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Q-projective transformations of an almost quaternion manifold: II

By Shigeyoshi FUJIMURA

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Continued to the previous paper ([8]), we shall study infinitesimal Q-projective transformations on the quaternion Kählerian manifold[3] and prove the following theorems:

THEOREM 4. *If a complete quaternion Kählerian manifold* (M, g, V) *with positive scalar curvature* S *admits an infinitesimal non-affine Q-projective transformation,* (M, g, V) *is isometric to the quaternion projective space of constant Q-sectional curvature* $S/4m(m+2)$.

THEOREM 5. *In a compact quaternion Kählerian manifold, each vector field which satisfies* (3.4) *is an infinitesimal Q-projective transformation.*

Concerning infinitesimal projective transformations of a Riemannian manifold or infinitesimal holomorphically projective transformations of a Kählerian manifold, we have known interesting analogous results, and we can see them in [9], [10], [11], and etc..

§ 5. Proof of Theorem 4.

From (3.4), $\cdots$, (3.7) and Ricci's formula, we get

$$(5.1)\qquad \begin{aligned} 4(m+1)\nabla_j\eta_i &= \nabla_j(\nabla_i\nabla_h X^h - \nabla_h\nabla_i X^h) + \nabla_j\nabla_h\nabla_i X^h \\ &\quad - \nabla_h\nabla_j\nabla_i X^h + \nabla_h\nabla_j\nabla_i X^h \\ &= -S(\nabla_j X_i + \nabla_i X_j)/4m + 2\nabla_j\eta_i - 2\Lambda_{ji}^{kh}\nabla_k\eta_h . \end{aligned}$$

Transvecting (5.1) by Λ_{fg}^{ji} and substituting it into (5.1), we have

$$(5.2)\qquad \nabla_j\eta_i = S\Big\{\Lambda_{ji}^{kh}(\nabla_k X_h + \nabla_h X_k) - (2m+3)(\nabla_j X_i + \nabla_i X_j)\Big\}\Big/32m^2(m+2)$$

where indices f and g run over the range $\{1, \cdots, 4m\}$. On the other hand, from (1.1) and (3.1), we have

3) We assume that the dimension $4m$ of $M \geqq 8$.

$$(5.3)\quad \begin{cases} \Lambda_{ji}^{kh} g_{kh} = 3g_{ji}\,, \\ \Lambda_{ji}^{kh} g_{hl} = -\Lambda_{jl}^{kh} g_{hi}\,, \\ \Lambda_{ji}^{kh} g_{hl} \Lambda_{kg}^{fl} = -3I_j^f g_{gi} + 2\Lambda_{jg}^{fh} g_{hi}\,, \\ \Lambda_{ji}^{kh} g_{hl} \Lambda_{gk}^{fl} = \Lambda_{gi}^{fh} g_{jh}\,. \end{cases}$$

Covariantly derivating (5.2) and using (3.4), (3.5) and (5.3), we get

$$(5.4)\quad \begin{aligned} \nabla_k \nabla_j \eta_i = -S(&2g_{ji}\eta_k + g_{kj}\eta_i + g_{ki}\eta_j \\ &-\Lambda_{jk}^{lh} g_{ih}\eta_l - \Lambda_{ik}^{lh} g_{jh}\eta_l)/16m(m+2)\,. \end{aligned}$$

η_h being a gradient 1-form, from (5.4) and Theorem D, we can complete the proof of Theorem 4.

Combining Theorems 3 and 4, we can obtain

COROLLARY 1. *If a compact quaternion Kählerian manifold (M, g, V) admits an infinitesimal non-affine Q-projective transformation, its scalar curvature S is positive and (M, g, V) is isometric to the quaternion projective space of constant Q-sectional curvature $S/4m(m+2)$.*

§ 6. Proof of Theorem 5.

We call a vector field X to be a Q-projective vector field if X satisfies (3.4). From (3.4), ⋯, (3.6) and (5.3), we have

$$(6.1)\quad \begin{aligned} &3m\left\{\nabla^h \nabla_h X^j + SX^j/4(m+2)\right\} - \Lambda^{kjih}\nabla_k \nabla_i X_h \\ &\quad = -3SX^j/2(m+2) + \Lambda^{kjih} R_{lkij} X^l \end{aligned}$$

because $\Lambda^{kjih}\Lambda_{ki}^{lf} g_{fh} = 12mg^{jl}$ and $\Lambda^{kjih}\Lambda_{ik}^{lf} g_{fh} = -3g^{jl}$, where $\Lambda^{kjih} = g^{kg}g^{ij}\Lambda_{gf}^{jh}$. On the other hand, from (3.3) and Ricci's formula, we have

$$(6.2)\quad R_{kjl}{}^h J_{p,i}{}^l - R_{kji}{}^l J_{p,l}{}^h = \gamma_{pq,kj} J_{q,i}{}^h$$

where $\beta_{pq,j}$ are components of β_{pq} and we put

$$\begin{aligned} &\gamma_{pq,kj} + \gamma_{qp,kj} = 0\,, \\ &\gamma_{12,kj} = \nabla_k \beta_{12,j} - \nabla_j \beta_{12,k} + \beta_{31,j}\beta_{23,k} - \beta_{31,k}\beta_{23,j}\,, \\ &\gamma_{31,kj} = \nabla_k \beta_{31,j} - \nabla_j \beta_{31,k} + \beta_{23,j}\beta_{12,k} - \beta_{23,k}\beta_{12,j}\,, \\ &\gamma_{23,kj} = \nabla_k \beta_{23,j} - \nabla_j \beta_{23,k} + \beta_{12,j}\beta_{31,k} - \beta_{12,k}\beta_{31,j}\,. \end{aligned}$$

Transvecting the three equations of (6.2) with $J_{1,hg}$, $J_{2,hg}$ and $J_{3,hg}$ respectively, we get

$$(6.3)\quad \begin{cases} -R_{kjlh} J_{1,i}{}^l J_{1,g}{}^h + R_{kjig} = \gamma_{12,kj} J_{3,ig} + \gamma_{31,kj} J_{2,ig}\,, \\ -R_{kjlh} J_{2,i}{}^l J_{2,g}{}^h + R_{kjig} = \gamma_{23,kj} J_{1,ig} + \gamma_{12,kj} J_{3,ig}\,, \\ -R_{kjlh} J_{3,i}{}^l J_{3,g}{}^h + R_{kjig} = \gamma_{31,kj} J_{2,ig} + \gamma_{23,kj} J_{1,ig} \end{cases}$$

where $J_{p,hg}=J_{p,h}^{\ j}g_{jg}$. Transvecting $(6.3)_1$ with $J_{2,}^{\ ig}$, $(6.3)_2$ with $J_{3,}^{\ ig}$ and $(6.3)_3$ with $J_{1,}^{\ ig}$ respectively, we obtain

$$(6.4)\qquad \begin{cases} R_{kjlh}J_{2,}^{\ lh}=2m\gamma_{31,kj}\,, \\ R_{kjlh}J_{3,}^{\ lh}=2m\gamma_{12,kj}\,, \\ R_{kjlh}J_{1,}^{\ lh}=2m\gamma_{23,kj} \end{cases}$$

where $J_{p,}^{\ ig}=J_{p,f}^{\ g}g^{fi}$. And we have

$$R_{kjih}J_{p,}^{\ ji}=-R_{khji}J_{p,}^{\ ji}\,,$$

from which, transvecting each equation of (6.3) with g^{ji}, we obtain

$$\begin{aligned} R_{kg}&=-m\gamma_{23,kj}J_{1,g}^{\ j}-\gamma_{31,kj}J_{2,g}^{\ j}-\gamma_{12,kj}J_{3,g}^{\ j}\,, \\ R_{kg}&=-\gamma_{23,kj}J_{1,g}^{\ j}-m\gamma_{31,kj}J_{2,g}^{\ j}-\gamma_{12,kj}J_{3,g}^{\ j}\,, \\ R_{kg}&=-\gamma_{23,kj}J_{1,g}^{\ j}-\gamma_{31,kj}J_{2,g}^{\ j}-m\gamma_{12,kj}J_{3,g}^{\ j}\,. \end{aligned}$$

Therefore, we have

$$(6.5)\qquad \begin{cases} \gamma_{23,kj}=R_{ki}J_{1,j}^{\ i}/(m+2)\,, \\ \gamma_{31,kj}=R_{ki}J_{2,j}^{\ i}/(m+2)\,, \\ \gamma_{12,kj}=R_{ki}J_{3,j}^{\ i}/(m+2) \end{cases}$$

(cf., (2.9) and (2.13) in [3]). From (6.4) and (6.5), we obtain

$$R_{lkih}J_{p,}^{\ ih}=SJ_{p,kl}/2(m+2)\,,$$

from which, we get

$$(6.6)\qquad \Lambda^{kjih}R_{lkih}=3SI_l^j/2(m+2)\,.$$

Thus, by virtue of (6.1), (6.6), Theorem 2 and the following Theorem E, we can prove Theorem 5:

THEOREM E ([4]). *In a compact quaternion Kählerian manifold, a vector field X is an infinitesimal Q-transformation if and only if X satisfies*

$$3m\left\{\nabla^h\nabla_h X^j+SX^j/4(m+2)\right\}-\Lambda^{kjih}\nabla_k\nabla_i X_h=0\,.$$

COROLLARY 2. *In a complete quaternion Kählerian manifold with positive scalar curvature, each Q-projective vector field is an infinitesimal Q-projective transformation.*

COROLLARY 3. *If a complete quaternion Kählerian manifold (M, g, V) with positive scalar curvature S admits a non-affine Q-projective vector field, (M, g, V) is isometric to the quaternion projective space of constant Q-sectional curvature $S/4m(m+2)$.*

COROLLARY 4. *If a compact quaternion Kählerian manifold* (M, g, V) *with scalar curvature* S *admits a non-affine* Q*-projective vector field,* S *is positive and* (M, g, V) *is isometric to the quaternion projective space of constant* Q*-sectional curvature* $S/4m(m+2)$.

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Department of Mathematics
Ritsumeikan University
Kyoto, Japan