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# Some considerations on fibred spaces with certain almost complex structures

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Fibred spaces with almost complex structures have been studied by M. Ako [1][1)] and B. Watson [2]. The interesting result on a fibred space with Kählerian structure was given in [1]. The purpose of the present paper is to study the analogous problem in fibred almost Kählerian and almost Tachibana spaces and give certain extensions of Theorem 5.1 in [1]. For the purpose we need to have the method in [1].

In section 1 we define fibred spaces $\tilde{M}$ and the additional conception. In section 2 we introduce a projectable Riemannian metric $\tilde{g}$ in $\tilde{M}$. In section 3 we give formulas for the covariant differentiation with respect to the Riemannian connection induced by $\tilde{g}$. In section 4 we give some lemmas which will be used to prove Theorems in section 5.

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## 1. Fibred spaces.

The manifolds, objects and mappings which we consider are assumed to be of class $C^{\infty}$. The notation used in this paper is the same as [1].

Let $\tilde{M}$ and $M$ be manifolds of dimension $n$ and $m$ respectively, where $n>m$. A mapping $\sigma$ from $\tilde{M}$ onto $M$ is called a submersion if the differential map $\sigma_*$ induced by $\sigma$ has the maximal rank $m$ everywhere in $\tilde{M}$. We assume the existence of such a submersion. $(\tilde{M}, M; \sigma)$ is then called a fibred space over $M$. Under the above assumption the inverse image $\mathscr{F}_P$ of $P \in M$ by $\sigma$ is an $(n-m)$-dimensional closed submanifold of $\tilde{M}$ and is called a fibre over $P$. Throughout this paper we assume that each fibre is connected.

A vector in $\tilde{M}$ at $\tilde{P} \in \tilde{M}$ is said to be vertical if it is tangent to the fibre over $\sigma(\tilde{P})$. A vector field of vertical vectors is called a vertical vector field.

Now, since the rank of $\sigma_*=m$, there are $(n-m)$ linearly independent vertical vector fields $C_\alpha(\alpha=m+1, \cdots, n)$ in a neighborhood of each point in

1) Numbers in brackets refer to references at the end of the paper.

$\tilde{M}$. $C_\alpha$ define an $(n-m)$-dimensional distribution $\tilde{P}\rightarrow T^V_{\tilde{P}}(M)$ which is completely integrable, where $T^V_{\tilde{P}}(\tilde{M})$ is the subspace spanned by $C_\alpha$ in the tangent space $T_{\tilde{P}}(\tilde{M})$ of $\tilde{M}$ at $\tilde{P}\in\tilde{M}$. Denoting by $T^H_{\tilde{P}}(\tilde{M})$ the complementary subspace of $T^V_{\tilde{P}}(\tilde{M})$ in $T_{\tilde{P}}(\tilde{M})$, we get an $m$-dimensional distribution $P\rightarrow T^H_{\tilde{P}}(\tilde{M})$ and we call it the horizontal distribution. Here such a distribution be fixed, we can choose $m$ linearly independent vector fields $E_a\,(a=1, 2, \cdots, m)$ in a neighborhood of every point $\tilde{P}$ such that at each point $\tilde{P}$ they span $T^H_{\tilde{P}}(\tilde{M})$.

Let $\begin{pmatrix} E^a \\ C^\alpha \end{pmatrix}$ be the inverse matrix of the matrix $(E_a, C_\alpha)$. Then each $(r, s)$-tensor $\tilde{T}$ in $M$ is expressed as

$$\begin{aligned}\tilde{T} = {} & T_{a_1\cdots a_s}{}^{b_1\cdots b_r}E^{a_1}\otimes\cdots\otimes E^{a_s}\otimes E_{b_1}\otimes\cdots\otimes E_{b_r}+\cdots\\ & +T_{a_1\cdots a_s}{}^{\beta_1\cdots\beta_r}E^{a_1}\otimes\cdots\otimes E^{a_s}\otimes C_{\beta_1}\otimes\cdots\otimes C_{\beta_r}+\cdots\\ & +T_{\alpha_1\cdots\alpha_s}{}^{b_1\cdots b_r}C^{\alpha_1}\otimes\cdots\otimes C^{\alpha_s}\otimes E_{b_1}\otimes\cdots\otimes E_{b_r}+\cdots\\ & +T_{\alpha_1\cdots\alpha_s}{}^{\beta_1\cdots\beta_r}C^{\alpha_1}\otimes\cdots\otimes C^{\alpha_s}\otimes C_{\beta_1}\otimes\cdots\otimes C_{\beta_r}\,.\end{aligned}$$

The first and the last terms in the right hand side are called the horizontal part and the vertical part of $\tilde{T}$ and denoted by $\tilde{T}^H$ and $\tilde{T}^V$ respectively. For (0, 0)-tensor $\tilde{f}$ in $\tilde{M}$ we define

$$\tilde{f}^H=\tilde{f}^V=\tilde{f}\,.$$

A tensor field $\tilde{T}$ in $\tilde{M}$ is said to be projectable if it satisfies

$$(\mathcal{L}_{\tilde{V}}\tilde{T}^H)^H=0$$

for any vertical vector $\tilde{V}$, where $\mathcal{L}_{\tilde{V}}$ denotes the Lie derivative with respect to $\tilde{V}$.

Let us denote by $\mathcal{T}^r_s(M)$ and $\mathcal{P}^{H}{}^r_s(\tilde{M})$ the space of all $(r, s)$-tensor fields in $M$ and that of all projectable and horizontal $(r, s)$-tensor fields in $\tilde{M}$ respectively. We have then by [1] isomorphisms $\pi$ from $\mathcal{P}^{H}{}^r_s(\tilde{M})$ onto $\mathcal{T}^r_s(M)$ and $L$ from $\mathcal{T}^r_s(M)$ onto $\mathcal{P}^{H}{}^r_s(\tilde{M})$ which are the inverse mappings each other. The former and the latter are called a projection and a lift respectively.

## 2. A projectable Riemannian metric.

We assume, here and in the sequel, that there is given a projectable Riemannian metric $\tilde{g}$ in $\tilde{M}$. Without loss of generality, we can furthermore assume that

$$\tilde{g}(E_a, C_\alpha)=0\,.$$

The Riemannian connection $\tilde{\nabla}$ with respect to $\tilde{g}$ and the Riemannian connection $\nabla$ with respect to $g=\pi\tilde{g}$ are related by

$$\nabla_X Y = \pi(\tilde{\nabla}_X LY^L) \qquad \text{for } X,\ Y \in \mathscr{T}_0^1 M).$$

On the other hand we have the induced Riemannian metric $'g$ and the induced Riemannian connection $'\nabla$ with respect to $'g$ in each fibre.

## 3. Expressions in terms of a local coordinate system.

From now on we discuss by means of a local coordinate system.

If $(\tilde{x}^i)$, $(x^a)$ and $('x^\alpha)$ $(i=1, 2, \cdots, m,\ m+1, \cdots, n;\ a=1, \cdots, m;\ \alpha=m+1, \cdots, n)$ are local coordinate systems of $\tilde{M}$, $M$ and each fibre respectively, the submersion $\sigma$ from $\tilde{M}$ onto $M$ is represented by equations $x^a = x^a(\tilde{x}^i)$ whose Jacobian matrix $\partial x^a/\partial \tilde{x}^i$ is of rank $m$ at any point of $\tilde{M}$. The vertical vector fields $C_\alpha$ and the horizontal covector fields $E^b$ may have $C_\alpha^i = \partial \tilde{x}^i/\partial' x^\alpha$ and $E_i^b = \partial x^b/\partial \tilde{x}^i$ as their components with respect to $(\tilde{x}^i)$ respectively. If the components of $E_a$ and $C^\beta$ are denoted by $E_a^i$ and $C_i^\beta$ respectively, we have

$$E_a^i E_i^b = \delta_a^b, \quad E_a^i C_i^\beta = 0, \quad E_i^b C_\alpha^i = 0, \quad C_i^\beta C_\alpha^i = \delta_\alpha{}^\beta,$$
$$E_i^a E_a^h + C_i^\alpha C_\alpha^h = \delta_i^h.$$

Since we may put $(\tilde{x}^i) = (x^a, 'x^\alpha)$, with respect to the natural frame the non-holonomic frame may have the following components:

$$E_a = \begin{pmatrix} \delta_a^b \\ -\Pi_a{}^\beta \end{pmatrix}, \qquad C_\alpha = \begin{pmatrix} 0 \\ \delta_\alpha^\beta \end{pmatrix}, \tag{3.1}$$
$$C^\beta = (\Pi_a{}^\beta, \delta_\alpha{}^\beta), \qquad E^b = (\delta_a^b, 0),$$

where $\Pi_a{}^\beta$ are functions in $\tilde{M}$.

Then by [1] we have the following formulas;

$$\tilde{\nabla}_j E_a^h = \left\{ {c \atop b\ a} \right\} E_j^b E_c^h + h_a{}^b{}_\alpha E_b^h C_j^\alpha + h_{ba}{}^\alpha E_j^b C_\alpha^h - 1_\beta{}^\alpha{}_a C_\alpha^h C_j^\beta, \tag{3.2}$$

$$\tilde{\nabla}_j E_i^a = -\left\{ {a \atop c\ b} \right\} E_j^c E_i^b + h_b{}^a{}_\alpha (E_j^b C_i^\alpha + E_i^b C_j^\alpha) - 1_{\beta\alpha}{}^a C_j^\beta C_i^\alpha, \tag{3.3}$$

$$\tilde{\nabla}_j C_\alpha^h = -h_b{}^a{}_\alpha E_a^h E_j^b - (1_\alpha{}^\beta{}_a - \Pi_a{}^\beta{}_\alpha)\, E_j^a C_\beta^h + 1_{\beta\alpha}{}^a E_a^h C_j^\beta + '\left\{ {\gamma \atop \beta\ \alpha} \right\} C_j^\beta C_\gamma^h, \tag{3.4}$$

$$\tilde{\nabla}_j C_i^\alpha = -h_{ba}{}^\alpha E_j^b E_i^a + (1_\beta{}^\alpha{}_a - \Pi_a{}^\alpha{}_\beta)\, E_j^a C_i^\beta + 1_\beta{}^\alpha{}_a E_i^a C_j^\beta - '\left\{ {\alpha \atop \gamma\ \beta} \right\} C_j^\gamma C_i^\beta, \tag{3.5}$$

where $h_{ba}{}^\alpha = h_b{}^c{}_\beta\, 'g^{\beta\alpha} g_{ca}$ is skew-symmetric in $b$ and $a$, $1_{\beta\alpha}{}^a = 1_\beta{}^\gamma{}_b\, 'g_{\gamma\alpha} g^{ba}$ is symmetric in $\beta$ and $\alpha$, $\Pi_a{}^\alpha{}_\beta = \partial_\beta \Pi_a{}^\alpha (\partial_\beta = \partial/\partial' x^\beta)$, $\left\{ {c \atop b\ a} \right\}$ and $'\left\{ {\gamma \atop \beta\ \alpha} \right\}$ are Christoffel symbols with respect to $g_{ba}$ and $'g_{\beta\alpha}$ respectively.

Since $E_b^j \tilde{\nabla}_j E_a^h - E_a^j \tilde{\nabla}_j E_b^h = 2h_{ba}{}^\alpha C_\alpha^h$, we have $h_{ba}{}^\alpha = 0$ as a necessary and sufficient condition for the horizontal distribution to be completely integrable. When the horizontal distribution is completely integrable we can choose a non-holonomic frame $(E_a, C_\alpha)$ such that $\Pi_a{}^\alpha = 0$.

On the other hand $C_\beta^j \tilde{\nabla}_j C_\alpha^h = 1_{\beta\alpha}{}^a E_a^h + '\left\{ {\gamma \atop \beta\ \alpha} \right\} C_\gamma^h$ being hold, we find $1_{\beta\alpha}{}^a$ are the components of the second fundamental tensor on $\mathscr{F}_P$ with respect to the normal vector $E_a$. Then we have $1_{\beta\alpha}{}^a = 0$ as a necessary and sufficient condition for each fiber to be totally geodesic.

## 4. Lemmas.

In this section we show some lemmas given in [1] which will be useful to prove Theorems in section 5.

First putting $j = a$ and $i = b$ in (3.5) and taking account of (3.1), we have

$$\partial_a \Pi_b{}^\alpha - \left\{ {c \atop a\ b} \right\} \Pi_c{}^\alpha = -h_{ab}{}^\alpha + (1_\beta{}^\alpha{}_a \Pi_b{}^\beta + 1_\beta{}^\alpha{}_b \Pi_a{}^\beta)$$
$$- \Pi_a{}^\alpha{}_\beta \Pi_b{}^\beta - '\left\{ {\alpha \atop \gamma\ \beta} \right\} \Pi_a{}^\gamma \Pi_b{}^\beta .$$

Then we get

$$h_{ab}{}^\alpha = \Pi_{[b}{}^\alpha \Pi_{a]}{}^\alpha{}_\beta + \partial_{[a} \Pi_{b]}{}^\alpha ,$$

where [ ] denotes the skew-symmetrization. Thus we have

LEMMA 4.1. *If $\Pi_a{}^\alpha$ are constant, then the horizontal distribution is integrable. Conversely, if the horizontal distribution is integrable, then we can choose a local coordinate system in which $\Pi_a{}^\alpha = 0$.*

$\tilde{M}$ is said to have isometric fibres if at each point of $\tilde{M}$ the equations

$$(\pounds_{E_a} \tilde{g})^V = 0 \qquad (a = 1, 2, \cdots, m)$$

are satisfied. By a straight forward computation we can see that $\tilde{M}$ has isometric fibres if and only if

$$\partial_a {}' g_{\beta\alpha} - \Pi_a{}^\gamma \partial_\gamma {}' g_{\beta\alpha} - {}' g_{\gamma\alpha} \Pi_a{}^\gamma{}_\beta - {}' g_{\beta\gamma} \Pi_a{}^\gamma{}_\alpha = 0 . \tag{4.1}$$

On the other hand by another computation we have

$$(\pounds_{E_a} \tilde{g})_{ji}^V = -2 1_{\beta\alpha a} C_j^\beta C_i^\alpha . \tag{4.2}$$

Now from Lemma 4.1 and (4.1) we have

LEMMA 4.2. *If $\tilde{M}$ has isometric fibres and the horizontal distribution is integrable, then $\tilde{M}$ is locally the Riemannian product of $\mathscr{F}_P$ and $\hat{M}$, where $\hat{M}$ is the integral submanifold of the horizontal distribution.*

Proof. Since the horizontal distribution is integrable, we can see that $\tilde{M}$ is locally the product of two submanifolds $\mathscr{F}_P$ and $\hat{M}$. Furthermore, in this case, from Lemma 4.1 we can choose a local coodinate system in which $\Pi_a{}^\alpha=0$. Then, from (4.1) we have

$$\partial_a{}' g_{\beta\alpha}=0\,,$$

with respect to such the local coordinate system. On the other hand we have also

$$\partial_\alpha g_{ba}=0\,,$$

because the Riemannian metric $\tilde{g}$ is projectable. Thus $\tilde{M}$ is locally the Riemannian product of $\mathscr{F}_P$ and $\hat{M}$.

Furthermore by means of (4.2) we have

LEMMA 4.3. *$\tilde{M}$ has isometric fibres if and only if the each fibre is totally geodesic submanifold of $\tilde{M}$.*

## 5. Fibred almost Kählerian and fibred almost Tachibana spaces.

We consider in this section an almost complex structure $\tilde{F}$ in $\tilde{M}$ which is assumed to be projectable, that is,

$$(\pounds_{\tilde{V}}\tilde{F}^H)^H=0$$

for any vertical vector field $\tilde{V}$. Furthermore we assume that $\tilde{F}$ is pure, that is, if we denote by $\tilde{F}_i{}^h$ the components of $\tilde{F}$ with respect to a local coordinate system, they are expressed by the non-holonomic frame $(E_a, C_\alpha)$ as follows;

$$\tilde{F}_i{}^h=f_b{}^a E_i^b E_a^h+f_\beta{}^\alpha C_i^\beta C_\alpha^h\,,$$

where $f_b{}^a$ are projectable functions by the assumption. Since we have

$$(5.1)\qquad f_b{}^a f_a{}^c=-\delta_b^c\,,\quad f_\beta{}^\alpha f_\alpha{}^\gamma=-\delta_\gamma^\beta\,,$$

we can see that $M$ and $\mathscr{F}_P$ admit almost complex structures respectively. An almost complex structure $\tilde{F}_i{}^h$ is said to be Kählerian, almost Kählerian and almost Tachibana if $\tilde{F}_i{}^h$ satisfies (i) $\tilde{\nabla}_j\tilde{F}_i{}^h=0$, (ii) $\tilde{\nabla}_j\tilde{F}_{ih}+\tilde{\nabla}_i\tilde{F}_{hj}+\tilde{\nabla}_h\tilde{F}_{ji}=0$ and (iii) $\tilde{\nabla}_j\tilde{F}_i{}^h+\tilde{\nabla}_i\tilde{F}_j{}^h=0$ respectively, where $\tilde{F}_{ih}=\tilde{F}_i{}^j\tilde{g}_{jh}$. Obviously (i) implies (ii) and (iii) and if $\tilde{F}_i{}^h$ satisfies (ii) and (iii) at the same time, then (i) is satisfied by $\tilde{F}_i{}^h$ [3].

In general by a straightforward computation we have

$$(5.2)\qquad \begin{aligned}\tilde{\nabla}_j\tilde{F}_i{}^h=&\nabla_c f_b{}^a E_j^c E_i^b E_a^h+(\Lambda_{c\beta}{}^\alpha-f_\beta{}^\gamma 1_\gamma{}^\alpha{}_c+f_\gamma{}^\alpha 1_\beta{}^\gamma{}_c)\,E_j^c C_i^\beta C_\alpha^h\\&+(f_b{}^a h_c{}^b{}_\beta-f_\beta{}^\gamma h_c{}^a{}_\gamma)\,E_j^c C_i^\beta E_a^h+(f_b{}^a h_{ca}{}^\alpha-f_\beta{}^\alpha h_{cb}{}^\beta)\,E_j^c E_i^b C_\alpha^h\end{aligned}$$

$$+(f_c^{\ a} H_b{}^c{}_\gamma - f_b^{\ c} h_c{}^a{}_\gamma)\, C_j^\gamma E_i^b E_a^h - (f_b^{\ a} 1_{\gamma\beta}{}^b - f_\beta^{\ \alpha} 1_{\gamma\alpha}{}^a)\, C_j^\gamma C_i^\beta E_a^h$$
$$+(f_\beta^{\ \alpha} 1_\gamma{}^\beta{}_b - f_b^{\ a} 1_\gamma{}^\beta{}_a)\, C_j^\gamma E_i^b C_\alpha^h + 'V_\gamma f_\beta^{\ \alpha} C_j^\gamma C_i^\beta C_\alpha^h\,,$$

where $\Lambda_{c\beta}{}^\alpha = (\pounds_{E_c} \tilde{F}_i^{\ h})\, C_\beta^i C_h^\alpha$. Using (5. 2) M. Ako [1] proved

THEOREM. *If $\tilde{M}$ is fibred Kählerian, then the horizontal distribution is integrable. In this case $\tilde{M}$ is locally the Riemannian product of $\hat{M}$ and $\mathscr{F}_P$ if and only if $\Lambda_{c\beta}{}^\alpha = 0$.*

Now we consider the case where the fibred space $\tilde{M}$ is almost Kählerian or almost Tachibana, and as extensions of the above theorem we have the following Theorem 5. 1 and Theorem 5. 2.

THEOREM 5. 1. *If $\tilde{M}$ is fibred almost Kählerian, the horizontal distribution is integrable. In this case $\tilde{M}$ is locally the Riemannian product of $\hat{M}$ and $\mathscr{F}_P$ if and only if $\Lambda_{c\beta}{}^\alpha = 0$.*

PROOF. When the almost complex structure is almost Kählerian, we substitute (5. 2) into (ii) and have

(5. 3) $\quad V_c f_{ba} + V_b f_{ac} + V_a f_{cb} = 0\,,$

(5. 4) $\quad h_{cb\alpha} = 0\,,$

(5. 5) $\quad \Lambda_{c\beta\alpha} = 2 f_\beta^{\ \gamma} 1_{\alpha\gamma c}\,,$

(5. 6) $\quad 'V_\gamma f_{\beta\alpha} + 'V_\beta f_{\alpha\gamma} + 'V_\alpha f_{\gamma\beta} = 0\,,$

where $f_{ba} = f_b^{\ c} g_{ca}$, $f_{\beta\alpha} = f_\beta^{\ \gamma}\, 'g_{\gamma\alpha}$ and $\Lambda_{c\beta\alpha} = \Lambda_{c\beta}{}^\gamma\, 'g_{\gamma\alpha}$. Obviously (5. 4) shows that the horizontal distribution is integrable. Furthermore from (5. 5) we find that $\Lambda_{c\beta\alpha} = 0$ if and only if $1_{\beta\alpha a} = 0$. Then by virtue of Lemma 4. 2 we can see that $\tilde{M}$ is locally the Riemannian product of $\hat{M}$ and $\mathscr{F}_P$ if and only if $\Lambda_{c\beta}{}^\alpha = 0$.

THEOREM 5. 2. *If $\tilde{M}$ is fibred almost Tachibana, then the horizontal distribution is integrable. In this case $\tilde{M}$ is locally the Riemannian product of $\hat{M}$ and $\mathscr{F}_P$ if and only if $\Lambda_{c\beta}{}^\alpha = 0$.*

PROOF. Since $\tilde{M}$ is almost Tachibana, we substitute (5. 2) into (iii) and have

(5. 7) $\quad V_c f_b^{\ a} + V_b f_c^{\ a} = 0\,,$

(5. 8) $\quad 2 f_c^{\ a} h_b{}^c{}_\gamma - h_c{}^a{}_\gamma f_b^{\ c} - h_b{}^a{}_\alpha f_\gamma^{\ \alpha} = 0\,,$

(5. 9) $\quad f_b^{\ a} h_{ca}{}^\alpha + f_c^{\ a} h_{ba}{}^\alpha = 0\,,$

(5. 10) $\quad 2 f_b^{\ a} 1_{\beta\gamma}{}^b - f_\beta^{\ \alpha} 1_{\gamma\beta}{}^a - f_\gamma^{\ \alpha} 1_{\beta\alpha}{}^a = 0\,,$

(5. 11) $\quad \Lambda_{c\beta}{}^\alpha + 2 1_\beta{}^\gamma{}_c f_\gamma^{\ \alpha} f_\gamma^{\ \alpha} - 1_\gamma{}^\alpha{}_c f_\beta^{\ \gamma} - 1_\beta{}^\alpha{}_a f_c^{\ a} = 0\,,$

(5. 12) $\quad 'V_\gamma f_\beta^{\ \alpha} + 'V_\beta f_\gamma^{\ \alpha} = 0\,.$

From (5.9) we have $f_b{}^c h_c{}^a{}_\gamma = -f_c{}^a h_b{}^c{}_\gamma$. Substituting this into (5.8) we have

(5.13) $$3f_c{}^\alpha h_b{}^c{}_\gamma = f_\gamma{}^\alpha h_b{}^a{}_\alpha .$$

Transvecting $f_a{}^d f_\beta{}^\gamma$ to each side of (5.13), taking account of (5.1) and renumbering indices, we have

(5.14) $$3f_\gamma{}^\alpha h_b{}^a{}_\alpha = f_c{}^a h_b{}^c{}_\gamma .$$

From (5.13) and (5.14) we get

$$f_c{}^a h_b{}^c{}_\gamma = 0 .$$

Then it follows that

$$h_b{}^a{}_\gamma = 0 ,$$

which shows the horizontal distribution is integrable.

It $\Lambda_{c\beta}{}^\alpha = 0$, we have from (5.11)

$$2 1_\beta{}^\delta{}_c f_\delta{}^\alpha - 1_\delta{}^\alpha{}_c f_\beta{}^\delta - 1_\beta{}^{\alpha b} f_c{}^b = 0 .$$

Transvecting $g^{ca'} g_{\alpha\beta}$, we have

(5.15) $$1_{\beta\gamma}{}^b f_b{}^a = 2 1_{\beta\alpha}{}^a f_\gamma{}^\alpha + 1_{\gamma\alpha}{}^a f_\beta{}^\alpha .$$

Substituting (5.15) into (5.10), we get

(5.16) $$3 1_{\beta\alpha}{}^a f_{\gamma\alpha} + 1_{\alpha\gamma}{}^a f_\beta{}^\alpha = 0 .$$

Interchanging the indices $\beta$ and $\gamma$ in (5.16) we have

(5.17) $$3 1_{\gamma\alpha}{}^a f_\beta{}^\alpha + 1_{\alpha\beta}{}^a f_{\gamma\alpha} = 0 .$$

From (5.16) and (5.17) we have

(5.18) $$1_{\gamma\beta}{}^a f_\beta{}^\alpha + 1_{\alpha\beta}{}^a f_\gamma{}^\alpha = 0 ,$$

and from (5.16) and (5.18) we have

$$1_{\beta\gamma}{}^a = 0 .$$

This means from Lemma 4.3 that $\tilde{M}$ has isometric fibres. Then by means of Lemma 4.2 $\tilde{M}$ is locally the Riemannian product of $\hat{M}$ and $\mathscr{F}_P$.

Conversely if we assume that $1_{\beta\gamma}{}^a = 0$, then clearly $\Lambda_{c\beta}{}^\alpha = 0$, which complete the proof.

## References

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