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The growth of entire and meromorphic functions

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1. Let $f(z)$ be an entire or meromorphic function. We shall denote by $\boldsymbol{C}$ the complex plane and by $\bar{\boldsymbol{C}}$, the extended complex plane. For $a\in\bar{\boldsymbol{C}}$, let $n(r, a)$ be the number of zeros of $f(z)-a$ in $|z|\leq r$, where for $a=\infty$, $n(r, \infty)$ stands for the number of poles of $f(z)$ in $|z|\leq r$. We shall assume, without loss of generality, that $f(z)$ has no zeros or poles at the origin. Let $T(r)=T(r,f)$ be the Nevanlinna characteristic function of $f(z)$. Let $n(0, a)=0$ and let

$$N(r, a,f)=N(r, a)=\int_{0}^{r}\frac{n(t,a)}{t}dt .$$

Let ρ be the order of $f(z)$. If

$$\limsup_{r\rightarrow\infty}\frac{\log^{+}n(r,a)}{\log r}=\rho_{1}(a,f)=\rho_{1}(a)<\rho ,$$

we call a an $e.\ v.\ B.$ (exceptional value in the sense of Borel). If

$$1-\limsup_{r\rightarrow\infty}\frac{N(r,a)}{T(r,f)}=\delta(a,f)=\delta(a)>0 ,$$

a is called $e.\ v.\ N.$ (exceptional value in the sense of Nevanlinna). Also, if

$$\tau=\limsup_{r\rightarrow\infty}\frac{T(r,f)}{r^{\rho}}\ (0<\rho<\infty) ,$$

then $f(z)$ is said to be of maximal, mean or minimal type according as $\tau=\infty$, $0<\tau<\infty$ or $\tau=0$. If $f(z)$ is an entire function, we denote as usual

$$M(r)=M(r,f)=\max_{|z|=r}|f(z)| .$$

2. We prove

THEOREM 1. *Let $f(z)$ be an entire function of order ρ, $(0\leq\rho<\infty)$. Then for every $\varepsilon>0$, as $r\rightarrow\infty$*

$$M\left(r+\frac{1}{r^{\rho-1+\varepsilon}}\right)\sim M(r) . \tag{1}$$

PROOF: Since $\log M(r)$ is a convex function of $\log r$,

$$\log M(r)=\int_0^r \frac{w(t)}{t}\,dt \qquad (2)$$

where $w(t)$ is an increasing function of t, see [1, 27]. From (2) it follows that

$$\limsup_{r\to\infty}\frac{\log\log M(r)}{\log r}=\rho=\limsup_{r\to\infty}\frac{\log w(r)}{\log r}\,.$$

Hence, for $r\geq r_0$,

$$w(r)<r^{\rho+\varepsilon/2}\,.$$

Now,

$$\begin{aligned}\log M\left(r+\frac{1}{r^{\rho-1+\varepsilon}}\right)&=\int_0^{r+\frac{1}{r^{\rho-1+\varepsilon}}}\frac{w(t)}{t}\,dt\\&=\int_0^r\frac{w(t)}{t}\,dt+\int_r^{r+\frac{1}{r^{\rho-1+\varepsilon}}}\frac{w(t)}{t}\,dt\\&=\log M(r)+\int_r^{r+\frac{1}{r^{\rho-1+\varepsilon}}}\frac{w(t)}{t}\,dt\,.\end{aligned}$$

Now, since $w(t)$ is increasing, we have

$$\begin{aligned}\int_r^{r+\frac{1}{r^{\rho-1+\varepsilon}}}\frac{w(t)}{t}\,dt&\leq w\left(r+\frac{1}{r^{\rho-1+\varepsilon}}\right)\int_r^{r+\frac{1}{r^{\rho-1+\varepsilon}}}\frac{1}{t}\,dt\\&=w\left(r+\frac{1}{r^{\rho-1+\varepsilon}}\right)\log\left(1+\frac{1}{r^{\rho+\varepsilon}}\right)\\&\leq w(2r)\frac{1}{r^{\rho+\varepsilon}}\\&<(2r)^{\rho+\varepsilon/2}\frac{1}{r^{\rho+\varepsilon}}\\&\to 0 \quad \text{as} \quad r\to\infty\,.\end{aligned}$$

Hence (1) follows.

We define the lower order of an entire function $f(z)$ by

$$\lambda=\liminf_{r\to\infty}\frac{\log\log M(r)}{\log r}\,.$$

For lower order λ, we have

THEOREM 2. *Let $f(z)$ be an entire function of lower order $\lambda(0<\lambda<\infty)$, then for every $\varepsilon>0$, as $r\to\infty$,*

$$M(r)=o\left(M\left(r+\frac{1}{r^{\lambda-1-\varepsilon}}\right)\right). \qquad (3)$$

PROOF: We may assume $\varepsilon<\lambda$. As in the proof of Theorem 1,

$$\log M(r)=\int_0^r \frac{w(t)}{t}\,dt$$

gives

$$\liminf_{r\to\infty}\frac{\log\log M(r)}{\log r}=\lambda=\liminf_{r\to\infty}\frac{\log w(r)}{\log r}.$$

Hence for, all $r\geq r_0$,

$$w(r)>r^{\lambda-\varepsilon/2}.$$

Now

$$\begin{aligned}\log M\left(r+\frac{1}{r^{\lambda-1-\varepsilon}}\right)&=\log M(r)+\int_r^{r+\frac{1}{r^{\lambda-1-\varepsilon}}}\frac{w(t)}{t}\,dt\\&\geq\log M(r)+w(r)\log\left(1+\frac{1}{r^{\lambda-\varepsilon}}\right)\\&>\log M(r)+\frac{w(r)}{2r^{\lambda-\varepsilon}}\end{aligned}$$

$$\left(\text{since, for } 0<x<1,\ \log(1+x)\geq\frac{x}{2}\right).$$

Hence,

$$\log\left\{\frac{M\left(r+\frac{1}{r^{\lambda-1-\varepsilon}}\right)}{M(r)}\right\}\to\infty \quad\text{as}\quad r\to\infty$$

and (3) follows.

Note: Theorem 2 can be generalized to the case $\lambda=\infty$. Precisely, we have

THEOREM 3. *If $f(z)$ is an entire function of lower order ∞, then for every real α, as $r\to\infty$,*

$$M(r)=o\left(M\left(r+\frac{1}{r^{\alpha}}\right)\right). \tag{4}$$

For the proof we may assume that $\alpha>0$. As in the proof of Theorem 2, we get

$$\log\left\{\frac{M\left(r+\frac{1}{r^{\alpha}}\right)}{M(r)}\right\}\geq\frac{w(r)}{2r^{\alpha+1}}.$$

But since $\lambda=\infty$, $w(r)\geq r^{\Delta}$ for $r\geq r_0$, where Δ is arbitrarily large. Choosing

$\Delta > \alpha+1$, we get (4).

Note: If $\rho=\infty$, $\lambda<\infty$, then using the same method we can show that for all real α,

$$\liminf_{r\to\infty} \frac{M(r)}{M\left(r+\frac{1}{r^{\alpha}}\right)} = 0 .$$

THEOREM 4. *Let $f(z)$ be an entire or meromorphic function for which $\rho_1(a)<\infty$. Then, as $r\to\infty$, for every $\alpha>\rho_1(a)$,*

$$N\left(r+\frac{1}{r^{\alpha-1}}, a\right) = N(r, a)+O(1) . \tag{5}$$

PROOF: We can assume that $n(0, a)=0$, then

$$\begin{aligned} N(r, a) \le N\left(r+\frac{1}{r^{\alpha-1}}, a\right) &= \int_0^{r+\frac{1}{r^{\alpha-1}}} \frac{n(t, a)}{t}\, dt \\ &= N(r, a)+\int_r^{r+\frac{1}{r^{\alpha-1}}} \frac{n(t, a)}{t}\, dt \\ &\le N(r, a)+n\left(r+\frac{1}{r^{\alpha-1}}, \mathrm{a}\right)\log\left(1+\frac{1}{r^{\alpha}}\right) \\ &\le N(r, a)+n\left(r+\frac{1}{r^{\alpha-1}}, a\right)\frac{1}{r^{\alpha}} . \end{aligned}$$

Now, since $\rho_1(a)<\alpha$, there exists $\varepsilon>0$, so that

$$\rho_1(a)+\varepsilon<\alpha , \qquad \text{also } n(r, a)<r^{\rho_1(a)+\varepsilon} \text{ for all } r\ge r_0 .$$

Hence

$$\begin{aligned} \frac{n\left(r+\frac{1}{r^{\alpha-1}}, a\right)}{r^{\alpha}} &< \frac{\left(r+\frac{1}{r^{\alpha-1}}\right)^{\rho(a)+\varepsilon}}{r^{\alpha}} \\ &\to 0 \quad \text{as} \quad r\to\infty , \end{aligned}$$

and (5) follows.

Putting $\alpha=\rho$, and taking $\rho_1(a)<\rho$, we get the following.

COROLLARY. *If $f(z)$ is an entire or meromorphic function of finite order ρ, with a as an e. v. B., then*

$$N\left(r+\frac{1}{r^{\rho-1}}, a\right) = N(r, a)+O(1) .$$

THEOREM 5. *Let $f(z)$ be an entire function of order $\rho(0<\rho<\infty)$. Suppose for distinct $a, b\in \boldsymbol{C}$, we have*

$$\delta(a)=1\,, \qquad n(r,b)=O(r^\rho)\,.$$

Then, as $r\to\infty$,

$$T(r,f)\sim N\left(r+\frac{1}{r^{\alpha-1}},\, b\right). \tag{6}$$

PROOF: By Nevanlinna second theorem,

$$T(r,f)\le N(r,a)+N(r,b)+O(\log r)\,.$$

Now, $f(z)$ is transcendental, hence $\log r=o(T(r,f))$. Also $N(r,a)=o(T(r,f))$ since $\delta(a)=1$. Hence

$$T(r,f)\sim N(r,b)\,. \tag{7}$$

Further,

$$\begin{aligned} N\left(r+\frac{1}{r^{\rho-1}},\, b\right) &= N(r,b)+\int_r^{r+\frac{1}{r^{\rho-1}}}\frac{n(t,b)}{t}\,dt \\ &\le N(r,b)+n\left(r+\frac{1}{r^{\rho-1}},\, b\right)\frac{1}{r^\rho} \\ &\le N(r,b)+A\left(r+\frac{1}{r^{\rho-1}}\right)^\rho\frac{1}{r^\rho} \\ &= N(r,b)+A\left(1+\frac{1}{r^\rho}\right)^\rho. \end{aligned}$$

$$N\left(r+\frac{1}{r^{\alpha-1}},\, b\right)-N(r,b)\le A\left(1+\frac{1}{r^\rho}\right)^\rho$$
$$\to A \quad \text{as} \quad r\to\infty\,.$$

Hence

$$N\left(r+\frac{1}{r^{\rho-1}},\, b\right)=N(r,b)+O(1)\,.$$

Hence (6) follows from (7).

THEOREM 6. *There does not exist any entire function satisfying simultaneously the conditions*:

(i) $M\left(r+\frac{1}{r^{\rho-1}}\right)=O\big(M(r)\big)$

(ii) $\frac{n(r,0)}{r^\rho}\to\infty$ *as* $r\to\infty$

where ρ *is the order of* $f(z)$. $(0<\rho<\infty)$.

PROOF: Suppose $M\left(r+\frac{1}{r^{\rho-1}}\right)=O(M(r))$. Then

$$\log M\left(r+\frac{1}{r^{\rho-1}}\right)=\log M(r)+\int_{r}^{r+\frac{1}{r^{\rho-1}}}\frac{w(t)}{t}\,dt$$

$$\geq \log M(r)+\frac{w(r)}{2r^{\rho}}\,.$$

Hence, using the hypothesis,

$$w(r)\leq Br^{\rho}\,.$$

So

$$\log M(r)=A+\int_{r_0}^{r}\frac{w(t)}{t}\,dt$$

$$\leq A+B\int_{r_0}^{r}t^{\rho-1}\,dt$$

$$\leq A_1+\frac{B}{\rho}r^{\rho}\,.$$

Hence

$$\limsup_{r\to\infty}\frac{\log M(r)}{r^{\rho}}\leq\frac{B}{\rho}\,.$$

Thus $f(z)$ is of minimal or mean. type Now, suppose $\frac{n(r,0)}{r^{\rho}}\to\infty$ as $r\to\infty$. We can assume $f(0)=1$. Then, by Jensen's theorem,

$$\log M(r)\geq\int_{0}^{r}\frac{n(t,0)}{t}\,dt$$

$$=A+\int_{r_0}^{r/2}\frac{n(t,0)}{t}\,dt+\int_{r/2}^{r}\frac{n(t,0)}{t}\,dt$$

$$\geq n\left(\frac{r}{2},0\right)\log 2\,.$$

Hence

$$\frac{\log M(r)}{r^{\rho}}\geq\frac{n\left(\frac{r}{2},0\right)\log 2}{\left(\frac{r}{2}\right)^{\rho}2^{\rho}}$$

$$\to\infty \quad\text{as}\quad r\to\infty\,.$$

Hence $f(z)$ is of maximal type. Thus (i) and (ii) are incompatible.

COROLLARY. *If $f(z)$ is an entire function of order $\rho(0<\rho<\infty)$ such that*

$$M\left(r+\frac{1}{r^{\rho-1}}\right)=O\big(M(r)\big)\,,$$

then $f(z)$ is of mean or minimal type.

References

[1] G. VALIRON: General Theory of Integral Functions (Chelsea Pub. Co., New York, 1949).

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