For general cases we shall prove in the next section that *if the mixed problem* (2. 1) *is not maximally non-positive for* A', *that is,*

$$(2.4) \qquad a_1|b|^2 + a_2 > 0\,,$$

then it is not L^2*-well posed.* As a preparation to prove this assertion, we drive the necessary conditions for L^2-well posedness in the rest of this section.

According to [5] we may assume by a change of variables that

$$(2.5) \qquad a_{33}^{(j)} = 0 \qquad \text{for all } j\,.$$

Such a change of variables is as follows:

$$t' = t,\ x' = x,\ y'_j = y_j - a_{33}^{(j)}t \qquad (j = 1, \cdots, n)\,.$$

We remark that a_1, a_2, $a_{12}^{(j)}$, $a_{13}^{(j)}$ and $a_{23}^{(j)}$ are invariant by the above change of variables.

We now define Lopatinski determinant and reflection coefficient for the problem (2. 1). Let $\hat{v}$ be a partial Fourier transform of v:

$$\hat{v}(\tau, x, \sigma) = \int e^{-i(t\tau + \sum_{j=1}^{n} y_j\sigma_j)} v(t, x, y)\, dtdy\,,$$

where $\operatorname{Im} \tau < 0$ and σ_j are real. Then we have from (2. 1) and (2. 5) that

$$-i\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \frac{\partial}{\partial x} \begin{pmatrix} \hat{v}_1 \\ \hat{v}_2 \end{pmatrix} + \begin{pmatrix} \tau + \sum a_{11}^{(j)}\sigma_j & \sum a_{12}^{(j)}\sigma_j \\ \sum \bar{a}_{12}^{(j)}\sigma_j & \tau + \sum a_{22}^{(j)}\sigma_j \end{pmatrix} \begin{pmatrix} \hat{v}_1 \\ \hat{v}_2 \end{pmatrix}$$
$$+ \begin{pmatrix} \sum a_{23}^{(j)}\sigma_j \\ \sum a_{13}^{(j)}\sigma_j \end{pmatrix} \hat{v}_3 = \begin{pmatrix} \widehat{-ig_1} \\ \widehat{-ig_2} \end{pmatrix},$$
$$\left(\sum \bar{a}_{13}^{(j)}\sigma_j, \sum \bar{a}_{23}^{(j)}\sigma_j\right) \begin{pmatrix} \hat{v}_1 \\ \hat{v}_2 \end{pmatrix} + \tau\hat{v}_3 = \widehat{-ig_3}\,,$$

where $\hat{v} = {}^t(\hat{v}_1, \hat{v}_2, \hat{v}_3)$ and $a_{kl}^{(j)} = \bar{a}_{lk}^{(j)}$ for $k > l$. By setting

$$\alpha_{kl}(\sigma) = \sum_{j=1}^{n} a_{kl}^{(j)}\sigma_j$$

and eliminating $\hat{v}_3$ in the equations above we then obtain the system of ordinary differential equations with parameter (τ, σ):

$$
(2.6)\qquad \begin{aligned} &\left(\frac{\partial}{i\partial x}+M(\tau,\sigma)\right)\begin{pmatrix}\hat{v}_1(\tau,x,\sigma)\\ \hat{v}_2(\tau,x,\sigma)\end{pmatrix}=\begin{pmatrix}h_1(\tau,x,\sigma)\\ h_2(\tau,x,\sigma)\end{pmatrix} \quad \text{in } x>0\,,\\ &\hat{v}_1(\tau,0,\sigma)+b\hat{v}_2(\tau,0,\sigma)=0\,. \end{aligned}
$$

Here $h_j(\tau, x, \sigma)=(\widehat{-ig_j})(\tau, x, \sigma)-\tau^{-1}\alpha_{j3}(\sigma)(\widehat{-ig_3})(\tau, x, \sigma)$ $(j=1, 2)$ and

$$
M(\tau,\sigma)=\begin{pmatrix}a_1 & 0\\ 0 & a_2\end{pmatrix}^{-1}\begin{pmatrix}\tau+\alpha_{11}(\sigma)-|\alpha_{13}(\sigma)|^2\tau^{-1} & \alpha_{12}(\sigma)-\alpha_{13}(\sigma)\overline{\alpha_{23}(\sigma)}\,\tau^{-1}\\ \overline{\alpha_{12}(\sigma)}-\overline{\alpha_{13}(\sigma)}\,\alpha_{23}(\sigma)\,\tau^{-1} & \tau+\alpha_{22}(\sigma)-|\alpha_{23}(\sigma)|^2\tau^{-1}\end{pmatrix}.
$$

If we set

$$
(2.7)\qquad s_1=a_1^{-1} \quad\text{and}\quad s_2=-a_2^{-1}\,,
$$

we obtain

$$
(2.8)\qquad M(\tau,\sigma)=\begin{pmatrix}s_1\left(\tau+\alpha_{11}(\sigma)-|\alpha_{13}(\sigma)|^2\tau^{-1}\right) & s_1\left(\alpha_{12}(\sigma)-\alpha_{13}(\sigma)\overline{\alpha_{23}(\sigma)}\,\tau^{-1}\right)\\ -s_2\left(\overline{\alpha_{12}(\sigma)}-\overline{\alpha_{13}(\sigma)}\,\alpha_{23}(\sigma)\,\tau^{-1}\right) & -s_2\left(\tau+\alpha_{22}(\sigma)-|\alpha_{23}(\sigma)|^2\tau^{-1}\right)\end{pmatrix}.
$$

Let $\lambda^+(\tau, \sigma)$ or $\lambda^-(\tau, \sigma)$ be a root with positive or negative imaginary part of the characteristic equation $\det(\lambda+M(\tau, \sigma))=0$ respectively. Furthermore, let $U^\pm(\tau, \sigma)$ be an eigenvector of $-M(\tau, \sigma)$ associated to $\lambda^\pm(\tau, \sigma)$ respectively. Therefore Lopatinski determinant $R(\tau, \sigma)$ and the reflection coefficient $C(\tau, \sigma)$ are defined by

$$
(2.9)\qquad R(\tau,\sigma)=B'\,U^+(\tau,\sigma)
$$

and

$$
(2.10)\qquad C(\tau,\sigma)=B'\,U^-(\tau,\sigma)/B'\,U^+(\tau,\sigma)
$$

(for instant see § 2 of [4], I). Through this paper we take $U^+(\tau, \sigma)$ as the following form:

$$
(2.11)\qquad U^+(\tau,\sigma)=\begin{pmatrix}\lambda^+(\tau,\sigma)-s_2\left(\tau+\alpha_{22}(\sigma)-|\alpha_{23}(\sigma)|^2\tau^{-1}\right)\\ s_2\left(\overline{\alpha_{12}(\sigma)}-\overline{\alpha_{13}(\sigma)}\,\alpha_{23}(\sigma)\,\tau^{-1}\right)\end{pmatrix}.
$$

From (2.9) and (2.11) we have

$$
(2.12)\qquad \begin{aligned} R(\tau,\sigma)&=\lambda^+(\tau,\sigma)-s_2\left(\tau+\alpha_{22}(\sigma)-|\alpha_{23}(\sigma)|^2\tau^{-1}\right)\\ &\quad+s_2 b\left(\overline{\alpha_{12}(\sigma)}-\overline{\alpha_{13}(\sigma)}\,\alpha_{23}(\sigma)\,\tau^{-1}\right). \end{aligned}
$$

Here we remark that, *in the second special cases mentioned above, the terms of $R(\tau, \sigma)$ involving b vanish identically.*

We next introduce the compensating matrix function $G(\tau, \sigma; x, s)$ considered in [1], [4] and [8] for the mixed problem (2. 1). Let v' be a unique solution of Cauchy problem:

$$Lv' = g \quad \text{and} \quad v'(0, x, y) = 0\,,$$

where $L=\partial/\partial t+A'\,\partial/\partial x+\sum A'_j\partial/\partial y$. We then see from this and (2. 1) that $w=v-v'$ satisfies the equations:

$$\begin{aligned} &Lw = 0 && \text{in } t>0,\ x>0,\ y\in \boldsymbol{R}^n\,, \\ &B'w = -B'v' && \text{in } t>0,\ x=0,\ y\in \boldsymbol{R}^n\,, \\ &w = 0 && \text{in } t=0,\ x>0,\ y\in \boldsymbol{R}^n\,. \end{aligned}$$

Thus the compensating matrix fnuction $G(\tau, \sigma;\ x, s)$ is defined by

$$\hat{w}(\tau, x, \sigma) = \int_0^\infty G(\tau, \sigma; x, s)\,\widehat{(-ig)}\,(\tau, s, \sigma)\,ds\,. \tag{2. 13}$$

According to the formula (4. 12) in § 4 of [4], I, the compensating matrix function is explicity expressed by

$$G(\tau, \sigma;\ x, s) = C(\tau, \sigma)\,e^{i(\lambda^+(\tau,\sigma)x-\lambda^-(\tau,\sigma)s)}\,\mathscr{B}(\tau, \sigma)\,.$$

Here

$$\begin{aligned} \mathscr{B}(\tau, \sigma) = &\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -\overline{\alpha_{13}(\sigma)}\,\tau^{-1} & -\overline{\alpha_{23}(\sigma)}\,\tau^{-1} \end{pmatrix} U^+(\tau, \sigma)\,T_2(\tau, \sigma) \\ &\times \begin{pmatrix} s_1 & 0 \\ 0 & -s_2 \end{pmatrix} \begin{pmatrix} 1 & 0 & -\alpha_{13}(\sigma)\,\tau^{-1} \\ 0 & 1 & -\alpha_{23}(\sigma)\,\tau^{-1} \end{pmatrix} \end{aligned} \tag{2. 14}$$

and $T_2(\tau, \sigma)$ denotes the second row of the inverse of 2×2 matrix $T(\tau, \sigma)$ such that

$$T(\tau, \sigma)^{-1}M(\tau, \sigma)\,T(\tau, \sigma) = \begin{pmatrix} -\lambda^+(\tau, \sigma) & 0 \\ 0 & -\lambda^-(\tau, \sigma) \end{pmatrix}. \tag{2. 15}$$

We shall prove that there exists the above matrix $T(\tau, \sigma)$ which is non-singular in a neighourhood of each (τ_0, σ_0) with Im $\tau_0<0$. *Hereafter a neighbourhood will be considered in* Im $\tau<0$. To do this, we seek concrete expressions of $\lambda^\pm(\tau, \sigma)$. It follows from (2. 8) that the characteristic equation $\det(\lambda+M(\tau, \sigma))=0$ is equivalent to

$$\begin{aligned} &\lambda^2+\Big\{s_1\big(\tau+\alpha_{11}(\sigma)\big)-s_2\big(\tau+\alpha_{22}(\sigma)\big)+\big(s_2|\alpha_{23}(\sigma)|^2-s_1|\alpha_{13}(\sigma)|^2\big)\,\tau^{-1}\Big\}\,\lambda \\ &\quad -s_1s_2\big(\tau+\alpha_{11}(\sigma)\big)\big(\tau+\alpha_{22}(\sigma)\big)+s_1s_2|\alpha_{12}(\sigma)|^2 \end{aligned}$$

$$+s_1 s_2 \tau^{-1}\Big\{|\alpha_{23}(\sigma)|^2\big(\tau+\alpha_{11}(\sigma)\big)+|\alpha_{13}(\sigma)|^2\big(\tau+\alpha_{22}(\sigma)\big)$$
$$-2\mathrm{Re}\big(\overline{\alpha_{12}(\sigma)}\,\alpha_{13}(\sigma)\,\overline{\alpha_{23}(\sigma)}\big)\Big\}=0\,.$$

Hence we have

$$2\lambda^{\pm}(\tau,\sigma)=-s_1\big(\tau+\alpha_{11}(\sigma)\big)+s_2\big(\tau+\alpha_{22}(\sigma)\big) \tag{2.16}$$
$$+\big(s_1|\alpha_{13}(\sigma)|^2-s_2|\alpha_{23}(\sigma)|^2\big)\,\tau^{-1}\pm D(\tau,\sigma)^{1/2}\,.$$

Here $D(\tau,\sigma)$ is the discriminant of the above equation and the branch $z^{1/2}$ will be determined by

$$z^{1/2}=\mathrm{sgn}\,\mathrm{Im}\,z\sqrt{\frac{|z|+\mathrm{Re}\,z}{2}}+i\sqrt{\frac{|z|-\mathrm{Re}\,z}{2}}\qquad(\sqrt{1}=1) \tag{2.17}$$

and $z^{1/2}$ for $z>0$ is defined by continuity. By definition of $D(\tau,\sigma)$ we have

$$D(\tau,\sigma)=\Big\{s_1\big(\tau+\alpha_{11}(\sigma)\big)-s_2\big(\tau+\alpha_{22}(\sigma)\big)+\big(s_2|\alpha_{23}(\sigma)|^2-s_1|\alpha_{13}(\sigma)|^2\big)\,\tau^{-1}\Big\}^2$$
$$+4s_1 s_2\big(\tau+\alpha_{11}(\sigma)\big)\,\big(\tau+\alpha_{22}(\sigma)\big)-4s_1 s_2|\alpha_{12}(\sigma)|^2$$
$$-4s_1 s_2\tau^{-1}\Big\{|\alpha_{23}(\sigma)|^2\big(\tau+\alpha_{11}(\sigma)\big)+|\alpha_{13}(\sigma)|^2\big(\tau+\alpha_{22}(\sigma)\big)$$
$$-2\,\mathrm{Re}\,\big(\overline{\alpha_{12}(\sigma)}\,\alpha_{13}(\sigma)\,\overline{\alpha_{23}(\sigma)}\big)\Big\}$$

which implies that

$$D(\tau,\sigma)=\Big\{s_1\big(\tau+\alpha_{11}(\sigma)\big)+s_2\big(\tau+\alpha_{22}(\sigma)\big)\Big\}^2-2\big(s_1|\alpha_{13}(\sigma)|^2$$
$$+s_2|\alpha_{23}(\sigma)|^2\big)\,\Big\{s_1\big(\tau+\alpha_{11}(\sigma)\big)+s_2\big(\tau+\alpha_{22}(\sigma)\big)\Big\}\,\tau^{-1} \tag{2.18}$$
$$-4s_1 s_2|\alpha_{12}(\sigma)|^2+8s_1 s_2\tau^{-1}\,\mathrm{Re}\,\big(\overline{\alpha_{12}(\sigma)}\,\alpha_{13}(\sigma)\,\overline{\alpha_{23}(\sigma)}\big)$$
$$+\Big(s_1|\alpha_{13}(\sigma)|^2-s_2\big(\alpha_{23}(\sigma)|^2\big)\Big)\,\tau^{-2}\,.$$

Put

$$S(\tau,\sigma)=s_1\big(\tau+\alpha_{11}(\sigma)\big)+s_2\big(\tau+\alpha_{22}(\sigma)\big)-\big(s_1|\alpha_{13}(\sigma)|^2+s_2|\alpha_{23}(\sigma)|^2\big)\tau^{-1}, \tag{2.19}$$

it then follows from (2.18) and (2.19) that

$$D(\tau,\sigma)=S(\tau,\sigma)^2-4s_1 s_2\Big\{|\alpha_{12}(\sigma)|^2-2\,\mathrm{Re}\,\big(\overline{\alpha_{12}(\sigma)}\,\alpha_{13}(\sigma)\,\overline{\alpha_{23}(\sigma)}\big)\tau^{-1}$$
$$+\big(|\alpha_{13}(\sigma)\,\alpha_{23}(\sigma)|^2\tau^{-2}\big)\Big\}=S(\tau,\sigma)^2-4s_1 s_2 \tag{2.20}$$
$$\times\big(\alpha_{12}(\sigma)-\alpha_{13}(\sigma)\,\overline{\alpha_{23}(\sigma)}\,\tau^{-1}\big)\,\big(\overline{\alpha_{12}(\sigma)}-\overline{\alpha_{13}(\sigma)}\,\alpha_{23}(\sigma)\,\tau^{-1}\big)\,.$$

Thus we obtain from (2. 11), (2. 12), (2. 16) and (2. 19) that

$$(2.21)\qquad 2U^{+}(\tau, \sigma)=\begin{pmatrix} -S(\tau, \sigma)+D(\tau, \sigma)^{1/2} \\ 2s_2\left(\overline{\alpha_{12}(\sigma)}-\overline{\alpha_{13}(\sigma)}\,\alpha_{23}(\sigma)\,\tau^{-1}\right)\end{pmatrix}$$

and

$$(2.22)\qquad 2R(\tau, \sigma)=-S(\tau, \sigma)+2bs_2\left(\overline{\alpha_{12}(\sigma)}-\overline{\alpha_{13}(\sigma)}\,\alpha_{23}(\sigma)\,\tau^{-1}\right)+D(\tau, \sigma)^{1/2}.$$

We now obtain the following

LEMMA 2. 1. *For every point* (τ_0, σ_0) *with* $\operatorname{Im}\tau_0<0$ *there exist a neighbourhood of* (τ_0, σ_0) *and a non-singular matrix* $T(\tau, \sigma)$ *defined there and satisfying the relation* (2. 15).

PROOF. We find from (2. 8) that an eigenvector $U^{-}(\tau, \sigma)$ associated to $\lambda^{-}(\tau, \sigma)$ may be taken as the following form:

$$(2.23)\qquad U^{-}(\tau, \sigma)=\begin{pmatrix} \lambda^{-}(\tau, \sigma)-s_2\left(\tau+\alpha_{22}(\sigma)-|\alpha_{23}(\sigma)|^2\tau^{-1}\right) \\ s_2\left(\overline{\alpha_{12}(\sigma)}-\overline{\alpha_{13}(\sigma)}\,\alpha_{23}(\sigma)\,\tau^{-1}\right)\end{pmatrix}$$

or

$$(2.24)\qquad U^{-}(\tau, \sigma)=\begin{pmatrix} -s_1\left(\alpha_{12}(\sigma)-\alpha_{13}(\sigma)\,\overline{\alpha_{23}(\sigma)}\,\tau^{-1}\right) \\ \lambda^{-}(\tau, \sigma)+s_1\left(\tau+\alpha_{11}(\sigma)-|\alpha_{13}(\sigma)|^2\tau^{-1}\right)\end{pmatrix}.$$

If we set

$$(2.25)\qquad T(\tau, \sigma)=\left(U^{+}(\tau, \sigma),\ U^{-}(\tau, \sigma)\right)$$

and $T(\tau, \sigma)^{-1}$ is well defined, we then see from (2. 8), (2. 11), (2. 23) and (2. 24) that $T(\tau, \sigma)$ satisfies the relation (2. 15). By a similar way as deriving (2. 21) we get

$$(2.26)\qquad 2U^{-}(\tau, \sigma)=\begin{pmatrix} -S(\tau, \sigma)-D(\tau, \sigma)^{1/2} \\ 2s_2\left(\overline{\alpha_{12}(\sigma)}-\overline{\alpha_{13}(\sigma)}\,\alpha_{23}(\sigma)\,\tau^{-1}\right)\end{pmatrix}$$

or

$$(2.27)\qquad 2U^{-}(\tau, \sigma)=\begin{pmatrix} -2s_1\left(\alpha_{12}(\sigma)-\alpha_{13}(\sigma)\,\overline{\alpha_{23}(\sigma)}\,\tau^{-1}\right) \\ S(\tau, \sigma)-D(\tau, \sigma)^{1/2}\end{pmatrix}$$

respectively. Hence it follows from (2. 21), (2. 25), (2. 26) and (2. 27) that

$$(2.28)\qquad \det T(\tau, \sigma)=-s_2 S(\tau, \sigma)\left(\overline{\alpha_{12}(\sigma)}-\overline{\alpha_{13}(\sigma)}\,\alpha_{23}(\sigma)\,\tau^{-1}\right)$$

or

$$
(2.29)\qquad \begin{aligned} 4\det T(\tau,\sigma) &= -\left(S(\tau,\sigma)-D(\tau,\sigma)^{1/2}\right)^2 \\ &+4s_1s_2\left(\alpha_{12}(\sigma)-\alpha_{13}(\sigma)\overline{\alpha_{23}(\sigma)}\,\tau^{-1}\right)\left(\overline{\alpha_{12}(\sigma)}-\overline{\alpha_{13}(\sigma)}\,\alpha_{23}(\sigma)\,\tau^{-1}\right) \end{aligned}
$$

respectively. Since s_1 and s_2 are positive in virtue of (2.2) and (2.7), we see from (2.17) and (2.19) that $\operatorname{Im} S(\tau,\sigma)$ and $\operatorname{Im}(S(\tau,\sigma)-D(\tau,\sigma)^{1/2})$ are negative for $\operatorname{Im}\tau<0$. In a neighbourhood of a point (τ_0,σ_0) at which $\overline{\alpha_{12}(\sigma_0)}-\overline{\alpha_{13}(\sigma_0)}\,\alpha_{23}(\sigma_0)\,\tau_0^{-1}$ is zero or non zero, we take $U^-(\tau,\sigma)$ as the form (2.25) or (2.24) respectively. Therefore we see from (2.28) and (2.29) that $T(\tau_0,\sigma_0)$ is non-singular. Thus the lemma is proved.

We remark here the followings used in the next section. If we set $U^+(\tau,\sigma)={}^t(u_1^+(\tau,\sigma),\ u_2^+(\tau,\sigma))$, namely,

$$
(2.30)\qquad \begin{aligned} 2u_1^+(\tau,\sigma) &= -S(\tau,\sigma)+D(\tau,\sigma)^{1/2}, \\ u_2^+(\tau,\sigma) &= s_2\left(\overline{\alpha_{12}(\sigma)}-\overline{\alpha_{13}(\sigma)}\,\alpha_{23}(\sigma)\,\tau^{-1}\right), \end{aligned}
$$

then we have by definition of $T_2(\tau,\sigma)$ that

$$
T_2(\tau,\sigma)=\left(\det T(\tau,\sigma)\right)^{-1}\left(-u_2^+(\tau,\sigma),\ u_1^+(\tau,\sigma)\right).
$$

Hence, by direct computation, we find from (2.14) that

$$
(2.31)\qquad \mathscr{B}(\tau,\sigma)=(\det T)^{-1}\begin{pmatrix} -s_1u_1^+u_2^+ & -s_2(u_1^+)^2 & u_1^+p_2\tau^{-1} \\ -s_1(u_2^+)^2 & -s_2u_1^+u_2^+ & u_2^+p_2\tau^{-1} \\ s_1u_2^+p_1\tau^{-1} & s_2u_1^+p_1\tau^{-}{}_1 & p_1p_2\tau^{-2} \end{pmatrix}
$$

where

$$
(2.32)\qquad \begin{aligned} p_1(\tau,\sigma) &= \overline{\alpha_{13}(\sigma)}\,u_1^+(\tau,\sigma)+\overline{\alpha_{23}(\sigma)}\,u_2^+(\tau,\sigma), \\ p_2(\tau,\sigma) &= s_1\alpha_{13}(\sigma)\,u_2^+(\tau,\sigma)+s_2\alpha_{23}(\sigma)\,u_1^+(\tau,\sigma). \end{aligned}
$$

By similar way as the proof of Theorem 3.1 in [1], we can verify that if the mixed problem (2.1) is L^2-well posed, then Hersh condition holds, that is,

$$
(2.33)\qquad R(\tau,\sigma)\neq 0 \qquad \text{for } \operatorname{Im}\tau<0 \text{ and real } \sigma
$$

(also see [3]). It follows from Hersh condition and Lemma 2.1 that the compensating matrix function $G(\tau,\sigma;\,x,s)$ is continuous in $\operatorname{Im}\tau<0$, real σ. On the other hand, we find from the estimate (1.4) that there are positive constants a and C such that

$$
(2.34)\qquad \int_0^\infty e^{-2at}\|v(t)\|^2dt\leqq C\int_0^\infty\|g(t)\|^2dt.
$$

Since v' is a solution of Cauchy problem, $w=v-v'$ must also satisfy the estimate (3. 34). Therefore the following proposition can be proved by the same way as proofs of Theorem 4. 1 and 5. 1 in [1] (also see [4] and [8]).

PROPOSITION 2. 2. *Assume that the mixed problem is L^2-well posed. Then Hersh condition holds and for every real point (η_0, σ_0) there exist a positive constant C and a neighbourhood $U(\eta_0, \sigma_0)$ of (η_0, σ_0) such that*

$$(2.35)\qquad \left|C(\tau, \sigma)\right|\left\|\mathscr{B}(\tau, \sigma)\right\| \leqq C\left|\operatorname{Im} \lambda^+(\tau, \sigma) \operatorname{Im} \lambda^-(\tau, \sigma)\right|^{1/2}\left|\operatorname{Im} \tau\right|^{-1}$$

for any $(\tau, \sigma) \in U(\eta_0, \sigma_0)$ with $\operatorname{Im} \tau < 0$. Here $\|\mathscr{B}(\tau, \sigma)\|$ denotes a matrix norm of $\mathscr{B}(\tau, \sigma)$.

REMARK 1. The converse of Proposition 2. 2 is also valid, but does not use this fact in this paper.

REMARK 2. $U^-(\tau, \sigma)$ will be taken as the form (2. 24) or (2. 25) according to $\overline{\alpha_{12}(\sigma_0)} - \overline{\alpha_{13}(\sigma_0)}\, \alpha_{23}(\sigma_0)\, \eta_0^{-1} \neq 0$ or $=0$, respectively.

Hereafter the functions in $\operatorname{Im} \tau < 0$ considered above will be continuously extended to $\operatorname{Im} \tau \leqq 0$, if it is possible. Thus we obtain from Proposition 2. 2 the following

COROLLARY 2. 3. *Assume that the mixed problem* (2. 1) *is L^2-well posed. Then Lopatinski determinant $R(\eta_0, \sigma_0)$ does not vanish for a real point (η_0, σ_0) at which $\eta(\overline{\alpha_{12}(\sigma)}\, \eta - \alpha_{13}(\sigma)\, \overline{\alpha_{23}(\sigma)})$ and $\|\mathscr{B}(\eta, \sigma)\|$ don't vanish and $\lambda^+(\eta, \sigma)$ is real simple.*

PROOF. Since $\lambda^+(\eta_0, \sigma_0)$ is real simple, we see from (2. 16) that $\lambda^-(\eta_0, \sigma_0)$ is also real simple and it holds with some constant $C>0$

$$(2.36)\qquad \left|\operatorname{Im} \lambda^{\pm}(\eta_0 - i\gamma, \sigma_0)\right| \leqq C\gamma$$

for any small $\gamma > 0$. Hence we obtain from (2. 35), (2. 36) and hypothesis $\|\mathscr{B}(\eta_0, \sigma_0)\| \neq 0$ that it holds with some constant $C>0$

$$(2.37)\qquad \left|C(\eta_0 - i\gamma, \sigma_0)\right| \leqq C$$

for any small $\gamma > 0$. On the other hand, from (2. 23) and hypothesis $\eta_0(\overline{\alpha_{12}(\sigma_0)}\, \eta_0 - \overline{\alpha_{13}(\sigma_0)}\, \alpha_{23}(\sigma_0)) \neq 0$, we have

$$\begin{aligned} B' U^-(\tau, \sigma) = {} & \lambda^-(\tau, \sigma) - s_2\left(\tau + \alpha_{22}(\sigma) - |\alpha_{23}(\sigma)|^2 \tau^{-1}\right) \\ & + s_2\, b\left(\overline{\alpha_{12}(\sigma)} - \overline{\alpha_{13}(\sigma)}\, \alpha_{23}(\sigma)\, \tau^{-1}\right). \end{aligned}$$

Since $\lambda^{\pm}(\eta_0, \sigma_0)$ is simple, we see from (2. 12) and this that if $R(\eta_0, \sigma_0) = 0$ then the numerator $B' U^-(\eta_0, \sigma_0)$ of reflection coefficient $C(\eta_0, \sigma_0)$ does not vanish. This contradicts (2. 37).

Except special cases we shall prove Theorem dividing the following two cases:

Case 1. $a_{12}^{(k)} \neq 0$ for some k or a $a_{12}^{(j)}=0$ for all j and $a_{13}^{(k)} a_{23}^{(k)} \neq 0$ for some k,

Case 2. $a_{12}^{(j)}=0$ for all j and $a_{13}^{(k)} a_{23}^{(l)} \neq 0$, $a_{13}^{(l)}=a_{23}^{(k)}=0$ for some $k \neq l$.

According to Case 1 or 2 we use Proposition 2.2 and Corollary 2.3 considering variables σ as $(0, \cdots, 0, \sigma_k, 0, \cdots, 0)$ or $(0, \cdots, 0, \sigma_k, 0, \cdots, 0, \sigma_l, 0, \cdots, 0)$ respectively.

§ 3. Proof of Theorem (genereal cases).

Through this section we assume that the mixed problem (2. 1) is not maximally non-positive. From (2. 4) and (2. 7) this assumption is equivalent to

$$s_1 - s_2 |b|^2 < 0 . \tag{3.1}$$

Thus we may assume that $b \neq 0$.

We shall first investigate zeroes of Lopatinski determinant $R(\tau, \sigma)$. Assume that $R(\tau, \sigma)=0$. We then find from (2. 19), (2. 22) and the second equality of (3. 20) that $\overline{\alpha_{12}(\sigma)} - \overline{\alpha_{13}(\sigma)}\, \alpha_{23}(\sigma)\, \tau^{-1}=0$ or

$$\begin{aligned} bS(\tau, \sigma) &= s_2 b^2 \left(\overline{\alpha_{12}(\sigma)} - \overline{\alpha_{13}(\sigma)}\, \alpha_{23}(\sigma)\, \tau^{-1}\right) + s_1 \left(\alpha_{12}(\sigma) - \alpha_{13}(\sigma)\, \overline{\alpha_{23}(\sigma)}\, \tau^{-1}\right) \\ &= \left(s_1 \alpha_{12}(\sigma) + s_2 \overline{\alpha_{12}(\sigma)}\, b^2\right) - \left(s_1 \alpha_{13}(\sigma)\, \overline{\alpha_{23}(\sigma)} + s_2 \overline{\alpha_{13}(\sigma)}\, \alpha_{23}(\sigma)\, b^2\right) \tau^{-1} . \end{aligned} \tag{3.2}$$

Using the first equality of (3. 2) we have

$$\begin{aligned} &-S(\tau, \sigma) + 2bs_2 \left(\overline{\alpha_{12}(\sigma)} - \overline{\alpha_{13}(\sigma)}\, \alpha_{23}(\sigma)\, \tau^{-1}\right) \\ &= s_2 b \left(\overline{\alpha_{12}(\sigma)} - \overline{\alpha_{13}(\sigma)}\, \alpha_{23}(\sigma)\, \tau^{-1}\right) - s_1 \left(\alpha_{12}(\sigma) - \alpha_{13}(\sigma)\, \overline{\alpha_{23}(\sigma)}\, \tau^{-1}\right) b^{-1} \\ &= \left(s_2 \overline{\alpha_{12}(\sigma)}\, b^2 - s_1 \alpha_{12}(\sigma)\right) b^{-1} \\ &\quad + \left(s_1 \alpha_{13}(\sigma)\, \overline{\alpha_{23}(\sigma)} - s_2 \overline{\alpha_{13}(\sigma)}\, \alpha_{23}(\sigma)\, b^2\right) \tau^{-1} b^{-1} . \end{aligned} \tag{3.3}$$

Therefore, from (2. 22), (3. 2) and (3. 3) we obtain the following

LEMMA 3. 1. *If a point* (τ, σ) *satisfies the relation* (3. 2), *then*

$$2R(\tau, \sigma) = Q(\tau, \sigma) + \left(Q(\tau, \sigma)^2\right)^{1/2} , \tag{3.4}$$

where

$$Q(\tau,\sigma)=\left(s_2\overline{\alpha_{12}(\sigma)}\,b^2-s_1\alpha_{12}(\sigma)\right)b^{-1}$$
(3.5)
$$+\left(s_1\alpha_{13}(\sigma)\,\overline{\alpha_{23}(\sigma)}-s_2\overline{\alpha_{13}(\sigma)}\,\alpha_{23}(\sigma)\,b^2\right)\tau^{-1}b^{-1}.$$

The following lemmas follows directly from Proposition 2.2, Corollary 2.3, Lemma 3.1 and the choice of branch (2.17).

LEMMA 3.2. *If there exists a point* (τ,σ) *with* $\mathrm{Im}\,\tau<0$ *satisfying* (3.2) *and* $\mathrm{Im}\ Q(\tau,\sigma)<0$, *then* $R(\tau,\sigma)$ *vanishes. Hence the mixed problem* (2.1) *is not* L^2*-well posed.*

LEMMA 3.3. *If there exists a real point* (η,σ) *satisfying* (3.2) *such that* $Q(\eta,\sigma)$ *is non-zero real and signs of* $Q(\eta,\sigma)$ *and* $\lim_{\gamma\downarrow 0}\mathrm{Im}\,D(\eta-i\gamma,\sigma)\,\gamma^{-1}$ *are different, then* $R(\eta,\sigma)$ *vanishes. Moreover, if* $\eta\left(\overline{\alpha_{12}(\sigma)}\,\eta-\overline{\alpha_{13}(\sigma)}\,\alpha_{23}(\sigma)\right)$ *and* $\|\mathscr{B}(\eta,\sigma\|$ *don't vanish for such a point* (η,σ), *then the mixed problem* (2.1) *is not* L^2*-well posed.*

We now consider the case that Theorem will be proved by using the above lemmas. However, an example mentioned in Introduction is not contained in this case (see Remark after Proposition 3.7).

To show the existence of a point (τ,σ) satisfying the assumptions of Lemma 3.2 or 3.3, we first compute the real and imaginary parts of $Q(\tau,\sigma)$. Using the relations:

$$\left(s_2\overline{\alpha_{12}(\sigma)}\,b^2-s_1\alpha_{12}(\sigma)\right)b^{-1}=-\mathrm{Re}\left(\alpha_{12}(\sigma)\,\bar b\right)(s_1-s_2|b|^2)|b|^{-2}$$
$$-i\,\mathrm{Im}\left(\alpha_{12}(\sigma)\,\bar b\right)(s_1+s_2|b|^2)|b|^{-2}$$

and

$$\left(s_1\alpha_{13}(\sigma)\,\overline{\alpha_{23}(\sigma)}-s_2\overline{\alpha_{13}(\sigma)}\,\alpha_{23}(\sigma)\,b^2\right)\tau^{-1}b^{-1}$$
$$=\left\{\eta\,\mathrm{Re}\left(\alpha_{13}(\sigma)\,\alpha_{23}(\sigma)\,b\right)(s_1-s_2|b|^2)\right.$$
$$\left.-\gamma\,\mathrm{Im}\left(\alpha_{13}(\sigma)\,\overline{\alpha_{23}(\sigma)}\,b\right)(s_1+s_2|b|^2)\right\}|\tau b|^{-2}$$
$$+i\left\{\gamma\,\mathrm{Re}\left(\alpha_{13}(\sigma)\,\overline{\alpha_{23}(\sigma)\,b}\right)(s_1-s_2|b|^2)\right.$$
$$\left.+\eta\,\mathrm{Im}\left(\alpha_{13}(\sigma)\,\overline{\alpha_{23}(\sigma)\,b}\right)(s_1+s_2|b|^2)\right\}|\tau b|^{-2},$$

where $\tau=\eta-i\gamma$ $(\gamma\geqq 0)$, we obtain from (3.5) that

$$|\tau b|^2\,\mathrm{Re}\,Q(\tau,\sigma)=-\gamma(s_1+s_2|b|^2)\,\mathrm{Im}\left(\alpha_{13}(\sigma)\,\overline{\alpha_{23}(\sigma)\,b}\right)$$
(3.6)
$$+(s_1-s_2|b|^2)\left\{\eta\,\mathrm{Re}\left(\alpha_{13}(\sigma)\,\overline{\alpha_{23}(\sigma)\,b}\right)-|\tau|^2\,\mathrm{Re}\left(\alpha_{12}(\sigma)\,\bar b\right)\right\}$$

and

$$|\tau b|^2 \operatorname{Im} Q(\tau, \sigma) = \gamma (s_1 - s_2 |b|^2) \operatorname{Re}\left(\alpha_{13}(\sigma)\, \overline{\alpha_{23}(\sigma)\, b}\right)$$
(3.7)
$$+ (s_1 + s_2 |b|^2) \left\{\eta \operatorname{Im}\left(\alpha_{13}(\sigma)\, \overline{\alpha_{23}(\sigma)\, b}\right) - |\tau|^2 \operatorname{Im}\left(\alpha_{12}(\sigma)\, \bar{b}\right)\right\}.$$

On the other hand, by a similar way as the above computation we find from the second equality of (3.2) that

$$S(\tau, \sigma) = \left\{(s_1 + s_2 |b|^2) \operatorname{Re}\left(\alpha_{12}(\sigma)\, \bar{b}\right) + i (s_1 - s_2 |b|^2) \operatorname{Im}\left(\alpha_{12}(\sigma)\, \bar{b}\right)\right\} |b|^{-2}$$
$$- \left\{(s_1 + s_2 |b|^2) \operatorname{Re}\left(\alpha_{13}(\sigma)\, \alpha_{23}(\sigma)\, b\right) + i (s_1 - s_2 |b|^2)\right.$$
$$\left. \times \operatorname{Im}\left(\alpha_{13}(\sigma)\, \overline{\alpha_{23}(\sigma)\, b}\right)\right\} \tau^{-1} |b|^{-2}.$$

By multiplying the above equality by $\tau = \eta - i\gamma$ and using (2.19), we obtain that

$$(\eta - i\gamma) \left\{s_1\left(\eta + \alpha_{11}(\sigma)\right) + s_2\left(\eta + \alpha_{22}(\sigma)\right) - (s_1 + s_2 |b|^2) \operatorname{Re}\left(\alpha_{12}(\sigma)\, \bar{b}\right) |b|^{-2}\right.$$
$$\left. - i\gamma (s_1 + s_2) - i (s_1 - s_2 |b|^2) \operatorname{Im}\left(\alpha_{12}(\sigma)\, \bar{b}\right) |b|^{-2}\right\}$$
(3.8)
$$= s_1 |\alpha_{13}(\sigma)|^2 + s_2 |\alpha_{23}(\sigma)|^2 - (s_1 + s_2 |b|^2) \operatorname{Re}\left(\alpha_{13}(\sigma)\, \overline{\alpha_{23}(\sigma)\, b}\right) |b|^{-2}$$
$$- i (s_1 - s_2 |b|^2) \operatorname{Im}\left(\alpha_{13}(\sigma)\, \overline{\alpha_{23}(\sigma)\, b}\right) |b|^{-2}.$$

To simplify the equation (3.8) we introduce a new variable ξ defined by

(3.9) $$\eta = \xi - \beta(\sigma)/2$$

where

(3.10) $$(s_1 + s_2)\, \beta(\sigma) = s_1 \alpha_{11}(\sigma) + s_2 \alpha_{22}(\sigma) - (s_1 + s_2 |b|^2) \operatorname{Re}\left(\alpha_{12}(\sigma)\, \bar{b}\right) |b|^{-2}.$$

Hence the left hand side of (3.8) is equal to

$$(s_1 + s_2) \left(\xi^2 - \beta(\sigma)^2/4 - \gamma^2\right) - \gamma (s_1 - s_2 |b|^2) \operatorname{Im}\left(\alpha_{12}(\sigma)\, \bar{b}\right) |b|^{-2}$$
$$- 2i\gamma\xi (s_1 + s_2) - i (s_1 - s_2 |b|^2) \operatorname{Im}\left(\alpha_{12}(\sigma)\, \bar{b}\right) \left(\xi - \beta(\sigma)/2\right) |b|^{-2}.$$

If we set

(3.11) $$G(\sigma) = \operatorname{Im}\left(\alpha_{13}(\sigma)\, \overline{\alpha_{23}(\sigma)\, b}\right) + \operatorname{Im}\left(\alpha_{12}(\sigma)\, \bar{b}\right) \beta(\sigma)/2,$$

then the equation (3.8) is equivalent to

(3.12) $$2\gamma\xi |b|^2 (s_1 + s_2) = (s_1 - s_2 |b|^2) \left(G(\sigma) - \xi \operatorname{Im}\left(\alpha_{12}(\sigma)\, \bar{b}\right)\right)$$

and

$$|b|^2(s_1+s_2)\left(\xi^2-\beta(\sigma)^2/4-\gamma^2\right)-\gamma(s_1-s_2|b|^2)\,\mathrm{Im}\left(\alpha_{12}(\sigma)\,\bar{b}\right)$$
$$(3.13)\qquad =|b|^2\left(s_1|\alpha_{13}(\sigma)|^2+s_2|\alpha_{23}(\sigma)|^2\right)-(s_1+s_2|b|^2)\,\mathrm{Re}\left(\alpha_{13}(\sigma)\,\overline{\alpha_{23}(\sigma)\,b}\right).$$

Substitute γ satisfying (3. 12) into the term $|b|^2(s_1+s_2)\gamma^2+\gamma(s_1-s_2|b|^2)\,\mathrm{Im}(\alpha_{12}(\sigma)\,\bar{b})$ in (3. 13), we see that this term is equal to

$$(s_1-s_2|b|^2)^2\left\{G(\sigma)^2\xi^{-2}-\left(\mathrm{Im}\left(\alpha_{12}(\sigma)\,\bar{b}\right)\right)^2\right\}/4|b|^2(s_1+s_2)\,.$$

Therefore we obtain the equation in ξ resulting from substituting γ of (3. 12) into (3. 13):

$$(3.14)\qquad 4|b|^4(s_1+s_2)^2\xi^4-F(\sigma)\,\xi^2-(s_1-s_2|b|^2)^2\,G(\sigma)^2=0$$

where

$$F(\sigma)=|b|^4(s_1+s_2)^2\beta(\sigma)^2-(s_1-s_2|b|^2)^2\left(\mathrm{Im}\left(\alpha_{12}(\sigma)\,\bar{b}\right)\right)^2$$
$$(3.15)\qquad +4|b|^4(s_1+s_2)\left(s_1|\alpha_{13}(\sigma)|^2+s_2|\alpha_{23}(\sigma)|^2\right)$$
$$-4|b|^2(s_1+s_2)\,(s_1+s_2|b|^2)\,\mathrm{Re}\left(\alpha_{13}(\sigma)\,\overline{\alpha_{23}(\sigma)\,b}\right).$$

For Case 1 mentioned in the last part of § 2 we shall prove the following propositions. In this case we set $a_{jl}=a_{jl}^{(k)}$ and $\sigma=(0,\cdots,\sigma_k,0,\cdots,0)$. Thus we may assume that σ is one variable.

PROPOSITION 3. 4. *Assume that* $s_1-s_2|b|^2<0$. *Then there exists a point* (τ,σ) *satisfying the assumptions of Lemma* 3. 2 *if one of the following conditions holds*:

(1) $G\neq 0$,
(2) $G=0$ *and* $\mathrm{Im}\,(a_{12}\bar{b})=0$,
(3) $\mathrm{Im}\,(a_{12}\bar{b})=\mathrm{Im}\,(a_{13}\overline{a_{23}b})=0$ *and* $F<0$.

Here $G=G(1)$ *and* $F=F(1)$.

PROPOSITION 3. 5. *Assume that* $s_1-s_2|b|^2<0$ *and* $\mathrm{Im}(a_{12}\bar{b})=\mathrm{Im}(a_{13}\overline{a_{23}b})=0$. *Then there exists a real point* (η,σ) *satisfying the assumptions of Lemma* 3. 3 *if one of the following conditions holds*:

(1) $a_{12}a_{23}=0$ *and* a_{13} *or* $a_{23}\neq 0$,
(2) $a_{13}a_{23}\neq 0$, $a_{12}=0$ *and* $F>0$,
(3) $a_{12}a_{13}a_{23}\neq 0$, $F>0$ *and* $I\neq 0$,
(4) $a_{12}a_{13}a_{23}\neq 0$, $F>0$, $I=0$ *and* $\beta+a_{13}\bar{a}_{23}/a_{12}\neq 0$,
(5) $a_{13}a_{23}\neq 0$, $a_{12}=0$, $F=0$ *and* $\beta\neq 0$,
(6) $a_{12}a_{13}a_{23}\neq 0$, $F=0$, $\beta\neq 0$ *and* $\beta+2a_{13}\bar{a}_{23}/a_{12}\neq 0$.

Here $\beta=\beta(1)$, $I=I(1)$ *and*

$$I(\sigma)=|b|^2(s_1|a_{13}|^2+s_2|a_{23}|^2)\sigma^2-(s_1+s_2|b|^2)\operatorname{Re}(a_{13}\sigma\overline{a_{23}\sigma b}). \tag{3.16}$$

PROFF OF PROPOSITION 3.4. First let $G\neq 0$. Then the equation (3.14) has two real roots having opposite signs and $G(\sigma)-\xi(\sigma)\operatorname{Im}(a_{12}\bar{b}\sigma)\neq 0$ for some root $\xi(\sigma)$. Hence we may choose σ such that

$$\xi(\sigma)\left\{G(\sigma)-\xi(\sigma)\operatorname{Im}(a_{12}\bar{b}\sigma)\right\}<0. \tag{3.17}$$

From (3.17) we see that there exist $\gamma(\sigma)>0$ satisfying (3.12). Here we use that $s_1>0$, $s_2>0$ and $s_1-s_2|b|^2<0$. Therefore it suffice to prove that $\operatorname{Im}Q(\tau(\sigma),\sigma)<0$ for such a point $(\tau(\sigma),\sigma)=(\eta(\sigma)-i\gamma(\sigma),\sigma)$, where $\eta(\sigma)=\xi(\sigma)-\beta(\sigma)/2$ by definition (3.9).

From (3.9) and (3.11) we have

$$\operatorname{Im}(a_{13}a_{23}b\sigma^2)-\eta\operatorname{Im}(a_{12}\bar{b}\sigma)=G(\sigma)-\xi\operatorname{Im}(a_{12}\bar{b}\sigma).$$

It follows from this, (3.7) and (3.9) that

$$\begin{aligned}|\tau b|^2\operatorname{Im}Q(\tau,\sigma)&=\left(\xi-\beta(\sigma)/2\right)(s_1+s_2|b|^2)\left(G(\sigma)-\xi\operatorname{Im}(a_{12}\bar{b}\sigma)\right)\\&\quad+\gamma\left\{(s_1-s_2|b|^2)\operatorname{Re}(a_{13}\overline{a_{23}b\sigma^2})-\gamma(s_1+s_2|b|^2)\operatorname{Im}(a_{12}\bar{b}\sigma)\right\}.\end{aligned} \tag{3.18}$$

Replacing the factor γ in the last term of (3.18) by $\gamma(\sigma)$ satisfying (3.12), we then find that

$$|b\tau(\sigma)|^2\operatorname{Im}Q\left(\tau(\sigma),(\sigma)=\frac{G(\sigma)-\xi(\sigma)\operatorname{Im}(a_{12}\bar{b}\sigma)}{2|b|^2(s_1+s_2)\xi(\sigma)}H(\sigma)\right. \tag{3.19}$$

where

$$\begin{aligned}H(\sigma)&=(s_1-s_2|b|^2)^2\operatorname{Re}(a_{13}\overline{a_{23}b\sigma^2})+|b|^2\left(2\xi(\sigma)^2-\beta(\sigma)\xi(\sigma)\right)\\&\quad\times(s_1+s_2)(s_1+s_2|b|^2)-\gamma(\sigma)(s_1^2-s_2^2|b|^4)\operatorname{Im}(a_{12}\bar{b}\sigma).\end{aligned} \tag{3.20}$$

In virtue of (3.17) and (3.19) our proof finishes if we prove $H(\sigma)>0$. To do this, we solve the equation (3.14), that is,

$$8|b|^4(s_1+s_2)^2\xi(\sigma)^2=F(\sigma)+\sqrt{J(\sigma)} \tag{3.21}$$

where

$$J(\sigma)=F(\sigma)^2+16|b|^4(s_1+s_2)^2(s_1-s_2|b|^2)^2G(\sigma)^2. \tag{3.22}$$

Substitute (3.21) into $4|b|^2(s_1+s_2)H(\sigma)$, it then follows from (3.15) and (3.20) that

$$
\begin{aligned}
4|b|^2(s_1+s_2)\,H(\sigma) &= 4|b|^2(s_1+s_2)\,(s_1-s_2|b|^2)^2\,\mathrm{Re}\,(a_{13}\overline{a_{23}b\sigma^2}) \\
&\quad -4\,|b|^2(s_1+s_2)\,(s_1+s_2\,|b|^2)^2\,\mathrm{Re}\,(a_{13}\overline{a_{23}b\sigma^2})+4|b|^4(s_1+s_2) \\
&\quad \times(s_1+s_2\,|b|^2)\,(s_1\,|a_{13}\,|^2+s_2\,|a_{23}|^2)\,\sigma^2+|b|^4(s_1+s_2)^2\beta(\sigma)^2 \\
&\quad \times(s_1+s_2\,|b|^2)-(s_1+s_2\,|b|^2)\,(s_1-s_2\,|b|^2)^2\Big(\mathrm{Im}\,(a_{12}\bar{b}\sigma)\Big)^2 \\
&\quad +(s_1+s_2\,|b|^2)\sqrt{J(\sigma)}\,-4\,|b|^4(s_1+s_2)^2\,(s_1+s_2\,|b|^2)\,\beta(\sigma)\,\xi(\sigma) \\
&\quad -4\gamma(\sigma)\,|b|^2(s_1+s_2)\,(s_1^2-s_2^2\,|b|^4)\,\mathrm{Im}\,(a_{12}\bar{b}\sigma)\,.
\end{aligned}
\tag{3.23}
$$

Using the identity which is easily proved:

$$
\begin{aligned}
&(s_1+s_2\,|b|^2)\,(s_1|\,a_{13}|^2+s_2\,|\,a_{23}|^2)-4s_1s_2\,\mathrm{Re}\,(a_{13}\overline{a_{23}b}) \\
&\qquad =|\,s_1\,a_{13}-s_2\,a_{23}\,b\,|^2+s_1\,s_2\,|\,a_{13}\,\bar{b}-a_{23}\,|^2
\end{aligned}
$$

and substituting $\gamma(\sigma)$ satisfying (3. 12) into (3. 23), we obtain that

$$
\begin{aligned}
&4\,|b|^2\,(s_1+s_2)\,H(\sigma) \\
&\quad =4\,|b|^4(s_1+s_2)\,(|\,s_1\,a_{13}-s_2\,a_{23}\,b\,|^2+s_1s_2|\,a_{13}\,\bar{b}-a_{23}\,|^2)\,\sigma^2 \\
&\quad +(s_1+s_2\,|b|^2)\,\Big\{|b|^4(s_1+s_2)^2\beta(\sigma)^2+(s_1-s_2|\,b|^2)^2\Big(\mathrm{Im}\,(a_{12}\bar{b}\sigma)\Big)^2 \\
&\quad +\sqrt{J(\sigma)}\,-4\,|b|^4(s_1+s_2)^2\beta(\sigma)\,\xi(\sigma)-2\,(s_1-s_2\,|b|^2)^2 \\
&\quad \times G(\sigma)\,\mathrm{Im}\,(a_{12}\bar{b}\sigma)\,\xi(\sigma)^{-1}\Big\}\,.
\end{aligned}
\tag{3.24}
$$

When the following quantity is negative:

$$
2\,|b|^4(s_1+s_2)^2\beta(\sigma)\,\xi(\sigma)+(s_1-s_2\,|b|^2)^2\,\mathrm{Im}\,(a_{12}\bar{b}\sigma)\,G(\sigma)\,\xi(\sigma)^{-1}\,.
$$

we see from (3. 24) that $H(\sigma)>0$. When it is non-negative, we shall show that

$$
\begin{aligned}
K(\sigma) &= \Big\{|b|^4(s_1+s_2)^2\,\beta(\sigma)^2+(s_1-s_2\,|b|^2)^2\,\Big(\mathrm{Im}\,(a_{12}\bar{b}\sigma)\Big)^2+\sqrt{J(\sigma)}\,\Big\}^2 \\
&\quad -4\Big\{2\,|b|^4(s_1+s_2)^2\beta(\sigma)\,\xi(\sigma)+(s_1-s_2|b|^2)^2\,\mathrm{Im}\,(a_{12}\bar{b}\sigma)\,G(\sigma)\,\xi(\sigma)^{-1}\Big\}^2
\end{aligned}
$$

is non-negative, where the above equality is the definition of $K(\sigma)$. In fact, it follows from (3. 21) and (3. 22) that

$$
16\,|b|^8(s_1+s_2)^4\beta(\sigma)^2\xi(\sigma)^2=2\,|b|^4(s_1+s_2)^2\beta(\sigma)^2\Big(F(\sigma)+\sqrt{J(\sigma)}\,\Big)
$$

and

$$
\begin{aligned}
&4(s_1-s_2\,|b|^2)^4\Big(\mathrm{Im}\,(a_{12}\bar{b}\sigma)\Big)^2G(\sigma)^2\xi(\sigma)^{-2} \\
&\qquad =\,-2(s_1-s_2\,|b|^2)^2\Big(\mathrm{Im}\,(a_{12}\bar{b}\sigma)\Big)^2\Big(F(\sigma)-\sqrt{J(\sigma)}\,\Big)\,.
\end{aligned}
$$

Hence we find from these and (3. 22) that

$$K(\sigma)=\left\{F(\sigma)-|b|^4(s_1+s_2)^2\beta(\sigma)^2+(s_1-s_2|b|^2)^2\left(\mathrm{Im}\,(a_{12}\bar{b}\sigma)\right)^2\right\}^2 \tag{3.25}$$
$$+4|b|^4(s_1+s_2)^2(s_1-s_2|b|^2)^2\left(2G(\sigma)-\beta(\sigma)\,\mathrm{Im}\,(a_{12}\bar{b}\sigma)\right)^2.$$

Assume that $a_{13}a_{23}\neq 0$, $s_1a_{13}-s_2a_{23}b=0$ and $a_{13}\bar{b}-a_{23}=0$. Then we have

$$s_1-s_2|b|^2=(s_1|a_{13}|^2-s_2|a_{23}|^2)/|a_{13}|^2=0$$

which contradicts the assumption of Proposition 3.4. Thus we conclude from (3.24) and (3.25) that $H(\sigma)>0$.

When $a_{13}a_{23}=0$, we find directly from (2.19), (3.2) and (3.5) that the equation (3.2) is equivalent to two equations:

$$(s_1+s_2)\,\eta+s_1a_{11}\sigma+s_2a_{22}\sigma-(s_1|a_{13}|^2+s_2|a_{23}|^2)\,\sigma^2\eta^{-1}$$
$$=(s_1+s_2|b|^2)\,\mathrm{Re}\,(a_{12}\bar{b}\sigma)\,|b|^{-2},$$
$$(s_1+s_2)\,\gamma=-(s_1-s_2|b|^2)\,\mathrm{Im}\,(a_{12}\bar{b}\sigma)\,|b|^{-2}$$

and it holds

$$\mathrm{Im}\,Q(\tau,\sigma)=-(s_1+s_2|b|^2)\,\mathrm{Im}\,(a_{12}\bar{b}\sigma)\,|b|^{-2}.$$

Remark that $\mathrm{Im}\,(a_{12}\bar{b})\neq 0$ because of $G\neq 0$. Therefore, if σ will be taken as $\mathrm{Im}\,(a_{12}\bar{b}\sigma)>0$, we see the existence of a point (τ,σ) satisfying the assumptions of Lemma 3.2.

We next assume that $G=0$ and $\mathrm{Im}\,(a_{12}b)\neq 0$. If $F>0$, then the equation (3.14); that is, $4|b|^2(s_1+s_2)^2\xi^2-F(\sigma)=0$, has non-zero real root $\xi(\sigma)$. Since (3.12) is equivalent to

$$2\gamma|b|^2(s_1+s_2)=-(s_1-s_2|b|^2)\,\mathrm{Im}\,(a_{12}\bar{b}\sigma),$$

there exists $\gamma(\sigma)>0$ satisfying (3.12) if σ is taken as $\mathrm{Im}\,(a_{12}\bar{b}\sigma)>0$. Therefore the same argument deriving $\mathrm{Im}\,Q(\tau(\sigma),\sigma)<0$ when $G\neq 0$ is directly applicable to this case.

Let $\xi=0$, then (3.12) holds automatically and from (3.13) we obtain the equation in γ:

$$4|b|^2(s_1+s_2)\,\gamma^2+4(s_1-s_2|b|^2)\,\mathrm{Im}\,(a_{12}\bar{b}\sigma)\,\gamma+|b|^2(s_1+s_2)\,\beta(\sigma)^2 \tag{3.26}$$
$$+4|b|^2(s_1|a_{13}|^2+s_2|a_{23}|^2)\,\sigma^2-4(s_1+s_2|b|^2)\,\mathrm{Re}\,(a_{13}\overline{a_{23}}\bar{b}\sigma^2)=0.$$

From (3.15) the discriminant of (3.26) is equal to $-16F(\sigma)$. If $F\leqq 0$ and we choose σ such that $\mathrm{Im}\,(a_{12}\bar{b}\sigma)>0$, then (3.26) has a positive root $\gamma(\sigma)$. Hence such a point $(\tau(\sigma),\sigma)=(-\beta(\sigma)/2-i\gamma(\sigma),\sigma)$ satisfies (3.2).

We now prove that $\mathrm{Im}\,Q(\tau(\sigma),\sigma)>0$. Since $\xi=0$ and $G=0$, we have from (3.18)

$$
(3.27)\quad \begin{aligned} |\tau(\sigma)b|^2 \operatorname{Im} Q\big(\tau(\sigma), \sigma\big) &= \gamma(\sigma)\Big\{(s_1 - s_2|b|^2) \operatorname{Re}(a_{13}\overline{a_{23}b\sigma^2}) \\ &\quad - \gamma(\sigma)(s_1 + s_2|b|^2) \operatorname{Im}(a_{12}\bar{b}\sigma)\Big\}. \end{aligned}
$$

Denoting the last factor of (3.27) by $L(\sigma)$ and using the inequality which follows from (3.26)

$$
2|b|^2(s_1+s_2)\gamma(\sigma) \geqq -(s_1 - s_2|b|^2) \operatorname{Im}(a_{12}\bar{b}\sigma),
$$

we obtain the following inequality:

$$
(3.28)\quad \begin{aligned} -2|b|^2(s_1+s_2) L(\sigma) &\geqq -(s_1 - s_2|b|^2) \\ &\quad \times \Big\{\big(\operatorname{Im}(a_{12}\bar{b}\sigma)\big)^2 (s_1 + s_2|b|^2) + 2|b|^2(s_1+s_2) \operatorname{Re}(a_{13}\overline{a_{23}b\sigma^2})\Big\}. \end{aligned}
$$

On the other hand, it follows from (3.15) and $F \leqq 0$ that

$$
(s_1 - s_2|b|^2)^2\big(\operatorname{Im}(a_{12}\bar{b}\sigma)\big)^2 + 4|b|^2(s_1+s_2)(s_1+s_2|b|^2) \operatorname{Re}(a_{13}\overline{a_{23}b\sigma^2}) \geqq 0,
$$

which implies that

$$
2|b|^2(s_1+s_2) \operatorname{Re}(a_{13}\overline{a_{23}b\sigma^2}) \geqq \frac{-(s_1 - s_2|b|^2)\big(\operatorname{Im}(a_{12}\bar{b}\sigma)\big)^2}{2(s_1+s_2|b|^2)}.
$$

Thus we see from this and (3.28) that

$$
\begin{aligned} -4|b|^2(s_1+s_2)(s_1+s_2|b|^2) L(\sigma) &\geqq -(s_1 - s_2|b|^2)\big(\operatorname{Im}(a_{12}\bar{b}\sigma)\big)^2 \\ &\quad \times \Big\{2(s_1+s_2|b|^2)^2 - (s_1 - s_2|b|^2)^2\Big\} \end{aligned}
$$

which shows that $\operatorname{Im} Q(\tau(\sigma), \sigma) < 0$.

Finally we assume that $\operatorname{Im}(a_{12}\bar{b}) = \operatorname{Im}(a_{13}\overline{a_{23}b}) = 0$ and $F < 0$. From (3.15) we have

$$
(3.29)\quad \begin{aligned} (s_1+s_2)^{-1}F(\sigma) &= |b|^2(s_1+s_2)\beta(\sigma)^2 + 4|b|^2(s_1|a_{13}|^2 + s_2|a_{23}|^2)\sigma^2 \\ &\quad - 4(s_1+s_2|b|^2) \operatorname{Re}(a_{13}\overline{a_{23}b\sigma^2}). \end{aligned}
$$

Let $\xi = 0$. Then (3.12) holds automatically and from (3.13) and (3.29) we have the equation in γ:

$$
4|b|^2\gamma^2 + F(\sigma) = 0.
$$

Since F is negative, this equation has a positive root $\gamma(\sigma)$. Hence there exists a point $(\tau(\sigma), \sigma) = (-\beta(\sigma)/2 - i\gamma(\sigma), \sigma)$ satisfying (3.2) and $\operatorname{Im}\tau(\sigma) < 0$. On the other hand, it follows from (3.18) that

$$
|\tau(\sigma)b|^2 \operatorname{Im} Q\big(\tau(\sigma), \sigma\big) = \gamma(\sigma)(s_1 - s_2|b|^2) \operatorname{Re}(a_{13}\overline{a_{23}b\sigma^2}).
$$

From (3. 29) and $F<0$ we obtain that $\mathrm{Re}\,(a_{13}\overline{a_{23}b\sigma^2})>0$ which shows that $\mathrm{Im}\,Q(\tau(\sigma),\sigma)<0$. Thus the proof is complete.

PROOF OF PROPOSITION 3. 5. If we set $\gamma=0$ in (3. 6), (3. 7) and (3. 13), we then obtain that $\mathrm{Im}\,Q(\eta,\sigma)=0$,

$$|\eta b|^2 Q(\eta,\sigma)=\eta(s_1-s_2|b|^2)(a_{13}\overline{a_{23}b\sigma^2}-\eta a_{12}\bar{b}\sigma) \tag{3. 30}$$

and

$$4|b|^2(s_1+s_2)\,\xi^2=|b|^2(s_1+s_2)\,\beta(\sigma)^2+4I(\sigma)\,. \tag{3. 31}$$

Here $I(\sigma)$ is defined in (3. 16) and $a_{12}\bar{b}$ and $a_{13}\overline{a_{23}b}$ are real by assumptions. From (3. 15) and (3. 16) the right hand side of (3. 31) is equal to $F(\sigma)/(s_1+s_2)$. Therefore the equation (3. 31) has a real root $\xi(\sigma)$.

We now compute the imaginary part of $D(\eta(\sigma)-i\gamma,\sigma)$ where $\eta(\sigma)=\xi(\sigma)-\beta(\sigma)/2$. It follows from (2. 19) and (2. 20) that

$$\begin{aligned}\mathrm{Im}\,D(\tau,\sigma)=&-2\gamma\left\{(s_1+s_2)+(s_1|a_{13}|^2+s_2|a_{23}|^2)\,\sigma^2|\tau|^{-2}\right\}\\&\times\left\{s_1(\eta+a_{11}\sigma)+s_2\big(\eta+a_{22}(\sigma)\big)-(s_1|a_{13}|^2+s_2|a_{23}|^2)\,\eta\sigma^2|\tau|^{-2}\right\}\\&+8s_1s_2\gamma\left\{\mathrm{Re}\,(\bar{a}_{12}a_{13}\bar{a}_{23})\,\sigma^3|\tau|^{-2}-|a_{13}a_{23}|^2\sigma^4\eta|\tau|^{-4}\right\}\end{aligned}$$

which implies that

$$\begin{aligned}\lim_{\gamma\downarrow 0}\mathrm{Im}\,D(\tau,\sigma)\,\gamma^{-1}=&-2S(\eta,\sigma)\left\{s_1+s_2+(s_1|a_{13}|^2+s_2|a_{23}|^2)\,\sigma^2\eta^{-2}\right\}\\&+8s_1s_2\left\{\mathrm{Re}\,(\bar{a}_{12}a_{13}\bar{a}_{23})\,\sigma^3\eta^{-2}-|a_{13}a_{23}|^2\sigma^4\eta^{-3}\right\}.\end{aligned} \tag{3. 32}$$

By definition of $\xi(\sigma)$, $\eta(\sigma)=\xi(\sigma)-\beta(\sigma)/2$ has to satisfy the equation (3. 2) in η, that is,

$$S\big(\eta(\sigma),\sigma\big)=-|b|^{-2}\eta(\sigma)^{-1}(s_1+s_2|b|^2)\left(a_{13}\overline{a_{23}b\sigma^2}-\eta(\sigma)\,a_{12}\bar{b}\sigma\right). \tag{3. 33}$$

Since $a_{12}\bar{b}$ and $a_{13}\overline{a_{23}b}$ are real, $\bar{a}_{12}a_{13}\bar{a}_{23}$ is also real and we have the following identity:

$$|a_{13}a_{23}|^2\sigma^4-(\bar{a}_{12}a_{13}\bar{a}_{23})\,\sigma^3\eta(\sigma)=|b|^{-2}a_{13}\overline{a_{23}b\sigma^2}\left(a_{13}\overline{a_{23}b\sigma^2}-\eta(\sigma)\,a_{12}\bar{b}\sigma\right). \tag{3. 34}$$

Substitute (3. 33) and (3. 34) into (3. 32), we then obtain that

$$\begin{aligned}\lim_{\gamma\downarrow 0}\mathrm{Im}\,D\big(\eta(\sigma)-i\gamma,\sigma\big)\,\gamma^{-1}=&\,2|b|^{-2}\eta(\sigma)^{-3}\left(a_{13}\overline{a_{23}b\sigma}-\eta(\sigma)\,a_{12}\bar{b}\sigma\right)\\&\times\left[(s_1+s_2|b|^2)\left\{(s_1+s_2)\,\eta(\sigma)^2+(s_1|a_{13}|^2+s_2|a_{23}|^2)\,\sigma^2\right\}-4s_1s_2a_{13}\overline{a_{23}b\sigma^2}\right]\end{aligned}$$

which implies that

(3. 35)
$$\lim_{\gamma\downarrow 0}\operatorname{Im}\Big(\eta(\sigma)-i\gamma,\sigma)\Big)\gamma^{-1}=2|b|^{-2}\eta(\sigma)^{-3}\Big(a_{13}\overline{a_{23}}\bar{b}\sigma^{2}-\eta(\sigma)\,a_{12}\bar{b}\sigma\Big)$$
$$\times\Big\{(s_1+s_2)\,(s_1+s_2|b|^2)\,\eta(\sigma)^2+|s_1a_{13}-s_2a_{23}b|^2\sigma^2+s_1s_2|a_{23}-a_{13}\bar{b}|^2\sigma^2\Big\}\,.$$

Therefore, if $\eta(\sigma)$ will be taken as

(3. 36) $$\eta(\sigma)\Big(a_{13}\overline{a_{23}}\sigma^2-\eta(\sigma)\,a_{12}\sigma\Big)\neq 0\,,$$

it then follows from (3. 30), (3. 35) and (3. 36) that $Q(\eta(\sigma),\sigma)$ is non-zero and signs of $Q(\eta(\sigma),\sigma)$ and $\lim_{\gamma\downarrow 0}\operatorname{Im}D(\eta(\sigma)-i\gamma,\sigma)\gamma^{-1}$ are different.

To prove the existence of such $\eta(\sigma)$ we remark that $\eta(\sigma)=0$ is equivalent to $\xi(\sigma)=\beta(\sigma)/2$. If $a_{13}a_{23}=0$ and a_{13} or a_{23} is non-zero, then $I>0$ and the equation (3. 31) has a root different from $\beta(\sigma)/2$. Hence (3. 36) is valid because of $a_{12}\neq 0$. If $F>0$, $I\neq 0$ and $a_{13}a_{23}\neq 0$, then the equation (3. 31) has two roots different from $\beta(\sigma)/2$. Hence we may choose $\eta(\sigma)$ satisfying (3. 36). If either $F>0$, $a_{13}a_{23}\neq 0$ and $I=a_{12}=0$ or $F>0$, $I=0$, $a_{12}a_{23}a_{13}\neq 0$ and $\beta+a_{13}\bar{a}_{23}/a_{12}\neq 0$, then $\beta\neq 0$ and (3. 36) valid for a root $\xi(\sigma)=-\beta(\sigma)/2$ of (3. 31), that is, $\eta(\sigma)=-\beta(\sigma)$. If either $F=a_{12}=0$, $\beta\neq 0$ and $a_{13}a_{23}\neq 0$ or $F=0$, $a_{12}a_{13}a_{23}\neq 0$, $\beta\neq 0$ and $\beta+2a_{13}\overline{a_{23}}/a_{12}\neq 0$, then (3. 36) is valid for a root $\xi(\sigma)=0$ of (3. 31), that is, $\eta(\sigma)=-\beta(\sigma)/2$.

Finally we shall prove that $\|\mathscr{B}(\eta(\sigma),\sigma)\|\neq 0$ for $\eta(\sigma)$ determined above. It follows from (2. 28), (2. 30) and (2. 31) that the (2. 1)-element of $\mathscr{B}(\eta(\sigma),\sigma)$ is equal to

$$s_1s_2\Big(\bar{a}_{12}\sigma\eta(\sigma)-\bar{a}_{13}a_{23}\sigma^2\Big)/\eta(\sigma)\,S\Big(\eta(\sigma),\sigma\Big)\,,$$

which does not vanish in virtue of (3. 33) and (3. 36). Thus the proof is complete.

When $a_{13}=a_{23}=0$ and $a_{12}\bar{b}$ is non-zero real, the quantities considered above have no terms involving τ^{-1}. Hence (3. 2) is equivalent to

(3. 37) $$(s_1+s_2)\,\eta+s_1a_{11}\sigma+s_2a_{22}\sigma=(s_1+s_2|b|^2)\,(a_{12}\bar{b}\sigma)|b|^{-2}$$

Moreover we obtain from (3. 5) and (3. 32) that

(3. 38) $$Q(\eta,\sigma)=-|b|^{-2}(s_1-s_2|b|^2)\,(a_{12}\bar{b}\sigma)$$

and

(3. 39) $$\lim_{\gamma\downarrow 0}\operatorname{Im}D(\eta-i\gamma,\sigma)\,\gamma^{-1}=-2\,(s_1+s_2)\,\Big\{(s_1+s_2)\,\eta+s_1a_{11}\sigma+s_2a_{22}\sigma\Big\}\,.$$

Substitute $\eta(\sigma)$ satisfying (3. 37) into (3. 39), we get

$$\lim_{\gamma\downarrow 0}\operatorname{Im}D\Big(\eta(\sigma)-i\gamma,\sigma\Big)\gamma^{-1}=-2|b|^{-2}(s_1+s_2)\,(s_1+s_2|b|^2)\,(a_{12}\bar{b}\sigma)\,.$$

Hence it follows from this and (3. 38) that signs of $Q(\eta(\sigma), \sigma)$ and $\lim_{\gamma\downarrow 0} \mathrm{Im}\, D(\eta(\sigma)-i\gamma, \sigma)$ are different. On the other hand, we find from (2. 28), (2. 30), (2. 31) and (3. 37) that the (2. 1)-element of $\mathscr{B}(\eta(\sigma), \sigma)$ is equal to the positive quantity $s_1 s_2|b|^2(s_1+s_2|b|^2)$. By the same method as the proof of Corollary 2. 3 we can conclude that $R(\eta(\sigma), \sigma)=0$ for such $\eta(\sigma)$ and the mixed problem (2. 1) then is not L^2-well posed.

For the remainder of Case 1 we shall show that under the assumption (3. 1) the condition (2. 35) for reflection coefficient does not hold in a neighbourhood of a real point (η, σ) such that $\eta=0$ or $\lambda^+(\eta, \sigma)$ is a real double root. To do this we remark the followings: If $\mathrm{Im}\,(a_{13}\overline{a_{23} b})=0$, then we see from (3. 16) that

$$I=(\bar{a}_{23}-\bar{a}_{13} b)\,(s_2 a_{23} b-s_1 a_{13})\,\bar{b}\,.$$

When $b=\bar{a}_{23}/\bar{a}_{13}$ or $s_1 a_{13}/s_2 a_{23}$, the condition that $\mathrm{Im}\,(a_{13}\overline{a_{23} b})=\mathrm{Im}\,(a_{12}\bar{b})=0$ is equivalent to $\mathrm{Im}\,(\bar{a}_{12} a_{13} \bar{a}_{23})=0$. Moreover, when $\mathrm{Im}\,(a_{13}\overline{a_{23} b})=\mathrm{Im}\,(a_{12}\bar{b})=0$, we see from (3. 15) and (3. 16) that $F=\beta=0$ implies $I=0$.

Our propositions are as follows:

Proposition 3. 6. *Assume that* $s_1-s_2|b|^2<0$, $\mathrm{Im}\,(\bar{a}_{12} a_{13} \bar{a}_{23})=0$ *and* $(\bar{a}_{23}-\bar{a}_{13} b)\,(s_2 a_{23} b-s_1 a_{13})=0$. *Then the condition* (2. 35) *for reflection coefficient does not hold in a neighbourhood of* (0, 1) *if either* $a_{13} a_{23}\neq 0$ *and* $\beta=0$ *or* $a_{12} a_{13} a_{23}\neq 0$ *and* $\beta+a_{13}\bar{a}_{23}/a_{12}=0$.

Proposition 3. 7. *Assume that* $s_1-s_2|b|^2<0$ *and* $\mathrm{Im}\,(a_{13}\overline{a_{23} b})=\mathrm{Im}\,(a_{12}\bar{b})=0$. *Then the condition* (2. 35) *for reflection coefficient does not hold in a neighourhood of a real point* $(\eta, 1)$ *if* $a_{12} a_{13} a_{23}\neq 0$, $F=0$ *and* $\beta+2a_{13}\bar{a}_{23}/a_{12}=0$

Remark. We find from the proofs of the propositions that $R(0, 1)\neq 0$ for the second case of Proposition 3. 6 and $\lambda^+(\eta, 1)$ is a real double root for η in Proposition 3. 7.

Proof of Proposition 3. 6. If $b=\bar{a}_{23}/\bar{a}_{13}$ or $s_1 a_{13}/s_2 a_{23}$, then we have

$$s_1-s_2|b|^2=(s_1|a_{13}|^2-s_2|a_{23}|^2)/|a_{13}|^2 \text{ or } -(s_1|a_{13}|^2-s_2|a_{23}|^2)/s_2|a_{23}|^2$$

respectively. Hence the assumption $s_1-s_2|b|^2<0$ is equivalent that

(3. 40) $\qquad s_1|a_{13}|^2-s_2|a_{23}|^2$ is negative or positive

respectively. Hereafter put

(3. 41) $\qquad d^{-1}=\bar{a}_{12}/\bar{a}_{13} a_{23}\,.$

Then d is real because $\bar{a}_{12} a_{13} \bar{a}_{23}$ is real. Moreover, we get that

(3. 42) $\qquad |b|^{-2}(s_1+s_2|b|^2)\, a_{12}\bar{b}=d^{-1}(s_1|a_{13}|^2+s_2|a_{23}|^2)$

for $b=\bar{a}_{23}/\bar{a}_{13}$ or $s_1 a_{13}/s_2 a_{23}$.

We first assume that $a_{12}a_{13}a_{23}\neq 0$ and $\beta+a_{13}\bar{a}_{23}/a_{12}=0$. It follows from (3.10), (3.41) and (3.42) that

$$s_1 a_{11}+s_2 a_{22}=d^{-1}(s_1|a_{13}|^2+s_2|a_{23}|^2)-d(s_1+s_2).$$

Hence we obtain from this and (2.19) that

$$S(\tau, 1)=(d^{-1}-\tau^{-1})\left\{d(s_1+s_2)\tau+s_1|a_{13}|^2+s_2|a_{23}|^2\right\}, \tag{3.43}$$

which implies that

$$\begin{aligned} &-S(\tau, 1)+2bs_2(\bar{a}_{12}-\bar{a}_{13}a_{23}\tau^{-1}) \\ &\quad=-(d^{-1}-\tau^{-1})\left\{d(s_1+s_2)\tau\pm(s_1|a_{13}|^2-s_2|a_{23}|^2)\right\}. \end{aligned} \tag{3.44}$$

Hereafter the upper sign or the lower sign corresponds to $b=\bar{a}_{23}/\bar{a}_{13}$ or $s_1 a_{13}/s_2 a_{23}$ respectively.

On the other hand, it follows from (2.20), (3.41) and (3.43) that

$$\begin{aligned} D(\tau, 1)=(d^{-1}-\tau^{-1})^2&\left\{d^2(s_1+s_2)^2\tau^2\right. \\ &\left.+2d(s_1+s_2)(s_1|a_{13}|^2+s_2|a_{23}|^2)\tau+(s_1|a_{13}|^2-s_2|a_{23}|^2)^2\right\}. \end{aligned}$$

Since $\operatorname{Im}(d^{-1}-\tau^{-1})<0$ for $\operatorname{Im}\tau<0$, se find from (3.40) and the choice of branch (2.17) that, in a neighbourhood $\tau=0$,

$$\begin{aligned} D(\tau, 1)^{1/2}=\pm(d^{-1}-\tau^{-1})&\left\{s_1|a_{13}|^2-s_2|a_{23}|^2\right. \\ &\left.+d(s_1+s_2)(s_1|a_{13}|^2+s_2|a_{23}|^2)(s_1|a_{13}|^2-s_2|a_{23}|^2)^{-1}\tau+O(\tau^2)\right\}. \end{aligned} \tag{3.45}$$

Thus we obtain from (2.22), (2.23), (3.44) and (3.45) that, in a neighbourhood of $\tau=0$,

$$\begin{aligned} 2R(\tau, 1)=(d^{-1}-\tau^{-1})&\left[d(s_1+s_2)\left\{-1\pm(s_1|a_{13}|^2+s_2|a_{23}|^2)\right.\right. \\ &\left.\left.\times(s_1|a_{13}|^2-s_2|a_{23}|^2)^{-1}\right\}\tau+O(\tau^2)\right] \end{aligned} \tag{3.46}$$

and

$$B'U^-(\tau, 1)=(d^{-1}-\tau^{-1})\left\{\mp(s_1|a_{13}|^2-s_2|a_{23}|^2)+O(\tau)\right\}. \tag{3.47}$$

From (3.46) *we see that* $R(0, 1)\neq 0$. Therefore we can conclude from (2.10) (3.46) and (3.47) that there exists a constant $C>0$ such that

$$C^{-1}\gamma^{-1}\leqq\left|C(-i\gamma, 1)\right|\leqq C\gamma^{-1} \tag{3.48}$$

for any small $\gamma>0$.

We treat the behaviors of $\operatorname{Im}\lambda^{\pm}(\tau, 1)$ in a neighbourhood of $\tau=0$. When $b=\bar{a}_{23}/\bar{a}_{13}$, it follows from (2. 16), (3. 40) and (3. 45) that, for any small $\gamma>0$,

$$\operatorname{Im}\lambda^{+}(-i\gamma, 1)=O(1)$$

and

$$\operatorname{Im}\lambda^{-}(-i\gamma, 1)=(s_1|a_{13}|^2-s_2|a_{23}|^2)\gamma^{-1}+O(1)\,.$$

When $b=s_1 a_{13}/s_2 a_{23}$, we obtain in the same way as the above case that

$$\operatorname{Im}\lambda^{+}(-i\gamma, 1)=(s_1|a_{13}|^2-s_2|a_{23}|^2)\gamma^{-1}+O(1)$$

and

$$\operatorname{Im}\lambda^{-}(-i\gamma, 1)=O(1)\,.$$

Therefore there exists a constant $C>0$ such that

(3. 49) $$\left|\operatorname{Im}\lambda^{+}(-i\gamma, 1)\operatorname{Im}(-i\gamma, 1)\right|\leqq C\gamma^{-1}$$

for any small $\gamma>0$.

We next treat the behavior of $\|\mathscr{B}(\tau, 1)\|$ in a neighourhood of $\tau=0$. It follows from (2. 28), (2. 30), (3. 43) and (3. 45) that, in a neighbourhood of $\tau=0$,

(3. 50) $$\det T(\tau, 1)=-s_2\bar{a}_{13}a_{23}(d^{-1}-\tau^{-1})^2\left\{s_1|a_{13}|^2+s_2|a_{23}|^2+O(\tau)\right\},$$

(3. 51) $$u_2^{+}(\tau, 1)=s_2\bar{a}_{13}a_{23}(d^{-1}-\tau^{-1})$$

and

(3. 52) $$u_1^{+}(\tau, 1)=-(d^{-1}-\tau^{-1})\left(s_2|a_{23}|^2+O(\tau)\right) \quad \text{or} \quad -(d^{-1}-\tau^{-1})\left(s_1|a_{13}|^2+O(\tau)\right)$$

according to $b=\bar{a}_{23}/\bar{a}_{13}$ or $b=s_1 a_{13}/s_2 a_{23}$, respectively. Moreover, from (2. 32), (3. 51) and (3. 52) we find that for $b=\bar{a}_{23}/\bar{a}_{13}$

(3. 53) $$p_2(\tau, 1)=s_2 a_{23}(d^{-1}-\tau^{-1})\left(s_1|a_{13}|^2-s_2|a_{23}|^2+O(\tau)\right)$$

and for $b=s_1 a_{13}/s_2 a_{23}$

(3. 54) $$p_1(\tau, 1)=\bar{a}_{23}(d^{-1}-\tau^{-1})\left(s_2|a_{23}|^2-s_1|a_{13}|^2+O(\tau)\right).$$

Therefore it follows from (2. 31), (2. 32) and (3. 50)-(3. 54) that there exists a constant $C>0$ such that

(3. 55) $$\left\|\mathscr{B}(-i\gamma, 1)\right\|\geqq C\gamma^{-1}$$

for any small $\gamma>0$.

If the condition (2. 35) holds in a neighbourhood of (0, 1), that is,

$$\left\|\mathscr{B}(-i\gamma, 1)\right\| \left|C(-i\gamma, 1)\right| \leqq C\gamma^{-1}\left|\operatorname{Im} \lambda^{+}(-i\gamma, 1) \operatorname{Im} \lambda^{-}(-i\gamma, 1)\right|^{1/2}$$

for small $\gamma>0$, then it follows from (3. 48), (3. 49) and (3. 55) that for some constant $C>0$ it must hold that

$$\gamma^{-1} \leqq C\gamma^{-1/2}.$$

However, this inequality is not valid for a small $\gamma>0$.

We next assume that $a_{13}a_{23} \neq 0$ and $\beta=0$. It follows from (2. 19), (3. 10) and (3. 42) that

$$S(\tau, 1)=(s_1+s_2)\tau+(d^{-1}-\tau^{-1})(s_1|a_{13}|^2+s_2|a_{23}|^2). \tag{3. 56}$$

Hence we have

$$\begin{aligned} &-S(\tau, 1)+2s_2 b(\bar{a}_{12}-\bar{a}_{13}a_{23}\tau^{-1}) \\ &\quad = -(s_1+s_2)\tau \mp (d^{-1}-\tau^{-1})(s_1|a_{13}|^2-s_2|a_{23}|^2). \end{aligned} \tag{3. 57}$$

On the other hand, it follows from (2. 20) and (3. 56) that

$$\begin{aligned} D(\tau, 1) = &(s_1+s_2)^2\tau^2+2(s_1+s_2)(s_1|a_{13}|^2+s_2|a_{23}|^2)(d^{-1}-\tau^{-1}) \\ &+(s_1|a_{13}|^2-s_2|a_{23}|^2)^2(d^{-1}-\tau^{-1})^2. \end{aligned}$$

By the same way as deriving (4. 45) we obtain that, in a neighbourhood of $\tau=0$,

$$\begin{aligned} D(\tau, 1)^{1/2} = &\pm(d^{-1}-\tau^{-1})\Big\{s_1|a_{13}|^2-s_2|a_{23}|^2+(s_1+s_2) \\ &\times(s_1|a_{13}|^2+s_2|a_{23}|^2)(s_1|a_{13}|^2-s_2|a_{23}|^2)^{-1}\tau^2+O(\tau^3)\Big\}. \end{aligned} \tag{3. 58}$$

Thus we find from (2. 22), (2. 23), (3. 57) and (3. 58) that, in a neighbourhood of $\tau=0$,

$$\begin{aligned} 2R(\tau, 1) = &-(s_1+s_2)\tau \pm (d^{-1}-\tau^{-1})\Big\{(s_1+s_2) \\ &\times(s_1|a_{13}|^2+s_2|a_{23}|^2)(s_1|a_{13}|^2-s_2|a_{23}|^2)^{-1}\tau^2+O(\tau^3)\Big\} \end{aligned}$$

and

$$B'U^{-}(\tau, 1) = -(s_1+s_2)\tau \mp (d^{-1}-\tau^{-1})\Big\{(s_1|a_{13}|^2-s_2|a_{23}|^2)+O(\tau^2)\Big\}.$$

Therefore there exists a constant $C>0$ such that

$$C^{-1}\gamma^{-2} \leqq \left|C(-i\gamma, 1)\right| \leqq C\gamma^{-2} \tag{3. 59}$$

for any small $\gamma<0$.

Since the coefficient in τ^{-1} of (3. 56) or (3. 58) is equal to one of (3. 43) or (3. 45) respectively, we find by the same way as the above case that the estimates (3. 49) and (3. 55) are also valid for this case. Therefore it follows from (3. 49), (3. 55) and (3. 58) that

$$\left\|\mathscr{B}(-i\gamma, 1)\right\| \left|C(-i\gamma, 1)\right| \geqq C\gamma^{-3}$$

and

$$\gamma^{-1}\left|\operatorname{Im} \lambda^{+}(-i\gamma, 1) \operatorname{Im} \lambda^{-}(-i\gamma, 1)\right|^{1/2} \leqq C\gamma^{-3/2}$$

for any small $\gamma>0$. This contradicts the condition (2. 35) of reflection coefficient.

Proof of Proposition 3. 7. It follows from (3. 10) and (3. 41) that the relation $\beta=-2a_{13}\bar{a}_{23}/a_{12}$ is equivalent to

$$s_1 a_{11}+s_2 a_{22}=-2d(s_1+s_2)+|b|^{-2}(s_1+s_2|b|^2)\, a_{12}\bar{b}\,. \tag{3. 60}$$

Substitute $\beta=-2d$ into $F=0$, we obtain from (3. 15) that

$$d^2|b|^2(s_1+s_2)+|b|^2(s_1|a_{13}|^2+s_2|a_{23}|^2)-(s_1+s_2|b|^2)\,(a_{13}\overline{a_{23}b})=0\,. \tag{3. 61}$$

Substituting again (3. 60) and (3. 61) into (2. 19), we then obtain

$$S(\tau, 1)=(s_1+s_2)\,(\tau-2d+d^2\tau^{-1})+|b|^{-2}(s_1+s_2|b|^2)\,(a_{12}\bar{b}-a_{13}\overline{a_{23}b}\tau^{-1})\,,$$

which implies that

$$S(\tau, 1)=(d^{-1}-\tau^{-1})\left\{d(s_1+s_2)\,(\tau-d)+|b|^{-2}(s_1+s_2|b|^2)\,(a_{13}\overline{a_{23}b})\right\}. \tag{3. 62}$$

Hence we have from (3. 62) that

$$\begin{aligned} &-S(\tau, 1)+2s_2 b(\bar{a}_{12}-\bar{a}_{13}a_{23}\tau^{-1}) \\ &\qquad =-(d^{-1}-\tau^{-1})\left\{d(s_1+s_2)\,(\tau-d)+|b|^{-2}(s_1-s_2|b|^2)\,(a_{13}\overline{a_{23}b})\right\} \end{aligned} \tag{3. 63}$$

and

$$\begin{aligned} &S(\tau, 1)-2s_1 b^{-1}(a_{12}-a_{13}\bar{a}_{23}\tau^{-1}) \\ &\qquad =(d^{-1}-\tau^{-1})\left\{d(s_1+s_2)\,(\tau-d)-|b|^{-2}(s_1-s_2|b|^2)\,(a_{13}\overline{a_{23}b})\right\}. \end{aligned} \tag{3. 64}$$

On the other hand, it follows from (2. 20) and (3. 62) that

$$\begin{aligned} D(\tau, 1)=(d^{-1}-\tau^{-1})^2 &\Big\{d^2(s_1+s_2)^2(\tau-d)^2+2d|b|^{-2}(s_1+s_2) \\ &\times(s_1+s_2|b|^2)\,(a_{13}\overline{a_{23}b})\,(\tau-d)+|b|^{-4}(s_1-s_2|b|^2)^2(a_{13}\overline{a_{23}b})^2\Big\}\,. \end{aligned}$$

Remark that $a_{13}\overline{a_{23}b}>0$ by (3. 61), $s_1-s_2|b|^2<0$ and $\mathrm{Im}\,(d^{-1}-\tau^{-1})<0$ for Im $\tau<0$. We then obtain from the choice of branch (2. 17) that

$$\begin{aligned} D(\tau, 1)^{1/2} &= (d^{-1}-\tau^{-1})\Big\{|b|^{-2}(s_1-s_2|b|^2)(a_{13}\overline{a_{23}b}) \\ &\quad + d(s_1+s_2)(s_1+s_2|b|^2)(s_1-s_2|b|^2)^{-1}(\tau-d)+O\big((\tau-d)^2\big)\Big\} \end{aligned} \tag{3. 65}$$

in a neighbourhood of $\tau=d$. From (2. 22), (3. 63) and (3. 65) Lopatinski determinant $R(\tau, \sigma)$ is expressed in a neighbourhood of $\tau=d$ by

$$\begin{aligned} 2R(\tau, 1) &= (d^{-1}-\tau^{-1})\Big[d(s_1+s_2)\Big\{-1+(s_1+s_2|b|^2) \\ &\quad \times(s_1-s_2|b|^2)^{-1}\Big\}(\tau-d)+O\big((\tau-d)^2\big)\Big]. \end{aligned} \tag{3. 66}$$

Since we treat the behavior of reflection coefficient $C(\tau, \sigma)$ in a neighbourhood of $\tau=d$, $U^-(\tau, \sigma)$ will be taken as the form (2. 24) in virtue of Remark 2 after Proposition 2. 2 in § 2. Hence we find from (2. 2), (2. 7), (3. 64) and (3. 65) that

$$\begin{aligned} 2b^{-1}B'U^-(\tau, 1) &= -2s_1b^{-1}(a_{12}-a_{13}\bar{a}_{23}\tau^{-1})+S(\tau, 1)-D(\tau, 1)^{1/2} \\ &= -2(d^{-1}-\tau^{-1})\Big\{|b|^{-2}(s_1-s_2|b|^2)+O(\tau-d)\Big\}. \end{aligned}$$

Therefore we obtain from this and (3. 66) that there exists a constant $C>0$ such that

$$C^{-1}\gamma^{-1}\leqq\big|C(d-i\gamma, 1)\big|\leqq C\gamma^{-1} \tag{3. 67}$$

for any small $\gamma>0$.

Now we shall prove that there exists a constant $C>0$ such that

$$\big\|\mathscr{B}(d-i\gamma, 1)\big\|\geqq C \tag{3. 68}$$

for any small $\gamma>0$. From (3. 62) and (3. 65) we have

$$S(\tau, 1)-D(\tau, 1)^{1/2}=(d^{-1}-\tau^{-1})\Big\{2s_2(a_{13}\overline{a_{23}b})+O(\tau-d)\Big\}.$$

Hence it follows from this, (2. 29) and (2. 30) that

$$u_2^+(\tau, 1)=s_2\bar{a}_{13}a_{23}(d^{-1}-\tau^{-1})$$

and

$$\det T(\tau, 1)=(d^{-1}-\tau^{-1})^2\Big\{s_2|a_{13}a_{23}|^2(s_1-s_2|b|^2)+O(\tau-d)\Big\}.$$

Using the above relations we obtain from (2. 31) that the (2. 1)-element of $\mathscr{B}(\tau, 1)$ is in a neighbourhood of $\tau=d$ equal to

$$-s_1 s_2 a_{13} \bar{a}_{23}(s_1 - s_2|b|^2) + O(\tau - d),$$

which shows that (3.68) is valid.

On the other hand, it follows from (3.65) that there exists a constant $C>0$ such that

(3.69) $$\left|\operatorname{Im} D(d-i\gamma, 1)^{1/2}\right| \leqq C\gamma$$

for any small $\gamma>0$. From (2.16) and (3.69) we see easily that there exists a constant $C>0$ such that

(3.70) $$\left|\operatorname{Im} \lambda^{\pm}(d-i\gamma, 1)\right| \leqq C\gamma$$

for any small $\gamma>0$. Therefore it follows from (3.67), (3.68) and (3.70) that

$$\left\|\mathscr{B}(d-i\gamma, 1)\right\| \left|C(d-i\gamma, 1)\right| \geqq C\gamma^{-1}$$

and

$$\gamma^{-1}\left|\operatorname{Im} \lambda^{+}(d-i\gamma, 1) \operatorname{Im} \lambda^{-}(d-i\gamma, 1)\right|^{1/2} \leqq C$$

for any small $\gamma>0$. This contradicts the condition (2.35) for reflection coefficient. Thus the proof is complete.

Finally we consider Case 2 mentioned in the last part of § 2. In this case, σ may be regarded as two variables (σ_k, σ_l), $a_{12}^{(k)}=a_{12}^{(l)}=a_{13}^{(l)}=a_{23}^{(k)}=0$ and $a_{13}^{(k)} a_{23}^{(l)} \neq 0$. By replacing $a_{13}\sigma$ or $a_{23}\sigma$ for Case 1 by $a_{13}^{(k)}\sigma_k$ or $a_{23}^{(l)}\sigma_l$ respectively, the assertions corresponding to Propositions 3.4, 3.5 and 3.6 can be proved in the same way as Case 1. Moreover, these proofs are carried out simpler than proofs for Case 1, because $a_{12}^{(k)}=a_{12}^{(l)}=0$.

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