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Stability of G-unfoldings

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§ 0. Introduction

In [4], R. Thom has presented the problem to study the bifurcation of singularities of G-invariant functions. (Where G is a compact Lie group). In realtion to this problem, G. Wassermann has classified singularities with compact abelian symmetry and their universal G-unfoldings ([6]). But, from the view point of "Catastrophe theory" we must classify stable G-unfoldings instead of universal G-unfoldings.

In this paper, we will prove the equivalence of these notions of G-unfoldings. Once this is proved, the list of universal G-unfoldings in [6] can be exchanged for stable G-unfoldings.

The main result of this paper will be formulated in § 1. Preliminary facts about G-invariant functions and jet bundles are contained in § 2. Proof of the main result will be given in § 3.

All functions and actions of Lie group should be smooth.

§ 1. Formulation of the result

Let G be a compact Lie group which acts linearly on $\boldsymbol{R}^n$. We shall denote $C^\infty(\boldsymbol{R}^n)$ the set of all C^∞-functions over $\boldsymbol{R}^n$; $C_0^\infty(\boldsymbol{R}^n)$ the set of all C^∞-function germs at 0. We shall set $\mathfrak{M}_0^\infty(\boldsymbol{R}^n):=\{f\in C_0^\infty(\boldsymbol{R}^n)|f(0)=0\}$. Then $C_0^\infty(\boldsymbol{R}^n)$ is an $\boldsymbol{R}$-algebra in the usual way, and $\mathfrak{M}_0^\infty(\boldsymbol{R}^n)$ is its unique maximal ideal.

A function $f\in C^\infty(\boldsymbol{R}^n)$ will be said to be *G-invariant* if $f(gx)=f(x)$ for any $x\in\boldsymbol{R}^n$ and $g\in G$. The set of G-invariant functions over $\boldsymbol{R}^n$ will be denoted by $C^G(\boldsymbol{R}^n)$ and the set of all G-invariant function germs at 0 denoted by $C_0^G(\boldsymbol{R}^n)$; it is a subalgebra of $C_0^\infty(\boldsymbol{R}^n)$, and $\mathfrak{M}_0^G(\boldsymbol{R}^n):=C_0^G(\boldsymbol{R}^n)\cap\mathfrak{M}_0^\infty(\boldsymbol{R}^n)$ is its unique maximal ideal.

Let $f:(\boldsymbol{R}^n, a)\dashrightarrow(\boldsymbol{R}, c)$ and $h:(\boldsymbol{R}^n, a')\dashrightarrow(\boldsymbol{R}, c')$ be germs of G-invariant functions at a and a' $(f(a)=c, f(a')=c')$. We shall say f is *G-right equivalent* to h (and we shall write $f\sim_G h$) if there is a equivariant diffeomorphism germ $\phi:(\boldsymbol{R}^n, a)\dashrightarrow(\boldsymbol{R}^n, a')$ such that $f=h\circ\phi+(c-c')$.

DEFNITION 1.1. Let $f\in\mathfrak{M}_0^G(\boldsymbol{R}^n)$. We say f is *strongly k-determined* if for any $h \in \mathfrak{M}_0^G(\boldsymbol{R}^n)$ such that $f-h \in \mathfrak{M}_0^\infty(\boldsymbol{R}^n)^{k+1} \cap G_0^G(\boldsymbol{R}^n)$ we have $f \sim_G h$. We say f is *strongly finitely determined* if f is strongly k-determined for some integer k.

Let f: $(\boldsymbol{R}^n, a) \dashrightarrow (\boldsymbol{R}, c)$ be a G-invariant function germ. An *r-dimensional G-unfolding* of f is a G-invariant function germ F: $(\boldsymbol{R}^n \times \boldsymbol{R}^r, (a, b)) \dashrightarrow (\boldsymbol{R}, c)$ such that $F(x, b) = f(x)$, (where G acts on $\boldsymbol{R}^r$ trivially).

Definition 1.2. Let $f \in \mathfrak{M}_0^G(\boldsymbol{R}^n)$, F be a r-dimensional G-unfolding of f, and H be a s-dimensional G-unfolding of f.

A *G-f-morphism* from H to F is a triple $\Phi = (\phi, \psi, \alpha)$, where ϕ: $(\boldsymbol{R}^n \times \boldsymbol{R}^s, (0, 0)) \dashrightarrow (\boldsymbol{R}^n, 0)$ is a G-equivariant map germ, ψ: $(\boldsymbol{R}^s, 0) \dashrightarrow (\boldsymbol{R}^r, 0)$ is a smooth map germ, and $\alpha \in \mathfrak{M}_0^\infty(\boldsymbol{R}^s)$ satisfying the following conditions:

(i) for $x \in \boldsymbol{R}^n$ we have $\phi(x, 0) = x$

(ii) for $x \in \boldsymbol{R}^n$, $u \in \boldsymbol{R}^s$ we have

$$H(x, u) = F\big(\phi(x, u), \psi(u)\big) + \alpha(u)\,.$$

We shall write $\Phi = (\phi, \psi, \alpha)$: $H \to F$.

The G-f-morphism $\Phi = (\phi, \psi, \alpha)$: $H \to F$ will be called a *G-f-isomorphism* if there is a G-f-morphism $\Phi' = (\phi', \psi', \alpha')$: $F \to H$ such that $\psi^{-1} = \psi'$, $-\alpha = \alpha'$, and $(\phi \times \psi)^{-1} = \phi' \times \psi'$.

Definition 1.3. Let $f \in \mathfrak{M}_0^G(\boldsymbol{R}^n)$, and let F be a G-unfolding of f. We say F is *universal* if for any G-unfolding H of f there exists a G-morphism of F to H.

Definition 1.4. Let f: $(\boldsymbol{R}^n, a) \dashrightarrow (\boldsymbol{R}, c)$ and h : $(\boldsymbol{R}^n, a') \dashrightarrow (\boldsymbol{R}, c')$ be G-invariant function germs. Let F : $(\boldsymbol{R}^n \times \boldsymbol{R}^r, (a, b)) \dashrightarrow (\boldsymbol{R}, c)$ and H: $(\boldsymbol{R}^n \times \boldsymbol{R}^r, (a', b')) \dashrightarrow (\boldsymbol{R}, c')$ be G-unfoldings of f and h respectively. We say F and H are *G-equivalent* if the following hold:

There exist

1) ϕ : $(\boldsymbol{R}^n, a') \dashrightarrow (\boldsymbol{R}^n, a)$: equivariant diffeomorphism germ
2) Φ : $(\boldsymbol{R}^n \times \boldsymbol{R}^r, (a', b')) \dashrightarrow (\boldsymbol{R}^n, a)$: equivariant map germ
3) ψ : $(\boldsymbol{R}^r, b') \dashrightarrow (\boldsymbol{R}^r, b)$: diffeomorphism germ
4) α : $(\boldsymbol{R}^r, b') \dashrightarrow (\boldsymbol{R}, c-c')$: smooth function germ such that
   a) $\Phi(x, b') = \phi(x)$ for $x \in \boldsymbol{R}^n$
   b) $H(x, u) = F(\Phi(x, u), \psi(u)) + \alpha(u)$ for $x \in \boldsymbol{R}^n$ and $u \in \boldsymbol{R}^r$.

Remark: Let f and h be G-invariant function germs which are G-right equivalent. Let F be a G-unfolding of f. Then there exist a G-unfolding H of h such that it is G-equivalent to F.

Definition 1.5. Let $f \in \mathfrak{M}_0^G(\boldsymbol{R}^n)$ and let $F \in C_0^G(\boldsymbol{R}^n \times \boldsymbol{R}^r)$ be a G-unfolding of f. We shall say F is *stable* if for every representative $\tilde{F}$ of F defined on U there is a neighbourhood $N_G(\tilde{F})$ of $\tilde{F}$ in $C^G(U)$ (with the C^∞-topology) such that for every $\tilde{H}\in N_G(\tilde{F})$ there is a point $(x_0, u_0)\in U$ such that $H: (\boldsymbol{R}^n\times\boldsymbol{R}^r, (x_0, u_0))\dashrightarrow(\boldsymbol{R}, \tilde{H}(x_0, u_0))$ is G-equivalent to F as a G-unfolding.

Now we are ready to state the main result of this paper.

THEOREM 1.6. *Let $f\in\mathfrak{M}_0^G(\boldsymbol{R}^n)$ and $F\in C_0^G(\boldsymbol{R}^n\times\boldsymbol{R}^r)$ be a G-unfolding of f. Suppose f is strongly k-determined, then the following statements are equivalent*:

(a) *F is a stable G-unfolding*

(b) *F is an universal G-unfolding.*

§ 2. Preliminaries

In which we recall some preliminary facts about G-invariant functions.

A) *Equivariant vector fields and the Jacobian ideal.*

Let $\Gamma_0^\infty(T\boldsymbol{R}^n)$ be the space of germs of vector field at 0 on $\boldsymbol{R}^n$. A germ $\xi\in\Gamma_0^\infty(T\boldsymbol{R}^n)$ is equivariant if it is equivariant with respect to the induced action on $T\boldsymbol{R}^n$ from the action on $\boldsymbol{R}^n$. Let $\Gamma_0^\infty(T\boldsymbol{R}^n)^G$ be the space of germs of equivariant vector field at 0 on $\boldsymbol{R}^n$.

We define $J(f):=df(\Gamma_0^\infty(T\boldsymbol{R}^n))$ and $J_G(f):=df(\Gamma_0^\infty(T\boldsymbol{R}^n)^G)$ for $f\in C_0^G(\boldsymbol{R}^n)$. It is easy to show these sets are ideals in $C_0^\infty(\boldsymbol{R}^n)$ and $C_0^G(\boldsymbol{R}^n)$ respectively.

The ideal $J(f)$ is called the *Jacobian ideal* of f, and $J_G(f)$ is called the *G-jacobian ideal* of f.

We also define an ideal $\tilde{J}_G(f):=\{df(\xi)|\xi\in\Gamma_0^\infty(T\boldsymbol{R}^n)^G$ *and* $\xi(0)=0\}$, which we call the *reduced G-jacobian ideal* of f.

B) *k-jets.*

Let k be a non-negative integer. We denote by $J^k(\boldsymbol{R}^n, \boldsymbol{R})$ the k-jet bundle over $\boldsymbol{R}^n\times\boldsymbol{R}$. Then we have a canonical decomposition $J^k(\boldsymbol{R}^n, \boldsymbol{R})\cong J^k(n, 1)\times\boldsymbol{R}^n\times\boldsymbol{R}$, where $J^k(n, 1)$ is the set of all k-jets at 0 of elements in $\mathfrak{M}_0^\infty(\boldsymbol{R}^n)$.

Let $\pi_k: \mathfrak{M}_0^\infty(\boldsymbol{R}^n)\to J^k(n, 1)$ be the natural map defined by $\pi_k(f):=j^kf(0)$.

We observe that $J^k(n, 1)$ is a finite dimensional vector space over $\boldsymbol{R}$ and G acts on $J^k(n, 1)$ by

$$g\left(j^kf(0)\right):=j^k(f\circ g^{-1})(0)$$

where $g\in G$ and $f\in\mathfrak{M}_0^\infty(\boldsymbol{R}^n)$.

Since the action of G on $J^k(n, 1)$ is defined by derivative, it is a linear action. Hence, the fixed point set $J_G^k(n, 1)$ of this action is a linear subspace of $J^k(n, 1)$.

Now let $J_G^k(\boldsymbol{R}^n, \boldsymbol{R})$ be the subspace of $J^k(\boldsymbol{R}^n, \boldsymbol{R})$ comprising k-jets of local invariant function, then we have $J_G(n, 1)\times(\boldsymbol{R}^n)^G\times\boldsymbol{R}\subset J_G^k(\boldsymbol{R}^n, \boldsymbol{R})$ via the canonical decomposition of $J^k(\boldsymbol{R}^n, \boldsymbol{R})$, (where $(\boldsymbol{R}^n)^G$ denotes a fixed poin-set of G on $\boldsymbol{R}^n$).

Defined $L_G^k(n) := \{j^k\phi(0)|\phi\colon (\boldsymbol{R}^n, 0)\dashrightarrow(\boldsymbol{R}^n, 0)$: *equivariant map germ, which is non-singular at* $0\}$. Then $L_G^k(n)$ is a Lie group; moreover we define an action of $L_G^k(n)$ on $J_G^k(n, 1)$ by

$$\left(j^k\phi(0)\right)\left(j^kf(0)\right) := j^k(f\circ\phi^{-1})(0)\,.$$

Let $z\in J_G^k(n, 1)$. We denote by $L_G(n)(z)$ the $L_G(n)$-orbit of z.

Remark: Let $f\in\mathfrak{M}_0^G(\boldsymbol{R}^n)$. Suppose f is strongly k-determined. Let $h\colon (\boldsymbol{R}^n, a)\dashrightarrow(\boldsymbol{R}, c)$ be a G-invariant function germ, where $a\in(\boldsymbol{R}^n)^G$. If $j^kh(a)\in L_G(n)(j^kf(0))\times(\boldsymbol{R}^n)^G\times\boldsymbol{R}$, then we have $f\sim_G h$.

We now have the formula for the tangent space at $z := j^kf(0)$ to the orbit $L_G(n)(z)$.

Lemma 2.1. (Beer [1]). *Let* $f\in\mathfrak{M}_0^G(\boldsymbol{R}^n)$ *and* $\pi_k^G := \pi_k|\mathfrak{M}_0^G(\boldsymbol{R}^n)$. *Then we have*

$$T_z\left(L_G(n)(z)\right) = \pi_k^G\left(\tilde{J}_G(f)\right).$$

Corollary 2.2. *Let* $f\in\mathfrak{M}_0^G(\boldsymbol{R}^n)$. *If* f *is strongly* k*-determined then*

$$\mathfrak{M}_0^\infty(\boldsymbol{R}^n)^{k+1}\cap C_0^G(\boldsymbol{R}^n)\subset\tilde{J}_G(f)\,.$$

C) *Infinitesimally universal G-unfoldings.*

Let $f\in\mathfrak{M}_0^G(\boldsymbol{R}^n)$ and let $F\in C_0^G(\boldsymbol{R}^n\times\boldsymbol{R}^r)$ be a G-unfolding of f. Denote the coordinate of $\boldsymbol{R}^r$ by $(u_1, \cdots, u_r)$. We shall say F is *infinitesimally universal* if 1, $\left\{\frac{\partial f}{\partial u_1}|\boldsymbol{R}^n\times 0\right\}, \cdots, \left\{\frac{\partial f}{\partial u_r}|\boldsymbol{R}^n\times 0\right\}$ generate $C_0^G(\boldsymbol{R}^n)/J_G(f)$ as an $\boldsymbol{R}$-vector space.

We now have the following fundamental result for G-unfoldings.

Theorem 2.5. (Beer [1], Poénaru [2]).

Let $f\in\mathfrak{M}_0^G(\boldsymbol{R}^n)$.

(i) *f has universal G-unfoldings if and only if f is strongly finitely determined.*

(ii) *Any two universal G-unfoldings of f of same unfolding dimension are G-f-isomorphic.*

(iii) *If* $F\in C_0^G(\boldsymbol{R}^n\times\boldsymbol{R}^r)$ *is a G-unfolding of f, then F is infinitesimally universal if and only if it is universal.*

As an easy consequence, if f is strongly finitely determined, and $b_1, \cdots,$

$b_s \in \mathfrak{M}_0^G(\boldsymbol{R}^n)$ are representatives of a basis of $\mathfrak{M}_0^G(\boldsymbol{R}^n)/(J_G(f)\cap \mathfrak{M}_0^G(\boldsymbol{R}^n))$, then the s-dimensional G-unfolding

$$H(x, u) := f(x) + u_1 b_1(x) + \cdots + u_s b_s(x)$$

$(x \in \boldsymbol{R}^n,\ (u_1, \cdots, u_s) \in \boldsymbol{R}^s)$ is an universal G-unfolding.

§ 3. Proof of Theorem 1. 6.

We shall say two G-unfoldings are *weakly G-equivalent* if all conditions in Definition 1. 4 hold except that ϕ, Φ are equivariant.

Let $f \in \mathfrak{M}_0^G(\boldsymbol{R}^n)$ and let $F \in C_0^G(\boldsymbol{R}^n \times \boldsymbol{R}^r)$ be a G-unfoldings of f. We say that F is *weakly stable* if for every invariant open neighbourhood U of $0 \in \boldsymbol{R}^n \times \boldsymbol{R}^r$ and every representative $\tilde{F}$ of F defined on U there is a neighbourhood $N_G(\tilde{F})$ of $\tilde{F}$ in $C^G(U)$ (with C^∞-topology) such that

$$H:\ \left(\boldsymbol{R}^n \times \boldsymbol{R}^r, (x_0, u_0)\right) \longrightarrow \left(\boldsymbol{R}, \tilde{H}(x_0, u_0)\right)$$

is weakly G-equivalent to F as a G-unfolding.

We will prove Theorem 1. 6 as the following form.

THEOREM 1. 6′. *Let $f \in \mathfrak{M}_0^G(\boldsymbol{R}^n)$ and $F \in C_0^G(\boldsymbol{R}^n \times \boldsymbol{R}^r)$ be a G-unfolding of f. Suppose f is strongly k-determined, then the following statements are equivalent:*

(a) *F is a stable G-unfolding.*

(b) *F is a weakly stable G-unfolding.*

(c) *F is an universal G-unfolding.*

(d) *F is an infinitesimally universal G-unfolding.*

It is clear that (a) implies (b). By Theorem 2. 5 (iii), (c) and (d) are equivalent.

Now we shall show first that (b) implies (d).

Let s be a non-negative integer. Let

$$j_1^s F:\ \boldsymbol{R}^n \times \boldsymbol{R}^r \longrightarrow J^s(\boldsymbol{R}^n, \boldsymbol{R})$$

be an extension of F defined by $j_1^s F(x, u) := j^s(F_u)(x)$, where $F_u:\ \boldsymbol{R}^n \to \boldsymbol{R}$ is a G-invariant function which is defined by $F_u(x) := F(x, u)$.

Let $O^s(f)$ be the orbit of $j^s f(0)$ defined by the action of invertible jets over $\boldsymbol{R}^n$ (not necessary equivariant jets).

We now have a canonical decompositions:

$$T_z\left(J^s(\boldsymbol{R}^n, \boldsymbol{R})\right) = J^s(\boldsymbol{R}^n, \boldsymbol{R}) = \boldsymbol{R}^n \times P(n, 1)\,,$$

where $z := j^s f(0)$ and $P^s(n, 1)$ denote the set of polynomial functions of degree s. Let $P_G^s(n, 1)$ be the set of G-invariant polynomial functions in $P^s(n, 1)$.

Then we need the following lemma.

LEMMA 3.1. *Let $f\in\mathfrak{M}_0^G(\boldsymbol{R}^n)$ and let $F\in C_0^G(\boldsymbol{R}^n\times\boldsymbol{R}^r)$ an weakly stable G-unfolding of f. For any non-negative integer s, we have*

$$(*)\qquad d(j_1^sF)_{(0,0)}\left(T_{(0,0)}(\boldsymbol{R}^n\times\boldsymbol{R}^r)\right)+T_z\left(O^s(f)\right)\supset\{0\}\times P_G^s(n,1)\,.$$

PROOF. Let $p\in P_G^s(n,1)$. Suppose that $p\notin Image(d(j_1^sF)_{(0,0)})+T_z(O^s(f))$, let V be a neighbourhood of z in $J^s(\boldsymbol{R}^n,\boldsymbol{R})$, let $D\subset Image(d(j_1^sF)_{(0,0)})$ be a complement of $Image(d(j_1^sF)_{(0,0)})\cap T_z(O^s(f))$ and M be a closed submanifold in V which contains $O^s(f)\cap V$ and transverse to $\{p\}\oplus D$, where $\{p\}$ denotes a line through p; M always exists for sufficiently small V.

Let $H(x,u,t)=F_t(x,u):=F(x,u)+tp(x)$, then j_1^sH is transverse to M at $(0,0,0)$.

Then, there exist a neighbourhood U of $0\in\boldsymbol{R}^n\times\boldsymbol{R}^r$ and a positive integer ε such that:

i) *j_1^sH is transversal to M over $M\times(-\varepsilon\ \varepsilon)$*

ii) $\dim(Image(d(j_1^sH)_{(x,u,t)})\geq\dim(Image(d(j_1^sF)_{(0,0)})+1$ for $(x,u,t)\in U\times(-\varepsilon\ \varepsilon)$.

Since F is weakly stable, there exists a positive number ε such that if $t\in(-\varepsilon\ \varepsilon)$, there exists $(x,u)\in U$ such that germ of F_t at (x,u) is weakly G-equivalent to germ of F at $(0,0)$ as G-unfoldings. Hence, germ of $(F_t)_u$ at x is right equivalent (not necessary G-right equivalent) to germ of f at 0; in particular

$$j_1^s(F_t)(x,u)\in O^s(f)\cap V\,.$$

Let $M':=(j_1^sH)^{-1}(M)$, which is a submanifold of $U\times(-\varepsilon\ \varepsilon)$ and let $t_0\in(-\varepsilon\ \varepsilon)$ be a regular value of a restriction to M' of the projection: $U\times(-\varepsilon\ \varepsilon)\to(-\varepsilon\ \varepsilon)$.

Then $j_1^s(F_{t_0})(x,u)\in M$ and

$$\dim\left(Image\left(d(j_1^sH)_{(x,u,t)}\right)=\dim\left(Image\left(d(j_1^sF_{t_0})_{(x,u)}\right)\,;\right.\right.$$

hence

$$\dim\left(Image\left(d(j_1^sF_{t_0})_{(x,u)}\right)\right)\geq\dim\left(Image\left(d(j_1^sF)_{(0,0)}\right)\right)+1\,.$$

This is imposible if a germ of (F_{t_0}) at (x,u) and a germ of F at $(0,0)$ are weakly G-equivalent.

This completes the proof. Q. E. D.

PROOF OF (d) FROM (b). Using the formula for the tangent space ([5], p 41, p 63～p 65), the relation (*) in Lemma 3.1 means the following:

$$df\left(\Gamma_0^\infty(T\boldsymbol{R}^n)\right)+V_F+\mathfrak{M}_0^\infty(\boldsymbol{R}^n)^S\supset C_0^G(\boldsymbol{R}^n)\,,$$

where V_F denote the $\boldsymbol{R}$-vector space generated by

$$1, \frac{\partial F}{\partial u_1}|\boldsymbol{R}^n\times 0, \cdots, \frac{\partial F}{\partial u_r}|\boldsymbol{R}^n\times 0\,.$$

Taking an average over G, we have:

$$J_G(f)+V_F+\mathfrak{M}_0^\infty(\boldsymbol{R}^n)^s\cap C_0^G(\boldsymbol{R}^n)=C_0^G(\boldsymbol{R}^n)\,.$$

Since f is strongly k-determined,

$$J_G(f)\supset\mathfrak{M}_0^\infty(\boldsymbol{R}^n)^{k+1}\cap C_0^G(\boldsymbol{R}^n)\,. \quad \text{(Corollary 2. 2).}$$

Hence, let s be a positive integer with $s\geq k+1$, then

$$J_G(f)+V_F=C_0^G(\boldsymbol{R}^n)\,.$$

This completes the proof. Q. E. D.

For the proof of (a) from (c) and (d), we need the following lemma.

LEMMA 3. 2. *Let $f\in\mathfrak{M}_0^G(\boldsymbol{R}^n)$ and let $F\in C_0^G(\boldsymbol{R}^n\times\boldsymbol{R}^r)$ be a G-unfolding of f. Suppose f is strongly k-determined, then the following statements are equivalent:*

(1) *F is an infinitesimally universal G-unfolding.*

(2) $d(j_1^k F)_{(0,0)}(T_{(0,0)}(\boldsymbol{R}^n\times\boldsymbol{R}^r))+T_z(J_G^k(n,1)^\perp\times L_G^k(n)(z)\times(\boldsymbol{R}^n)^G\times\boldsymbol{R})=T_z(J^k(n,1)\times\boldsymbol{R}^n\times\boldsymbol{R})$.

Where $z:=j^kf(0)$, and $J_G^k(n,1)^\perp$ is the orthogonal complement of $J_G^k(n,1)$ in $J^k(n,1)$, (in certain invariant Riemannian metric).

PROOF. First, we prove that (1) implies (2).

Denote the coordinate of $\boldsymbol{R}^n\times\boldsymbol{R}$ by $(x_1, \cdots, x_n, u_1, \cdots, u_r)$.

Since $T_{(0,0)}(\boldsymbol{R}^n\times\boldsymbol{R}^r)$ is generated by $\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial u_1}, \cdots, \frac{\partial}{\partial u_r}$ over $\boldsymbol{R}$, then $d(j_1^k F)_{(0,0)}(T_{(0,0)}(\boldsymbol{R}^n\times\boldsymbol{R}^r))$ is generated by

$$j^k\left(\frac{\partial f}{\partial x_i}\right)(0)\,, \qquad i=1, \cdots, n\,,$$

$$j^k\left(\frac{\partial F}{\partial u_j}|\boldsymbol{R}^n\times 0\right)(0)\,, \qquad j=1, \cdots, r\,,$$

$$\frac{\partial}{\partial x_i}|_0\,, \qquad i=1, \cdots, n\,,$$

$$\frac{\partial}{\partial u_j}|_0\,, \qquad i=1, \cdots, r$$

over $\boldsymbol{R}$. (See [5], p 63～p 64).

Now, the space $T_z(J_G^k(n,1)^{\perp}\times L_G^k(n)(z)\times(\boldsymbol{R}^n)^G\times\boldsymbol{R})$ is generated by

$$T_z\Big(J_G^k(n,1)^{\perp}\Big),\ T_{(0,0)}\Big((\boldsymbol{R}^n)^G\times\boldsymbol{R}\Big),\quad\text{and}\quad \pi_k^G\Big(\tilde{J}_G(f)\Big)\quad\text{over}\quad \boldsymbol{R}.$$

Since F is infinitesimally universal, we have

$$\pi_k^G\Big(\tilde{J}(f)+V_F+\Big\langle\frac{\partial f}{\partial x_i}\,|\,i=1,\cdots,n\Big\rangle\Big)\supset T_zJ_G^k(n,1)\oplus\boldsymbol{R},$$

where $\Big\langle\frac{\partial f}{\partial x_i}\,|\,i=1,\cdots,n\Big\rangle$ denotes the vector space which is generated by $\frac{\partial f}{\partial x_1},\cdots,\frac{\partial f}{\partial x_n}$ over $\boldsymbol{R}$.

Hence, we have

$$d(j_1^kF)_{(0,0)}\Big(T_{(0,0)}(\boldsymbol{R}^n\times\boldsymbol{R}^r)\Big)+T_z\Big(J_G^k(n,1)^{\perp}\times L_G^k(n)(z)\times(\boldsymbol{R}^n)^G\times\boldsymbol{R}\Big)$$
$$=T_z\Big(J^k(n,1)\times\boldsymbol{R}^n\times\boldsymbol{R}\Big).$$

Using the same method as above, we can also proved the converse.

Q. E. D.

REMARK: i) The condition (2) in Lemma 3. 2 means that the mapping $j_1^kF:\boldsymbol{R}^n\times\boldsymbol{R}^r\to J^k(\boldsymbol{R}^n,\boldsymbol{R})$ is transeverse to the submanifold

$$J_G^k(n,1)^{\perp}\times L_G^k(n)(z)\times(\boldsymbol{R}^n)^G\times\boldsymbol{R}\quad\text{at}\quad(0,0)\in\boldsymbol{R}^n\times\boldsymbol{R}^r.$$

ii) Let $(x_0,u_0)\in(\boldsymbol{R}^n)^G\times\boldsymbol{R}^r$. Since the local situation about (x_0,u_0) as a G-space is same as about $(0,0)$, the assertion of Lemma 3. 2 is still valid for (x_0,u_0).

PROOF OF (a) FROM (c) AND (d). Let U be an invariant neighbourhood of $O\in\boldsymbol{R}^n\times\boldsymbol{R}^r$, and let $\tilde{F}\in C^G(U)$ be a representative of F.

We now define the neighbourhood $N(\tilde{F})$ of $\tilde{F}$ in $C^\infty(U)$ as follows:

$$N(\tilde{F}):=\Big\{\tilde{H}\in C^\infty(U)\Big|$$

There exists $(x_0,u_0)\in U$ *such that*

$$j_1^k\tilde{H}(x_0,u_0)\in J_G^k(n,1)^{\perp}\times L_G^k(n)(z)\times(\boldsymbol{R}^n)^G\times\boldsymbol{R}$$

and

$$d(j_1^k\tilde{H})_{(x_0,u_0)}\Big(T_{(x_0,u_0)}(\boldsymbol{R}^n\times\boldsymbol{R}^r)\Big)+T_w\Big(J_G^k(n,1)^{\perp}\times L_G(n)(z)\times$$
$$\times(\boldsymbol{R}^n)^G\times\boldsymbol{R}\Big)=T_w\Big(J_k(n,1)\times\boldsymbol{R}^n\times\boldsymbol{R}\Big),\quad\textit{where}\quad w:=j_1^k\tilde{H}(x_0,u_0).\Big\}.$$

By the above remark i), $N(\tilde{F})$ is an open neighbourhood of $\tilde{F}$ in $C^\infty(U)$ (with C^∞-topology).

Let $N_G(\tilde{F}) := N(\tilde{F}) \cap C_G(U)$.

If $\tilde{H} \in N_G(\tilde{F})$, then there exists $(x_0, u_0) \in U$ such that

$$j_1^k \tilde{H}(x_0, u_0) \in J_G^k(n, 1)^{\perp} \times L_G(n)(z) \times (\boldsymbol{R}^n)^G \times \boldsymbol{R}.$$

Hence, $(x_0, u_0) \in (\boldsymbol{R}^n)^G \times \boldsymbol{R}$. Since $\tilde{H}$ is G-invariant then

$$j_1^k \tilde{H}(x_0, u_0) \in \{0\} \times L_G^k(n)(z) \times (\boldsymbol{R}^n)^G \times \boldsymbol{R}.$$

Let $h := \tilde{H}_{u_0}$, then $j^k h(0) \in L_G^k(n)(z) \times (\boldsymbol{R}^n)^G \times \boldsymbol{R}$. Since f is strongly k-determined, then we have $f \sim_G h$. Hence, there exists an equivariant diffeomorphism germ

$$\phi : (\boldsymbol{R}^n, 0) \longrightarrow (\boldsymbol{R}^n, x_0)$$

such that

$$f(x) = h \circ \phi(x) - x_0.$$

We now define a G-unfolding

$$H' : \big(\boldsymbol{R}^n \times \boldsymbol{R}^r, (x_0, u_0)\big) \longrightarrow \big(\boldsymbol{R}, h(x_0)\big)$$

by

$$H'(x, u) := \tilde{H}\big(\phi(x), u_0 + u\big) - x_0$$

for $(x, u) \in U$.

Then H' is a G-unfolding of f such that H' and H are G-equivalent. By the above remark ii), H is infinitesimally universal at $(x_0, u_0) \in \boldsymbol{R}^n \times \boldsymbol{R}$. Hence, H' is an infinitesimally universal G-unfolding of f. On the other hand, by the uniqueness of infinitesimally universal G-unfoldings of same unfolding dimension (Theorem 2.5 ii)), H' and F are G-f-isomorphic. Hence, H and F are G-equivalent.

This completes the proof. Q. E. D.

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