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On a parametrix for a weakly hyperbolic operator

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§ 1. Introduction.

In this paper we consider the Cauchy problem in the domain $[0, T]\times R^{n}$ for the weakly hyperbolic partial differential operator

(1.1) $$P(t, x, D_{t}, D_{x})=D_{t}^{2}+2a(t, D_{x})D_{t}+b(t, D_{x})+P_{1}(t, x, D_{t}, D_{x}),$$

where $D_{t}=\frac{1}{i}\frac{\partial}{\partial t}$ and $D_{t}=\left(\frac{1}{i}\frac{\partial}{\partial x_{1}}, \cdots, \frac{1}{i}\frac{\partial}{\partial x_{n}}\right)$.
Here $a(t, D_{t})$ and $b(t, D_{x})$ are respectively the first and the second order homogeneous partial differential operators depending smoothly on t such that, for any $\xi\in R^{n}$, $a^{2}(t, \xi)-b(t, \xi)\geqslant 0$ if $t\geqslant 0$. $P_{1}(t, x, D_{t}, D_{x})$ is an arbitrary first order term with smooth coefficients which are constant for large $|x|$.

Now we impose the following condition for the principal symbol $P_{2}(t, \tau, \xi)=\tau^{2}+2a(t, \xi)\tau+b(t, \xi)$ of P. For any (t, τ, ξ), $(\tau, \xi)\neq 0$,

(1.2) $$grad_{(t,\tau,\xi)}P_{2}\neq 0 .$$

Note that if $P_{2}\neq 0$, $grad_{(t,\tau,\xi)}P_{2}\neq 0$ from the homogeneity of P_{2}. Examples of such P_{2} are $D_{t}^{2}-t\Delta$, $D_{t}^{2}-t\Delta_{x'}-\Delta_{x''}$ etc. Here Δ is the Laplacian. Furthermore $x=(x', x'')$ and $\Delta_{x'}$, $\Delta_{x''}$ are the corresponding Laplacians.

In the following we shall discuss the Cauchy problem

(1.3) $$\begin{cases} P(t, x, D_{t}, D_{x})u=f & \text{in } [0, T]\times R^{n}, \\ D_{t}^{j}u(0, x)=v_{j}(x),\ j=0,1, & \text{in } R^{n}, \end{cases}$$

for given f and v_{j}. The correctness of (1.3) can be shown by the energy estimate if we reduce P to a simple form by the change of variable of § 2 (see [11]). On the other hand Ivrii discussed the correctness of the Cauchy problem of a weakly hyperbolic operator whose principal symbol has smooth coefficients depending on (t, x) and does not have critical points with respect to (t, x, τ, ξ) ([7]). He called these operators completely regularly hyperbolic. By the energy estimate, he proved that the Cauchy problem for such an operator is correct, regardless of its lower order terms. The regularity property of its solution does not depend on the lower order terms.

In this paper we shall construct a parametrix of the Cauchy problem

(1. 3). However due to difficulty of the estimate of amplitude functions, we discuss only the case that the coefficients of the principal symbol are constant with respect to x. The construction of a phase function in general case will be given in § 5 as appendix. From the properties of this parametrix we can show the propagations of the singularities along null bicharacteristc strips of P_2 (see Lemma 3. 3).

In [1] Alinhac constructed a parametrix of the operator

$$D_t^2 - t^2 D_x^2 - D_y^2 + \text{lower order terms}. \tag{1.4}$$

For the construction of the parametrix, he used the solutions of the ordinary differential equation $u'' - t^2 u + Au = 0$ where A is some constant. Their asymptotic behaviors at infinty was crucial. On the other hand in [5] we treated an operator of the following form;

$$D_t^2 - tA(t, x, D_x) + \text{lower order terms}, \tag{1.5}$$

where $A(t, x, D_x)$ is an elliptic second order operator with a positive symbol. In this case for the construction of parametrices, the Airy function, which is a solution of the ordinary differential equation $u'' - tu = 0$, plays an important role. The properties of the Airy function are well known ([2], [10]). They were used in the diffractive boundary value problems for the strictly hyperbolic operators ([3], [6], [8], [12]). In this paper we also construct the following parametrices for (1. 3) using the Airy function.

THEOREM 1. 1. *There exist operators $G^k(t, s)$, $k = 0, 1$, with parameters $0 \leqslant s, t \leqslant T$ such that*

$$\left(G^k(t, s)\, V\right)(x) \in C^\infty\left([0, T] \times [0, T] \times R^n\right) \quad \textit{for } V \in C_0^\infty(R^n)$$

$$\left(G^k(t, s)\, V\right)(x) \in C^\infty\left([0, T] \times [0, T];\, \mathscr{D}'(R^n)\right) \quad \textit{for } V \in \mathscr{E}'(R^n)$$

and

$$\begin{cases} P(t, x, D_t, D_x)\left(G^k(t, s)\, V\right) = R^k(t, s) V, \\ G^k(t, s)\, V\big|_{t=s} = \left(\delta_0^k I + R_0^k(s)\right) V, \\ D_t G^k(t, s)\, V\big|_{t=s} = \left(\delta_1^k I + R_1^k(s)\right) V, \end{cases} \tag{1.6}$$

where δ_j^k is the Kronecker's delta, $R^k(t, s)$ and $R_j^k(s)$ are operators with C^∞-kernels of (x, y) depending smoothly on (t, s) and s respectively.

This theorem will be proved in § 2 and § 3.

Let

$$\mathscr{H}_q\Big([0, T]\times R^n\Big)=\Big\{u\,;\ D_t^k u(t, \cdot)\in L^2\Big([0, T]\,;\ H_{q-k}(R^n)\Big),\ k\leqslant q\Big\}\,,$$

where q is some positive real number and $H_q(R^n)$ is the Sobolev space. From Theorem 1. 1 we have the following

THEOREM 1. 2. *Let* $f\in\mathscr{H}_q([0, T]\times R^n)$, $v_0\in H_{q+\frac{4}{3}}(R^n)$ and $v_1\in H_{q+\frac{1}{3}}(R^n)$ *be functions such that for some compact set* K *in* R^n, *supp* $f\subset[0, T]\times K$ *and supp* $v_0\cup$ *supp* $v_1\subset K$. *Then there exists a unique solution of* (1. 3) *which satisfies the estimate*

$$(1.7)\qquad \begin{aligned}\sum_{k\leqslant q+1}\Big\|D_t^k u(t, \cdot)\Big\|_{q+1-k} &\leqslant C\|v_0\|_{q+\frac{4}{3}}+C\|v_1\|_{q+\frac{1}{3}}\\ &+C\sum_{k\leqslant q-1}\Big\|D_t^k f(0, \cdot)\Big\|_{q-k}+C\sum_{k\leqslant q}\int_0^t\Big\|D_t^k f(s, \cdot)\Big\|_{q-k}s^{-\frac{1}{2}}ds\,.\end{aligned}$$

Here q *is an arbitrary positive number,* $\|\cdot\|_q$ *is the norm of* $H_q(R^n)$ *and* C *depends only on* K *and* q.

In Theorem 1. 2 of [5] we assumed that $v_0\in H_{q+1}$ and the first term of the right hand side of (1. 7) is replaced by $\|v_0\|_{q+1}$. Such difference of the estimate arises from the fact that the characteristic roots of (1. 5) degenerate at $t=0$, while, those of (1. 1) do not in general. Finally we remark that in [7] it is assumed that $v_0\in H_{q+2}$, $v_1\in H_{q+1}$ and $\|v_0\|_{q+\frac{4}{3}}$, $\|v_1\|_{q+\frac{1}{3}}$ are replaced by $\|v_0\|_{q+2}$, $\|v_1\|_{q+1}$ respectively in (1. 7).

Theorem 1. 2 will be proved in § 4.

§ 2. Phase functions and amplitude functions.

We shall reduce the operator (1. 1) to a more conveient forms. In doing so we examine the condition (1. 2) more precisely. At a double characteristic point we see that $a^2(t, \xi)-b(t, \xi)=0$ and $\tau=-a(t, \xi)$. Thus we have the equations

$$\begin{cases}\partial P_2/\partial t=2\tau a_t+b_t=-2aa_t+b_t=\partial(-a^2+b)/\partial t\,,\\ \partial P_2/\partial\tau=2\tau+2a=0\\ \nabla_\xi P_2=2\tau\nabla_\xi a+\nabla_\xi b=-2a\nabla_\xi a+\nabla_\xi b=\nabla_\xi(-a^2+b)\,.\end{cases}$$

If $\nabla_\xi(-a^2+b)\neq 0$, the function $-a^2+b$ changes the sign at the duoble characteristic point. This contradicts the hyperbolicity of P and hence $\nabla_\xi(-a^2+b)=0$. Therefore the condition (1. 2) means that $\partial(-a^2+b)/\partial t\neq 0$.

Now let $\alpha(t)$ be a vector with components $(\alpha_1(t), \cdots, \alpha_n(t))$ such that $\alpha_i(t)=\int_0^t a_i(\sigma)\,d\sigma$, where $a_i(t)$ is the coefficient of $a(t, D_x)=\sum_{i=1}^n a_i(t)\,D_{x_i}$. We define the operator U for $\phi\in C^\infty([0, T]\times R^n)$ as follows;

$$U\phi(t, x) = \phi\big(t, x-\alpha(t)\big),$$

and denote by U^{-1} its inverse operator. Then we have that

$$U^{-1}\big(D_t + a(t, D_x)\big)\, U = D_t\,.$$

Noting that $P_2(t, \tau, \xi) = (\tau + a(t, \xi))^2 - (a^2(t, \xi) - b(t, \xi))$ we reduce P to the form

$$D_t^2 - \big(a^2(t, D_x) - b(t, D_x)\big) + P_1'(t, x, D_t, D_x)\,, \tag{2.1}$$

where $P_1'(t, x, D_t, D_x)$ is some first order term.

Hence in the following we assume that the principal symbol of P satisfies the conditions;

$$P_2(t, \tau, \xi) = \tau^2 - A(t, \xi)\,, \tag{2.2}$$

where $A(t, \xi) > 0$ if $t > 0$ and

$$A_t(t, \xi) \neq 0 \quad \text{if} \quad A(t, \xi) = 0\,. \tag{2.3}$$

Now we solve the eikonal equation $P_2(t, \varphi_t, \varphi_x) = 0$. In a conic neighborhood of a simple root this equation can be solved easily. Thus we shall consider only in a conic neighborhood of the double characteristic point. In general it is difficult to solve at the double root (see § 5). But in this case we find a solution by integration with respect to t since the principal symbol does not depend on x.

Let $A(0, \xi_0) = 0$ for some $\xi_0 \in R^n$. Since $A_t(0, \xi_0) \neq 0$, by the implicit function theorem, there exists a conic neighborhood $\Gamma \subset R^n \backslash 0$ of ξ_0 and in Γ, P_2 is written as follows;

$$P_2(t, \tau, \xi) = \tau^2 - \big(t + \zeta(\xi)\big)\, A'(t, \xi)\,, \tag{2.4}$$

where $\zeta(\xi)$ is a real non-negative function of homogeneous degree 0, $\zeta(\xi_0) = 0$, and $A'(t, \xi)$ is a positive elliptic symbol of homogeneous degree 2. For example if $P_2 = \tau^2 - t|\xi'|^2 - |\xi''|^2$, $\Gamma = \{\xi\,;\ |\xi''| \leqslant c|\xi'|\}$ and $\zeta(\xi) = |\xi''|^2/|\xi'|^2$.

We remark that the characteristic roots $\pm\sqrt{(t+\zeta)\, A'}$ have singularity on a surface $t + \zeta(\xi) = 0$. Nevertheless the null-bicharacteristic strips of P_2 have no singular points in hyperbolic region, and they are tangent to the surface $t = -\zeta(\xi)$ of the first order of contact there.

Now we state the following

LEMMA 2.1. *There exists a positive number T and a real positive C^∞-function $\rho(t, \xi)$ of homogeneous degree $\frac{2}{3}$ defined in $[0, T] \times \Gamma$ such that $\varphi_\pm(t, x, \xi) = \langle x, \xi \rangle \pm \frac{2}{3} \rho^{\frac{3}{2}}(t, \xi)$ satisfies the eikonal equation*

(2. 5) $$P_2(t, \varphi_{\pm t}, \varphi_{\pm x}) = 0\,.$$

Moreover

(2. 6) $$\rho(t, \xi) = 0\Big(t+\zeta(\xi)\Big)$$

and

(2. 7) $$\rho_t(0, \xi) \gtrneqq 0\,.$$

PROOF. Let $d'(\sigma, \xi)=\sqrt{A'}(\sigma, \xi)$. Integrating the characteristic root with respect to σ we obtain

(2. 8) $$\lambda(t, \xi) = \int_{-\zeta(\xi)}^{t} \sqrt{\sigma+\zeta(\xi)}\, d'(\sigma, \xi)\, d\sigma\,.$$

Then $\varphi_\pm(t, x, \xi)=\langle x, \xi\rangle \pm \lambda(t, \xi)$ solves (2. 5). Now put $\rho(t, \xi)=\left(\frac{3}{2}\lambda\right)^{\frac{2}{3}}(t, \xi)$. By a change of variable we see that

$$\lambda(t, \xi) = \int_0^{t+\zeta} \sqrt{\sigma}\, d'(\sigma-\zeta, \xi)\, d\sigma = 0\left((t+\zeta)^{\frac{3}{2}}\right).$$

This proves the lemma since $d'(0, \xi_0) \neq 0$.

Let $x \in R^n$ and U some open neighborhood of x. Let us construct the amplitude functions $g(t, x, \xi)$ and $h(t, x, \xi)$ in $[0, T]\times U \times \Gamma$ such that;

(2. 9) $$P(t, x, D_t, D_x)\int_{C_\pm} \Big(g(t, x, \xi) - zh(t, x, \xi)\Big)\, e^{i(\frac{z^2}{3}-z\rho+\langle x, \xi\rangle)}\, dz = 0(|\xi|^{-\infty})$$

where $C_\pm$ is the complex contour;

$$C_\pm = \begin{cases} |t| e^{(\frac{\pi}{2}\pm\frac{\pi}{3})i} & \text{for } t\to\pm\infty\,, \\ |t| e^{-\frac{\pi}{2}i} & \text{for } t\to\mp\infty\,. \end{cases}$$

Moreover

(2. 10) $$\begin{cases} g(t, x, \xi) \sim \sum_{j=0} g_{-j}(t, x, \xi), \ \underset{\xi}{ord}\, g_{-j} = -j,\ g|_{t=0} \neq 0\,, \\ h(t, x, \xi) \sim \sum_{j=0} h_{-j}(t, x, \xi), \ \underset{\xi}{ord}\, h_{-j} = -\dfrac{1}{3} - j\,. \end{cases}$$

We constructed these amplitude functions in [5]. However, for the sake of completeness we shall review the process in the following (see § 2 of [5] and also § 3, § 4 of [3]).

By taking $a(t, x, \xi, z)=g(t, x, \xi)-zh(t, x, \xi)$ the equation (2. 9) follows from

(2. 11) $$e^{-i(\frac{z^3}{3}-z\rho+\langle x, \xi\rangle)}\, P\Big\{a(t, x, \xi, z)\, e^{i(\frac{z^3}{3}-z\rho+\langle x, \xi\rangle)}\Big\} = 0(|\xi|^{-\infty})$$

Now let $B(t, x, \xi, z)$ be a polynomial in z and smooth with respect to

(t, x, ξ) for $\xi \neq 0$. We shall call B of homogeneous degree m with respect to (ξ, z) if

$$B(t, x, k\xi, k^{\frac{1}{3}} z) = k^m B(t, x, \xi, z) \qquad \text{for} \quad k>0 .$$

In the sense of this homogeneity we expand the left hand side of (2.11) asymptoticaly. By usual calculation we obtain that

$$\begin{aligned} (2.12) \qquad & e^{-i(\frac{z^3}{3}-z\rho+\langle x,\xi\rangle)} P\Big\{a(t, x, \xi, z)\, e^{i(\frac{z^3}{3}-z\rho+\langle x,\xi\rangle)}\Big\} \\ & = P_2(t, -z\rho_t, \xi)\, a_0 + \Big\{-\frac{1}{i} 2z\rho_t \frac{\partial a_0}{\partial t} - \frac{1}{i}\sum_{k=1}^{n} \frac{\partial A}{\partial \xi_k}(t, \xi) \frac{\partial a_0}{\partial x_k} \\ & \quad -\frac{1}{i} z\rho_{tt} a_0 + P_1(t, x, -z\rho_t, \xi)\, a_0 + P_2(t, -z\rho_t, \xi)\, a_{-1}\Big\} \\ & \quad + \cdots , \end{aligned}$$

where $a_{-j} = g_{-j} - zh_{-j}$ and $P_1(t, x, \tau, \xi)$ is the principal symbol of $P_1(t, x, D_t, D_x)$.

Let $B(t, x, \xi, z)$ be homogeneous of degree m with respect to (ξ, z) and

$$(2.13) \qquad B(t, x, \xi, \pm\sqrt{\rho}) = 0 .$$

Then there exists a polynomial $B_1(t,x,\xi,z)$ and $B(t,x,\xi,z) = i(z^2-\rho)\, B_1(t,x,\xi,z)$. Remarking $i(z^2-\rho) = \dfrac{\partial}{\partial z}\Big(i\Big(\dfrac{z^3}{3} - z\rho\Big)\Big)$ we obtain by integration by parts that

$$\int_{C_\pm} i(z^2-\rho)\, B_1 e^{i(\frac{z^3}{3}-z\rho)} dz = -\int_{C_\pm} \frac{\partial B_1}{\partial z} e^{i(\frac{z^3}{3}-z\rho)} dz ,$$

where the order of $\partial B_1/\partial z$ is $m-1$.

Note that $P_2(t, -z\rho_t, \xi) = z^2\rho_t^2 - A(t, \xi) = (z^2-\rho)\,\rho_t^2$, since $A(t, \xi) = \rho\rho_t^2$ from (2.5). Thus from the above argument we can eliminate the second order term of (2.12). The first order term becomes

$$\begin{aligned} (2.14) \qquad & -\frac{2}{i} z\rho_t \frac{\partial a_0}{\partial t} - \frac{1}{i}\sum_{k=1}^{n} \frac{\partial A}{\partial \xi_k} \frac{\partial a_0}{\partial x_k} - \frac{1}{i} z\rho_{tt} a_0 + P_1(t, x, -z\rho_t, \xi)\, a_0 \\ & -\frac{\partial}{\partial z}\Big(\frac{1}{i}\rho_t^2 a_0\Big) . \end{aligned}$$

Substituting $\pm\sqrt{\rho}$ for z in (2.14), we obtain from (2.13) the transport equation;

$$\begin{aligned} (2.15)^{\pm} \qquad & \pm\frac{2}{i}\sqrt{\rho}\,\rho_t\Big(\frac{\partial g_0}{\partial t} \pm\sqrt{\rho}\frac{\partial h_0}{\partial t}\Big) - \frac{1}{i}\sum_{k=1}^{n} \frac{\partial A}{\partial \xi_k}\Big(\frac{\partial g_0}{\partial x_k} \pm\sqrt{\rho}\frac{\partial h_0}{\partial x_k}\Big) \\ & + \frac{1}{i}\Big(\pm\sqrt{\rho}\,\rho_{tt} + iP_1(t, x, \pm\sqrt{\rho}\,\rho_t, \xi)\Big)(g_0 \pm \sqrt{\rho}\, h_0) + \frac{1}{i}\rho_t^2 h_0 = 0 . \end{aligned}$$

Now let $a_0^{\pm} = g_0 \pm \sqrt{\rho}\, h_0$. Since $\dfrac{\partial g_0}{\partial t} \pm\sqrt{\rho}\dfrac{\partial h_0}{\partial t} = \dfrac{\partial a_0^{\pm}}{\partial t} \mp \dfrac{1}{2\sqrt{\rho}}\rho_t h_0$, the equatioon (2.14) is equivalent to

$$(2.16)^{\pm} \qquad \pm 2\sqrt{\rho}\,\rho_t \frac{\partial a_0^{\pm}}{\partial t} + X a_0^{\pm} + c_{\pm} a_0^{\pm} = 0\,,$$

where

$$X = -\sum_{k=1}^{n} \frac{\partial A}{\partial \xi_k}(t, \xi) \frac{\partial}{\partial x_k}$$

and

$$c_{\pm} = \pm\sqrt{\rho}\,\rho_{tt} + iP_1(t, x, \pm\sqrt{\rho}\,\rho_t, \xi)\,.$$

Note that the sign $\pm$ is taken according to $\pm\sqrt{\rho}$, but is independent of the contour $C_{\pm}$. Succesively we obtain similar equations for $j \geqslant 0$;

$$(2.16)_j^{\pm} \qquad \pm 2\sqrt{\rho}\,\rho_t \frac{\partial a_{-j}^{\pm}}{\partial t} + X a_{-j}^{\pm} + c_{\pm} a_{-j}^{\pm} = f_{-j}^{\pm}\,,$$

where $f_{-j}^{\pm}$ is determined by $a_0^{\pm}, \cdots, a_{-j+1}^{\pm}$.

In order to solve the equation $(2.16)_j^{\pm}$ we must eliminate $\sqrt{\rho}$ in the coefficient of $\partial/\partial t$. Since $\rho_t \neq 0$ we may regard ρ as independent variable. Thus we have $\pm 2\sqrt{\rho}\,\rho_t \frac{\partial a_{-j}^{\pm}}{\partial t} = \pm 2\sqrt{\rho}\,\rho_t^2 \frac{\partial a_{-j}^{\pm}}{\partial \rho}$. Making a change of variables $\rho = \sigma^2$, $x = x$, we may rewrite $(2.16)_j^{\pm}$ in the form;

$$(2.17)_j^{\pm} \qquad \pm \rho_t^2 \frac{\partial \hat{a}_{-j}^{\pm}}{\partial \sigma} + X \hat{a}_{-j}^{\pm} + c_{\pm} \hat{a}_{-j}^{\pm} = \hat{f}_{-j}^{\pm}\,.$$

For $\rho_t^2 \neq 0$, $(2.17)_j^{\pm}$ is solved and let $\hat{a}_{-j}^{+}$ be a solution of $(2.17)_j^{+}$. Decompose $\hat{a}_{-j}^{+}$ into odd ande ven functions

$$\hat{a}_{-j}^{+}(x, \xi, \sigma) = \hat{g}_{-j}(x, \xi, \sigma^2) + \sigma \hat{h}_{-j}(x, \xi, \sigma^2)\,.$$

Note that if we replace σ by $-\sigma$ in $\hat{a}_{-j}^{+}$, $\hat{a}_{-j}^{+}(x, \xi, -\sigma)$ is a solution of $(2.17)_j^{-}$. Thus we obtain $g(t, x, \xi)$ and $h(t, x, \xi)$ as desired, if we give initial conditions $\hat{a}_0^{+}(x, \xi, 0) = 1$, $\hat{a}_{-j}^{+}(x, \xi, 0) = 0$ $(j > 0)$ and put $g_{-j}(t, x, \xi) = \hat{g}_{-j}(x, \xi, \sigma^2)$, $h_{-j}(t, x, \xi) = \hat{h}_{-j}(x, \xi, \sigma^2)$.

Now let $A_{\pm}(x)$ be the integral

$$(2.18) \qquad A_{\pm}(x) = \int_{C_{\pm}} \exp\left(i\left(\frac{z^3}{3} - zx\right)\right) dz\,.$$

Then the integral of the left hand side of (2.9) becomes in the form;

$$(2.19) \qquad \left\{g(t, x, \xi)\, A_{\pm}\big(\rho(t, \xi)\big) - ih(t, x, \xi)\, A'_{\pm}\big(\rho(t, \xi)\big)\right\} e^{i\langle x, \xi\rangle}\,,$$

where $A'_{\pm}$ is the derivative of $A_{\pm}$.

For the usual Airy function $Ai(x)$, it holds that

$$A_{\pm}(x)=2\pi e^{\pm\frac{2}{3}\pi i}Ai\left(e^{\pm\frac{2}{3}\pi i}(-x)\right). \tag{2.20}$$

Since $Ai(x)=xAi''(x)$, $A_{\pm}(x)$ solves the ordinary differential equation $y''=-xy$.

Furthermore it is well known that the Airy function has the asymptotic expansion;

$$Ai(x)\sim\exp\left(-\frac{2}{3}x^{\frac{3}{2}}\right)\left(\sum_{\nu=0}^{\infty}a_{\nu}x^{-\frac{1}{4}-\frac{3}{2}\nu}\right) \quad \text{for} \quad x\in C,\ |x|\geqq\varepsilon, \tag{2.21}$$
$$\text{and} \quad -\pi+\varepsilon\leqq\arg x\leqq\pi-\varepsilon,$$

where $a_0\neq 0$ and ε is a small positive number (sse [7], [10]). Thus for real $x\geqq\varepsilon$ we may represent $A_{\pm}(x)$ as follows;

$$A_{\pm}(x)=\Phi_{\pm}(x)\exp\left(\pm i\frac{2}{3}x^{\frac{3}{2}}\right), \tag{2.22}$$

where

$$\Phi_{\pm}(x)\sim\sum_{\nu=0}^{\infty}b_{\pm,\nu}x^{-\frac{1}{4}-\frac{3}{2}\nu} \quad \text{with} \quad b_{\pm,0}\neq 0. \tag{2.23}$$

By the termwise differentiation we see that for real $x\geqq\varepsilon$,

$$A'_{\pm}(x)=\tilde{\Phi}_{\pm}(x)\exp\left(\pm i\frac{2}{3}x^{\frac{3}{2}}\right), \tag{2.22$'$}$$

where

$$\tilde{\Phi}_{\pm}(x)\sim\sum_{\nu=0}^{\infty}\tilde{b}_{\pm,\nu}x^{\frac{1}{4}-\frac{3}{2}\nu} \quad \text{with} \quad \tilde{b}_{\pm,0}\neq 0. \tag{2.23$'$}$$

Finally we remark that $A_{\pm}(x)\neq 0$ and $A'_{\pm}(x)\neq 0$ for all $x\in R$.

§ 3. Construction of the parametrices for the Cauchy problem.

It is well known that in a conic neighborhood of the simple root the parametrices (1. 6) are constructed by the usual method. Therefore we shall consider only in a conic neighborhood of the double characteristic point $t=0$ and $\xi=\xi_0$.

Let $U\times\Gamma$ is the open conic set mentioned in the previous section. Extend $\rho(t,\xi)$ smoothly to $[0,T]\times R^n\backslash 0$ preserving the homogeneity. Now we define the operator $G_{\pm}(t,s)$ for $V\in C_0^{\infty}(R^n)$ such that

$$(3.1)\qquad \left(G_{\pm}(t,s)\,V\right)(x)=(2\pi)^{-n}\int e^{i\langle x-y,\xi\rangle}\left(g(t,x,\xi)\frac{A_{\pm}(\rho(t))}{A'_{\pm}(\rho(s))}\right.$$
$$\left.-ih(t,x,\xi)\frac{A'_{\pm}(\rho(t))}{A'_{\pm}(\rho(s))}\right)V(y)\,dyd\xi\,,$$

where $\rho(t)=\rho(t,\xi)$ and $\rho(s)=\rho(s,\xi)$.

Let A and B are operators from $\mathcal{E}'(R^n)$ to $\mathcal{D}'(R^n)$. We denote $A\equiv B$ in $U\times\Gamma$ if

$$(3.2)\qquad WF'(A-B)\cap(U\times\Gamma)\times(U\times\Gamma)=\phi\,,$$

where WF' is the wave front set of the distribution kernel of the operator. From (2.9)

$$(3.3)\qquad P(t,x,D_t,D_x)\,G_{\pm}(t,s)\equiv 0 \quad \text{in} \quad U\times\Gamma\,,$$

where we regard (t,s) as parameters.

To examine the continuity of $G_{\pm}(t,s)$ we state some properties of its symbols. We denote by $S^m_{\frac{1}{3},0,\frac{2}{3}}([0,T]\times R^n_x\times R^n_\xi)$ the set of all $a(t,x,\xi)\in C^\infty([0,T]\times R^n_x\times R^n_\xi\backslash 0)$ such that for all multi-indices α, β, γ the estimate

$$(3.4)\qquad \left|\partial^\alpha_\xi\partial^\beta_x\partial^\gamma_t a(t,x,\xi)\right|\leqslant \underset{\alpha,\beta,\gamma}{C}\,|\xi|^{m-\frac{1}{3}|\alpha|+\frac{2}{3}\gamma}$$

is valid for $|\xi|\geqslant 1$ and some constant $\underset{\alpha,\beta,\gamma}{C}$.

The following lemma is fundamental.

LEMMA 3.1. *Let $r(\theta)$ be a function in $C^\infty(R)$. Assume that if $\theta\geqslant 1$, $r(\theta)\in S^m_{1,0}(R)$ for some positive number m. Then $r(\rho(t))\in S^{m\frac{2}{3}}_{\frac{1}{3},0,\frac{2}{3}}([0,T]\times R^n_x\times R^n_\xi)$.*

PROOF. When $\rho\geqslant 1$ we have that

$$\begin{aligned}
\left|\partial^\alpha_\xi\partial^\gamma_t r(\rho)\right| &= \left|\sum_{\gamma_1+\cdots+\gamma_\nu=\gamma}\frac{\gamma!}{\nu!\,\gamma_1!\cdots\gamma_\nu!}\partial^\alpha_\xi\left(r^{(\nu)}(\rho)\,\partial^{\gamma_1}_t\rho\cdots\partial^{\gamma_\nu}_t\rho\right)\right| \\
&= \left|\sum_{\alpha^1+\alpha^2=\alpha}\binom{\alpha}{\alpha^1}\left(\sum_{\substack{\gamma_1+\cdots+\gamma_\nu=\gamma\\ \alpha^1_1+\cdots+\alpha^1_\mu=\alpha^1}}\frac{\gamma!\,\alpha^1!}{\nu!\,\mu!\,\gamma_1!\cdots\gamma_\nu!\,\alpha^1_1!\cdots\alpha^1_\mu!}r^{(\nu+\mu)}(\rho)\,\partial^{\alpha^1_1}_\xi\rho\cdots\partial^{\alpha^1_\mu}_\xi\rho\right.\right. \\
&\qquad\left.\times\,\partial^{\alpha^2}_\xi(\partial^{\gamma_1}_t\rho\cdots\partial^{\gamma_\nu}_t\rho)\right| \\
&\leqslant C\sum_{\substack{\nu\leqslant\gamma\\ \mu\leqslant|\alpha^1|}}\rho^{m-\nu-\mu}|\xi|^{\frac{2}{3}\mu-|\alpha^1|}|\xi|^{\frac{2}{3}\nu-|\alpha^2|} \\
&\leqslant C\sum_{\mu\leqslant|\alpha|}\rho^m|\xi|^{\frac{2}{3}\mu-|\alpha|}|\xi|^{\frac{2}{3}\gamma} \\
&\leqslant C|\xi|^{\frac{2}{3}m-\frac{1}{3}|\alpha|+\frac{2}{3}\gamma}\,.
\end{aligned}$$

When $\rho\leqslant 2$, by the boundedness of $r^{(\nu)}(\rho)$ and the above computations the assertion follows easily.

Let $\chi(\sigma)$ be a function in $C^\infty(R)$ such that $\chi(\sigma)\equiv 1$ for $\sigma\leqslant 1$, $\chi(\sigma)\equiv 0$ for $\sigma\geqslant 2$. Set $\chi_1(\sigma)=\chi(\sigma)$ and $\chi_2(\sigma)=1-\chi(\sigma)$. Now apply the above lemma to the Airy function $A_\pm(x)$. Then we have

LEMMA 3.2.

$$\left.\begin{array}{l} A_\pm(\rho(t))\,\chi_2(\rho(t)) \\ (1/A'_\pm(\rho(t)))\,\chi_1\rho(t)) \end{array}\right\} \in S^0_{\frac{1}{3},0,\frac{2}{3}}\left([0,T]\times R^n_x\times R^n_\xi\right) \tag{3.5}$$

and

$$\left.\begin{array}{l} \Phi_\pm(\rho(t))\,\chi_2(\rho(t)) \\ (1/\tilde{\Phi}_\pm(\rho(t)))\,\chi_2(\rho(t)) \end{array}\right\} \in S^0_{\frac{1}{3},0,\frac{2}{3}}\left([0,T]\times R^n_x\times R^n_\xi\right). \tag{3.6}$$

PROOF. Since $\Phi_\pm(\theta)\,\chi_2(\theta)$, $1/\tilde{\Phi}_\pm(\theta)\,\chi_2(\theta)\in S^{-\frac{1}{4}}_{1,0}(R)\subset S^0_{1,0}(R)$ by (2.23), (2.23)′, (3.6) is valid from Lemma 3.1. (3.5) is also valid, since $\chi_1(\theta)\in S^0_{1,0}(R)$ and vanishes for $\theta\geqslant 2$.

We divide (3.1) into four parts as follows;

$$\begin{aligned} &\left(G_\pm(t,s)\,V\right)(x) \\ &= \sum_{\mu,\nu=1,2}(2\pi)^{-n}\int e^{i\langle x-y,\xi\rangle}\left(g\frac{A_\pm(\rho(t))}{A'_\pm(\rho(s))}-ih\frac{A'_\pm(\rho(t))}{A'_\pm(\rho(s))}\right) \\ &\qquad \chi_\mu\left(\rho(t)\right)\chi_\nu\left(\rho(s)\right)V(y)\,dyd\xi \\ &= \sum_{\mu,\nu=1,2}\left(G_{\pm,\mu\nu}(t,s)\,V\right)(x). \end{aligned} \tag{3.7}$$

We denote by $S^m_{\frac{1}{3},0,\frac{2}{3}}([0,T]\times[0,T]\times R^n_x\times R^n_\xi)$ the set of all $a(t,s,x,\xi)\in C^\infty([0,T]\times[0,T]\times R^n_x\times R^n_\xi\backslash 0)$ such that for all multi-indices α, β, γ_1, γ_2 the estimate

$$\left|\partial^\alpha_\xi\partial^\beta_x\partial^{\gamma_1}_t\partial^{\gamma_2}_s a(t,s,x,\xi)\right|\leqslant C_{\alpha,\beta,\gamma_1,\gamma_2}|\xi|^{m-\frac{1}{3}|\alpha|+\frac{2}{3}(\gamma_1+\gamma_2)} \tag{3.8}$$

is valid for $|\xi|\geqslant 1$ and some constant $C_{\alpha,\beta,\gamma_1,\gamma_2}$.

Then from Lemma 3.2 and (3.7) we regard (3.1) as the sum of integral operators with amplitudes contained in $S^0_{\frac{1}{3},0,\frac{2}{3}}([0,T]\times[0,T]\times R^n_x\times R^n_\xi)$ and phase functions $\langle x-y,\xi\rangle\pm\left(\frac{2}{3}\rho^{\frac{3}{2}}(t)\,\delta^2_\mu-\frac{2}{3}\rho^{\frac{3}{2}}(s)\,\delta^2_\nu\right)$ respectively.
Hence by the oscillatory integral method (see [4]), we conclude that

$$\left(G_\pm(t,s)\,V\right)(x)\in C^\infty\left([0,T]\times[0,T]\times R^n\right) \qquad \text{for} \quad V\in C^\infty_0(R^n)$$

and

$$\left(G_{\pm}(t, s)\ V\right)(x)\in C^{\infty}\left([0, T]\times[0, T]\ ;\ \mathscr{D}'(R^{n})\right) \qquad \text{for} \quad V\in\mathscr{E}'(R^{n}).$$

Our next step is to determine the wave front set of $G_{\pm}(t, s)$ and L^2 estimates.

Let $\Lambda_{t,s}^{\pm}$ be a conic set in $T^*(R_x^n\times R_y^n)$ such that

$$\Lambda_{t,s}^{\pm}=\left\{\left((x, \eta)\ (y, \eta)\right):\ x\pm\sqrt{\rho}\ \rho_{\xi}(t, \eta)=y\pm\sqrt{\rho}\ \rho_{\xi}(s, \eta),\ \eta\in R^{n}\right\}.$$

With this notation we have

LEMMA 3.3. *Let* $a(t,s,x,\xi)\in S^{m}_{\frac{1}{3},0,\frac{2}{3}}([0, T]\times[0, T]\times R_x^n\times R_\xi^n)$. *Define the operator* $A_{\pm}(t, s)$ *for* $V\in C_0^{\infty}(R^n)$ *by*

$$\begin{aligned}(3.9)\qquad \left(A_{\pm}(t, s)\ V\right)(x)=\int\exp\left(i\left(\langle x-y, \xi\rangle\pm\frac{2}{3}\left(\rho^{\frac{3}{2}}(t)-\rho^{\frac{3}{2}}(s)\right)\right)\right)\\ \times a\chi_2\left(\rho(t)\right)\chi_2\left(\rho(s)\right)V(y)\,dyd\xi\,.\end{aligned}$$

Then we see that

$$(3.10)\qquad WF'\left(A_{\pm}(t, s)\right)\subset\Lambda_{t,s}^{\pm}\,.$$

PROOF. We shall prove only for the case that the sign is $+$. If $((x_0, \theta)\ (y_0, \eta))\notin\Lambda_{t,s}^{+}$, two cases occur; $\theta\neq\eta$ or $\theta=\eta$ and $x_0+\sqrt{\rho}\ \rho_{\xi}(t, \eta)\neq y_0+\sqrt{\rho}\ \rho_{\xi}(s, \eta)$. Now we must show that

$$\left\langle A_{+}(t, s)\,(\phi e^{i\tau\langle y,\eta\rangle}),\ \psi e^{-i\tau\langle x,\theta\rangle}\right\rangle=0(\tau^{-\infty}) \qquad \text{for} \quad \tau>0\,,$$

where $\psi, \phi\in C_0^{\infty}(R^n)$ are supported near x_0 and y_0 respectively.

At first assume that $x_0+\sqrt{\rho}\ \rho_{\xi}(t, \eta)\neq y_0+\sqrt{\rho}\ \rho_{\xi}(s, \eta)$ for some $\eta\in R^n$. Let Γ_1, Γ_2 be conic neighborhoods of η and $\Gamma_1\Subset\Gamma_2$. Let $\omega_{\eta}(\xi)\in C^{\infty}(R^n\backslash 0)$ be a function of homogeneous degree 0 and assume that $\operatorname{supp}\omega_{\eta}\subset\Gamma_2$ and $\omega_{\eta}(\xi)\equiv 1$ on Γ_1. Put $\lambda(t, s, x, \xi)=\frac{2}{3}\left(\rho^{\frac{3}{2}}(t)-\rho^{\frac{3}{2}}(s)\right)$. Then we have

$$\begin{aligned}&\left\langle A_{+}(t, s)\,(\phi e^{i\tau\langle y,\eta\rangle}),\ \psi e^{-i\tau\langle x,\eta\rangle}\right\rangle\\ &\quad=\tau^{n}\int e^{i\tau(\langle x-y,\xi-\eta\rangle+\lambda)}a'(t, s, x, \tau\xi)\,\omega_{\eta}(\xi)\,\phi\psi dxdyd\xi\\ &\quad\quad+\tau^{n}\int e^{i\tau(\langle x-y,\xi-\eta\rangle+\lambda)}a'(t, s, x, \tau\xi)\,(1-\omega_{\eta})\,(\xi)\,\phi\psi dxdyd\xi\,,\end{aligned}$$

where $a'(t, s, x, \xi)=a(t, s, x, \xi)\,\chi_2(\rho(t))\,\chi_2(\rho(s))$.

If Γ_2 is sufficiently small, then the second term of the right hand side is rapidly decreasing, since on the support of $1-\omega_{\eta}$, $\xi-\eta\neq 0$. Furthermore on the support of ω_{η} we can define the operator

$$L=\left\langle\frac{x+\sqrt{\rho}\,\rho_\xi(t,\xi)-y-\sqrt{\rho}\,\rho_\xi(s,\xi)}{|x+\sqrt{\rho}\,\rho_\xi(t,\xi)-y-\sqrt{\rho}\,\rho_\xi(s,\xi)|^2},\ D_\xi\right\rangle$$

since $x+\sqrt{\rho}\,\rho_\xi(t,\xi)-y-\sqrt{\rho}\,\rho_\xi(s,\xi)\neq 0$. Moreover the coefficients of L are smooth on the support of $\chi_2(\rho(t))\,\chi_2(\rho(s))$. For $Le^{i(\langle x-y,\xi-\eta\rangle+\lambda)}=e^{i(\langle x-y,\xi-\eta\rangle+\lambda)}$ and $({}^tL)^N d'\in S^{m-\frac{1}{3}N}_{\frac{1}{3},0,\frac{2}{3}}$, by integration by parts the Lemma is proved. With a similar argument we can prove for the case that $\theta\neq\eta$.

From this lemma we conclude that $WF((G_\pm(t,s)\,V)(\cdot)\subset\Lambda^\pm_{t,s}\circ WF(V)$. In fact since the symbols of $G_{\pm,21}(t,s)$ and $G_{\pm,12}(t,s)$ vanish if $\rho(s)\neq 0$ or $\rho(t)\neq 0$ and ξ is sufficiently large, wave front sets of $G_{\pm,21}$, $G_{\pm,12}$ do not appear. Hence the propagtions of singularities of $G_\pm(t,s)$ occur along null-bicharacteristic strips of $P_2(t,\tau,\xi)$ passing through $WF(V)$.

Now we prove the H_q-continuity of the operator $G_\pm(t,s)$. Let $a(t,s,x,\xi)$ be an element of $S^m_{\frac{1}{3},0,\frac{2}{3}}([0,T]\times[0,T]\times R^n_x\times R^n_\xi)$. Let $\lambda(\xi)$ be one of the functions $\frac{2}{3}(\rho^{\frac{3}{2}}(t)-\rho^{\frac{3}{2}}(s))$, $\frac{2}{3}\rho^{\frac{3}{2}}(t)$ or $\frac{2}{3}\rho^{\frac{3}{2}}(s)$ and $\phi(x)$ a function in $C_0^\infty(R^n)$. Then we have

$$\left|\int e^{\pm i\lambda(\xi)}\phi(x)\,a(t,s,x,\xi)\,e^{-i\langle x,\eta\rangle}dx\right|\leqslant C_N(1+|\xi|)^m(1+|\eta|)^{-N}, \tag{3.11}$$

for any integer N and some constant C_N. Indeed by integration by parts we see that

$$\begin{aligned}\eta^\alpha&\int e^{\pm i\lambda(\xi)}\phi(x)\,a(t,s,x,\xi)\,e^{-i\langle x,\eta\rangle}dx\\&=\int e^{\pm i\lambda(\xi)}(D_x^\alpha\phi a)(t,s,x,\xi)\,e^{-i\langle x,\eta\rangle}dx,\end{aligned}$$

where α is any multi-index. The right hand side can be estimated by a constant times $(1+|\xi|)^m$. Thus we obtain (3.11). Form this estimate we have the following

LEMMA 3.4. *We denote by $A_\pm(t,s)$ the operator*

$$A_\pm(t,s)\,V(x)=\int e^{i(\langle x-y,\xi\rangle\pm\lambda(\xi))}a(t,s,x,\xi)\,V(y)\,dyd\xi\,. \tag{3.12}$$

Let ϕ be a function in $C_0^\infty(R^n)$. Then we see that

$$\left\|\phi A_\pm(t,s)\,V\right\|_{q-m}\leqslant C\|V\|_q\qquad\textit{for}\quad V\in C_0^\infty(R^n)\,.$$

PROOF. We shall show that

$$\left|\left\langle A_\pm(t,s)\,V,\ \phi W\right\rangle\right|\leqslant C\|V\|_q\|W\|_{m-q}\qquad\text{for}\quad V,\ W\in C_0^\infty(R^n)\,. \tag{3.13}$$

Set $b_\pm(t, s, x, \xi)=e^{\pm i\lambda(\xi)}\psi(x)\, a(t, s, x, \xi)$ and denote by $\hat{b}_\pm(t, s, \eta, \xi)$ the Fourier transform of $b_\pm$ with respect to x. Then we have

$$\begin{aligned}\langle A_\pm(t, s)\, V, \psi W\rangle &= \iint W(x)\, e^{i\langle x,\xi\rangle \pm i\lambda(\xi)}\, \psi a(t, s, x, \xi)\, \hat{V}(\xi)\, d\xi dx \\ &= \iint \hat{W}(-\eta)\, b_\pm(t, s, \eta-\xi, \xi)\, \hat{V}(\xi)\, d\eta d\xi \\ &= \iint \hat{b}_\pm(t, s, \eta-\xi, \xi)\, (1-|\xi|^2)^{-\frac{q}{2}}(1+|\eta|^2)^{\frac{q-m}{2}}\, v(\xi)\, w(\eta)\, d\eta d\xi\,,\end{aligned}$$

where $v(\xi)=(1+|\xi|^2)^{\frac{q}{2}}\hat{V}(\xi)$ and $w(\eta)=(1+|\eta|^2)^{\frac{m-q}{2}}\hat{W}(-\eta)$. From (3. 11)

$$\begin{aligned}&\left|\hat{b}_\pm(t, s, \eta-\xi, \xi)\, (1+|\xi|^2)^{-\frac{q}{2}}(1+|\eta|^2)^{\frac{q-m}{2}}\right| \\ &\qquad \leqslant C_N(1+|\eta-\xi|)^{-N}(1+|\xi|^2)^{\frac{m-q}{2}}(1+|\eta|^2)^{\frac{q-m}{2}}\end{aligned}$$

for any integer N.
Since $(1+|\eta|^2)^{\frac{q-m}{2}}\leqslant(1+|\xi|^2)^{\frac{q-m}{2}}(1+|\eta-\xi|)^{|q-m|}$,

$$\begin{aligned}&\left|\iint \hat{b}_\pm(t, s, \eta-\xi, \xi)\, (1+|\xi|^2)^{-\frac{q}{2}}(1+|\eta|^2)^{\frac{q-m}{2}}\, v(\xi)\, w(\eta)\, d\xi d\eta\right| \\ &\qquad \leqslant \int C_N(1+|\eta-\xi|)^{-N+|q-m|}\left|v(\xi)\right|\left|w(\eta)\right| d\xi d\eta\,.\end{aligned}$$

Thus from Hausdorff-Young's inequality, (3. 13) follows.

Note that in the above proof it is sufficient to assume that the symbol depends on t, s continuously. This lemma shall be used in the next section.

In order to prove Theorem 1. 1 we shall consider the initial conditions (1. 6). We denote by $\gamma_\pm^0(s)$ and $\gamma_\pm^1(s)$ the operators $G_\pm(s, s)$ and $D_t G_\pm(s, s)$ respectively. Since the phase function of $G_\pm(s, s)$ and $D_t G_\pm(s, s)$ is $\langle x-y, \xi\rangle$, $\gamma_\pm^0(s)$ and $\gamma_\pm^1(s)$ are pseudo-differential operators with symbols of type $\rho=\frac{1}{3}$, $\delta=0$ in Hörmander's sense ([4]) with parameter s. Note that the order of $\gamma_\pm^0(s)$ is zero by lemma 3. 2 and at the point $s=0$ and $\xi=\xi_0$, its principal part is $g_0 e^{\pm i\pi/3} Ai(0)/Ai'(0)$, since $\zeta(\xi_0)=0$.

We shall prove that $\gamma_\pm^1(s)$ is of order $\frac{2}{3}$ and its principal part is $-i\rho_t g_0$ at $t=0$, $\xi=\xi_0$. By a direct computation we see that

$$D_t G_\pm(s, s)\, V=(2\pi)^{-n}\int e^{i\langle x-y,\xi\rangle}\Big\{\Big(\rho_t\rho h(s, x, \xi)-ig_t(s, x, \xi)\Big)\times \frac{A_\pm(\rho(s))}{A_\pm'(\rho(s))}+\Big(-i\rho_t g(s, x, \xi)-h_t(s, x, \xi)\Big)\Big\}\, V(y)\, dy d\xi\,,$$

where we used the relation $A''_{\pm}(x)=-xA_{\pm}(x)$. We find that $h\rho_t\rho\frac{A_{\pm}}{A'_{\pm}}(\rho)\in S^{\frac{2}{3}}_{\frac{1}{3},0}(R^n_x\times R^n_\xi)$. Indeed since $\theta\frac{A_{\pm}}{A'_{\pm}}(\theta)=\theta\frac{\Phi_{\pm}}{\tilde{\Phi}_{\pm}}(\theta)\in S^{\frac{1}{2}}_{1,0}$ from (2. 23), (2. 23)′, applying Lemma 3. 1 we conclude that $\rho\frac{A_{\pm}}{A'_{\pm}}(\rho)\in S^{\frac{1}{3}}_{\frac{1}{3},0}$ and hence $h\rho_t\rho\frac{A_{\pm}}{A'_{\pm}}(\rho)\in S^{\frac{2}{3}}_{\frac{1}{3},0}$. Therefore the principal part is $-ig(0,x,\xi_0)$ at $s=0$, $\xi=\xi_0$ since $\rho(0,\xi_0)=0$.

We denote by γ and $\tilde{\Lambda}$ matrices

$$\gamma=\begin{pmatrix}\gamma^0_+(s), & \gamma^0_-(s)\\ \gamma^1_+(s), & \gamma^1_-(s)\end{pmatrix},\qquad \tilde{\Lambda}=\begin{pmatrix}\Lambda^{\frac{2}{3}}, & 0\\ 0, & I\end{pmatrix},$$

where $\Lambda^{\frac{2}{3}}$ is the pseudo-differential operator with symbol $(1+|\xi|^2)^{1/3}$. Then the principal part of $\tilde{\Lambda}\gamma$ is written in the form

$$g_0(0,x,\xi_0)\begin{pmatrix}|\xi_0|^{\frac{2}{3}}e^{\frac{\pi}{3}i}Ai(0)/Ai(0), & |\xi_0|^{\frac{2}{3}}e^{-\frac{\pi}{3}i}Ai(0)/Ai(0)\\ -i\rho_t(0,\xi_0) & , \quad -i\rho_t(0,\xi_0)\end{pmatrix}$$

at $s=0$ and $\xi=\xi_0$. From (2. 7), (2. 10), $\tilde{\Lambda}\gamma$ is an elliptic operator of order $\frac{2}{3}$ for any $s\in[0,T]$, if T is sufficiently small. Let $K(s)$ be its parametrix of order $-\frac{2}{3}$. Define the operator $G^k(t,s)$, $k=0,1$, by the form

$$G^k(t,s)V=\big(G_+(t,s),G_-(t,s)\big)K(s)\begin{pmatrix}\delta^k_0\Lambda^{\frac{2}{3}}V\\ \delta^k_1V\end{pmatrix}\qquad\text{for}\quad V\in C^\infty_0(R^n). \tag{3.15}$$

In $U\times\Gamma$ it follows from (3. 3) that

$$\begin{cases}P(t,x,D_t,D_x)G^k(t,s)\equiv0,\\ G^k(s,s)\equiv\delta^k_0I,\\ D_tG^k(s,s)\equiv\delta^k_1I,\end{cases} \tag{3.16}$$

where t, s are considered as parameters.

Finally we shall construct a global parametrix $G^k(t,s)$ which satisfies (1. 6) in $R^n_x\times R^n_\xi\backslash0$. Since the lower order terms of $P(t,x,D_t,D_x)$ are constant for large $|x|$, it suffices to construct a finite number of microlocal parametrices. Thus there exist a finite open covering of $R^n_x\times R^n_\xi\backslash0$ and a partition of unity $\{U_j\times\Gamma_j,\alpha_j,\beta_j\}_{j\in J}$ such that;

i) U_j is an open set and Γ_j is an open cone such that $\bigcup_{j\in J}U_j\times\Gamma_j=R^n_x\times R^n_\xi\backslash0$.

ii) For each $j\in J$ there exists an operator $G^k_j(t,s)$ which satisfies (3 16) in $U_j\times\Gamma_j$.

iii) $\alpha_j \in C^\infty(U_j \times \Gamma_j)$ is a function of homogeneous degree 0 in ξ such that; $\operatorname{supp} \alpha_j \Subset U_j \times \Gamma_j$ and $\sum_{j \in J} \alpha_j(x, \xi) \equiv 1$.

iv) $\beta_j \in C^\infty(U_j \times \Gamma_j)$ is a function of homogeneous degree 0 in ξ such that; $\operatorname{supp} \beta_j \Subset U_j \times \Gamma_j$ and denoting by σ the parameter of the bicharacteristic strip of P_2, any bicharacteristic strip which starts from some point (x, ξ) in $\operatorname{supp} \alpha_j$ at $\sigma = 0$, does not intersect supp $(1-\beta_j)$ for $0 \leqslant \sigma \leqslant T$.

Now we define the global parametrix as follows;

$$G^k(t, s) = \sum_{j \in J} \beta_j(x, D_x)\, G_j^k(t, s)\, \alpha_j(x, D_x)\,, \tag{3.17}$$

where $\alpha_j(x, D_x)$, $\beta_j(x, D_x)$ are pseudo-differential operators with symbols $\alpha_j(x, \xi)$, $\beta_j(x, \xi)$ respectively. We see easily that (3.17) satisfies (1.6), since $[P, \beta_j]\, G_j^k(t, s)\, \alpha_j \equiv 0$ by Lemma 3.3 and iv).

§ 4. *Estimate of the solution of the Cauchy problem.*

We shall give the formula of the solution in order to obtain the uniqueness of the solution and the estimate (1.7).

We denote by G' the operator

$$G'f(t, x) = \int_0^t \Big(G^1(t, s) - R_0^1(s) - i(t-s)\, R_1^1(s)\Big) f(s, x)\, ds\,, \tag{4.1}$$

where $f \in C^\infty([0, T]; C_0^\infty(R^n))$. From (1.6) we obtain

$$\begin{cases} PG'f = f - Wf\,, \\ D_t^j G'f|_{t=0} = 0\,, \qquad \text{for} \quad j = 0, 1\,. \end{cases}$$

Here W is an operator with a C^∞-kernel $k(t, s, x, y)$ such that

$$Wf(t, x) = \int_0^t \int k(t, s, x, y) f(s, y)\, dy ds\,.$$

In order to construct the inverse of $I - W$ by the Nrumann series, we must insert suitable cut off functions. Let v_0, $v_1 \in C_0^\infty(R^n)$ and K a compact set in R^n such that K contains the set $\bigcup_{0 \leqslant t \leqslant T} \operatorname{supp} f(t, x) \cup \operatorname{supp} v_0 \cup \operatorname{supp} v_1$ and any bicharacteristic curve $x(\sigma)$, $0 \leqslant \sigma \leqslant T$, starting from some point of this set at $\sigma = 0$. Let α_1, $\alpha_2 \in C_0^\infty(R^n)$ be functions such that $\alpha_1 \equiv 1$ on K and $\alpha_2 \equiv 1$ on $\operatorname{supp} \alpha_1$ and $W' = \alpha_2 W$. Then $(I - W')^{-1}$ is given by $\sum_{k=0}^\infty W'^k$. Setting $f' = (I - W')^{-1} f$ we obtain from (4.2) that

$$\alpha_1 PG'f' = \alpha_1 (I - W) f' = \alpha_1 (I - W') f' = f$$

since $\alpha_1 W = \alpha_1 \alpha_2 W$. Thus $u = G'f'$ is a solution of the Cauchy problem

$$(4.3)\qquad \begin{cases} Pu=f & \text{in}\quad [0,T]\times K, \\ D_t^j u|_{t=0}=0\,, & j=0,1\,. \end{cases}$$

Now we shall prove the uniqueness of (4.3). Let P^* be the adjoint operator of P then the principal symbol of P^* is the same as that of P. Thus we can also construct a parametrix of P^* with smooth parameters $0\leqslant t\leqslant s\leqslant T$, and solve the equation (4.3) for given $f\in C^\infty([0,T]\times K)$ in $[0,T]\times K$ with the initial surface $t=T$ instead of $t=0$. Note that P^* is strictly hyperbolic in $t>0$ and the domain of influence of f is finite because the bicharacteristic curve does not tangent to the surface $t=0$. Hence for the equation (4.3) with initial surface $t=T$ we obtain a smooth solution which vanishes identically for large $|x|$. Therefore by the usual dual argument we conclude the uniqueness of (4.3). Choose K sufficiently large for given $f\in C^\infty([0,T]\,;\,C_0^\infty(R^n))$. Then from the above facts there exists a unique solution $u\in C^\infty([0,T]\,;\,C_0^\infty(R^n))$ of (4.3) in $[0,T]\times R^n$.

For v_0, v_1 the solution of (1.3) is given by

$$(4.4)\qquad \begin{aligned} u=&\sum_{k=0}^{1}\alpha_1 G^k(t,0)\,v_k-\Big(\sum_{k=0}^{1}\alpha_1 R_0^k(0)\,v_k+it\sum_{k=0}^{1}\alpha_1 R_1^k(0)\,v_k\Big)\\ &+G'(I-W')^{-1}f'\,, \end{aligned}$$

where

$$f'=f-P\Big\{\sum_{k=0}^{1}\alpha_1 G^k(t,0)\,v_k-\Big(\sum_{k=0}^{1}\alpha_1 R_0^k(0)\,v_k+it\sum_{k=0}^{1}\alpha_1 R_1^k(0)\,v_k\Big)\Big\}\,.$$

Now we shall estimate the operator

$$\begin{aligned}\int_0^t G^1(t,s)f(s,\cdot)\,ds&=\int_0^t\Big(G^1(t,s)\,s^{\frac{1}{2}}\Big)f(s,\cdot)\,s^{-\frac{1}{2}}ds\\ &=\int_0^t\Big(G_+(t,s),\ G_-(t,s)\Big)\,s^{\frac{1}{2}}K(s)\begin{pmatrix}0\\ f(s,\cdot)\end{pmatrix}s^{-\frac{1}{2}}ds\,.\end{aligned}$$

Let $t\geqslant s$ and χ, $\tilde{\chi}\in C^\infty(R)$ functions such that $\chi(\sigma)\equiv 1$ if $\sigma\leqslant 1$, $\chi(\sigma)\equiv 0$ if $\sigma\geqslant 2$ and $\tilde{\chi}(\sigma)=\chi(\sigma-1)$. Put $\chi_1=\chi(\rho(t))$, $\chi_2=1-\chi(\rho(t))$, $\tilde{\chi}_1=\tilde{\chi}(\rho(s))$ and $\tilde{\chi}_2=1-\tilde{\chi}(\rho(s))$. Decompose $G_\pm(t,s)\,s^{\frac{1}{2}}$ as follows;

$$\begin{aligned}G_\pm(t,s)\,s^{\frac{1}{2}}\,V=&\sum_{j=1}^{2}\int e^{i\langle x-y,\xi\rangle}\Big(g\frac{A_\pm(\rho(t))}{A'_\pm(\rho(s))}-ih\frac{A'_\pm(\rho(t))}{A'_\pm(\rho(s))}\Big)s^{\frac{1}{2}}\chi_2\tilde{\chi}_j\,Vdyd\xi\\ &+\int e^{i\langle x-y,\xi\rangle}\Big(g\frac{A_\pm(\rho(t))}{A'_\pm(\rho(s))}-ih\frac{A'_\pm(\rho(t))}{A'_\pm(\rho(s))}\Big)s^{\frac{1}{2}}\chi_1\tilde{\chi}_1\,Vdyd\xi\\ =&\sum_{j=1}^{2}G_{\pm,2j}(t,s)\,s^{\frac{1}{2}}\,V+G_{\pm,11}(t,s)\,s^{\frac{1}{2}}\,V\,.\end{aligned}$$

Then we have

LEMMA 4.1. *Let $t \geqslant s$. Then $G_{\pm,22}(t,s)\,s^{\frac{1}{2}}$, $G_{\pm,21}(t,s)\,s^{\frac{1}{2}}$ and $G_{\pm,11}(t,s)\,s^{\frac{1}{2}}$ become operators of the form* (3.12) *with symbols in $S^{-1/3}_{1/3,0}(R^n_x \times R^n_\xi)$ depending continuously on (t,s). Here $\lambda(\xi)$ stands for the phase functions $\frac{2}{3}(\rho^{\frac{3}{2}}(t)-\rho^{\frac{3}{2}}(s))$, $\frac{2}{3}\rho^{\frac{3}{2}}(t)$ and* 0.

PROOF. We prove that $\frac{\Phi_\pm(\rho(t))}{\tilde{\Phi}_\pm(\rho(s))}s^{\frac{1}{2}}\chi_2\tilde{\chi}_2$, $\frac{\Phi_\pm(\rho(t))}{A'_\pm(\rho(s))}s^{\frac{1}{2}}\chi_2\tilde{\chi}_1$, and $\frac{A_\pm(\rho(t))}{A'_\pm(\rho(s))}s^{\frac{1}{2}}\chi_1\tilde{\chi}_1 \in S^{-\frac{1}{3}}_{\frac{1}{3},0}$. Note that from (2.6), (2.7) $C_1(t+\zeta)|\xi|^{\frac{2}{3}} \leqslant \rho(t) \leqslant C_2(t+\zeta)|\xi|^{\frac{2}{3}}$ for some constants C_1, C_2. Differentiating with respect to ξ we obtain that

$$\left|\partial_\xi^\alpha s^{\frac{1}{2}}\Phi_\pm\left(\rho(t)\right)/\tilde{\Phi}_\pm\left(\rho(s)\right)\right| = \left|s^{\frac{1}{2}}\sum_{\alpha^1+\alpha^2=\alpha}\binom{\alpha}{\alpha_1}\partial_\xi^{\alpha^1}\Phi_\pm\left(\rho(t)\right)\partial_\xi^{\alpha^2}(1/\tilde{\Phi}_\pm)\left(\rho(s)\right)\right|$$

$$=\left|s^{\frac{1}{2}}\sum_{\alpha^1+\alpha^2=\alpha}\binom{\alpha}{\alpha^1}\sum_{\substack{\alpha_1^1+\cdots+\alpha_\nu^1=\alpha^1\\ \alpha_1^2+\cdots+\alpha_\mu^2=\alpha^2}}\frac{\alpha^1!\,\alpha^2!}{\nu!\,\mu!\,\alpha_1^1!\cdots\alpha_\nu^1!\,\alpha_1^2!\cdots\alpha_\mu^2!}\times\right.$$

$$\left.\Phi_\pm^{(\nu)}\left(\rho(t)\right)\partial_\xi^{\alpha_1^1}\rho(t)\cdots\partial_\xi^{\alpha_\nu^1}\rho(t)(1/\tilde{\Phi}_\pm)^{(\mu)}\left(\rho(s)\right)\partial_\xi^{\alpha_1^2}\rho(s)\cdots\partial_\xi^{\alpha_\mu^2}\rho(s)\right|$$

$$\leqslant C_\alpha s^{\frac{1}{2}}\rho^{-\frac{1}{4}}(t)\,\rho^{-\frac{1}{4}}(s)|\xi|^{-\frac{1}{3}|\alpha|}.$$

Since $s^{\frac{1}{2}}\rho^{-\frac{1}{4}}(t)\,\rho^{-\frac{1}{4}}(s) \leqslant s^{\frac{1}{2}}\rho^{-\frac{1}{2}}(s) \leqslant \frac{1}{C_1}\left(\frac{s}{s+\zeta}\right)^{\frac{1}{2}}|\xi|^{-\frac{1}{3}} \leqslant \frac{1}{C_1}|\xi|^{-\frac{1}{3}}$, the assertion for the first function is proved. From the boundedness of $\left(\frac{1}{A'_\pm}\right)^{(\nu)}(\rho(s))$ on the support of $\tilde{\chi}_1$ we see that

$$\left|\partial_\xi^\alpha s^{\frac{1}{2}}\Phi_\pm\left(\rho(t)\right)/A'_\pm\left(\rho(s)\right)\right| \leqslant C_\alpha s^{\frac{1}{2}}\rho^{-\frac{1}{4}}(t)|\xi|^{-\frac{1}{3}|\alpha|} \leqslant C s^{\frac{1}{2}}|\xi|^{-\frac{1}{3}|\alpha|}.$$

Since on the support of $\tilde{\chi}_1$, $s \leqslant (s+\zeta) \leqslant \frac{3}{C_1}|\xi|^{-\frac{2}{3}}$, the assertion holds for the second function. By the same argument we can prove for the last one. Note also that if we replace $\Phi_\pm(\rho(t))$, $A_\pm(\rho(t))$ by $\tilde{\Phi}_\pm(\rho(t))$, $A'_\pm(\rho(t))$ in the above functions, the symbol class $S^{-1/3}_{1/3,0}$ must be replaced by $S^0_{1/3,0}$. This proves the lemma.

Now remarking the degree of $K(s)$, we apply Lemma 3.4 and Lemma 4.1 to (4.4). Then we obtain

$$\left\|u(t,\cdot)\right\|_{q+1} \leqslant C\left\|\sum_{k=0}^{1}G^k(t,0)\,v_k\right\|_{q+1} + C\int_0^t\left\|G^1(t,s)f(s,\cdot)\right\|_{q+1}s^{-\frac{1}{2}}ds$$

$$\leqslant C\left\|\Lambda^{\frac{2}{3}}v_0\right\|_{q+1-\frac{2}{3}} + C\|v_1\|_{q+1-\frac{2}{3}} + C\int_0^t\left\|f(s,\cdot)\right\|_q s^{-\frac{1}{2}}ds$$

$$\leqslant C\|v_0\|_{q+1} + C\|v_1\|_{q+\frac{1}{3}} + C\int_0^t\left\|f(s,\cdot)\right\|_q s^{-\frac{1}{2}}ds$$

and

$$\left\|D_t u(t,\cdot)\right\|_q \leqslant C\left\|\sum_{k=0}^{1} D_t G^k(t,s)\,v_k\right\|_q + C\int_0^t \left\|D_t G^1(t,s)\,s^{\frac{1}{2}} f(s,\cdot)\right\|_q s^{-\frac{1}{2}}\,ds$$
$$\leqslant C\|v_0\|_{q+1} + C\|v_1\|_{q+\frac{1}{3}} + C\int_0^t \left\|f(s,\cdot)\right\|_q s^{-\frac{1}{2}}\,ds\,,$$

since $D_t G_{\pm}(t,s)$ is of degree 1. Moreover from the equation

$$PD_t u = D_t P u + [P, D_t]\,u = D_t f + B_1 u\,,$$

where B_1 is a second order operator, we see that

$$\left\|D_t u(t,\cdot)\right\|_q + \left\|D_t^2 u(t,\cdot)\right\|_{q-1} \leqslant C\left\|D_t u(0,\cdot)\right\|_q + C\left\|D_t^2 u(0,\cdot)\right\|_{q+\frac{1}{3}-1}$$
$$+ C\int_0^t \left\|F(s,\cdot)\right\|_{q-1} s^{-\frac{1}{2}}\,ds$$
$$\leqslant C\left\|u(0,\cdot)\right\|_{q+1} + C\sum_{k=0}^{1}\left\|D_t^{k+1} u(0,\cdot)\right\|_{q+\frac{1}{3}-k} + C\sum_{k=0}^{1}\int_0^t \left\|D_t^k f(s,\cdot)\right\|_{q-k} s^{-\frac{1}{2}}\,ds\,,$$

where $F = D_t f + B_1 u$. Indeed apply the above argument to $(D_t u, F)$ instead of (u, f). Iterating this procedure for k times, $k \leqslant q$, we see that

$$\sum_{k\leqslant q+1}\left\|D_t^k u(t,\cdot)\right\|_{q+1-k} \leqslant C\left\|u(0,\cdot)\right\|_{q+1} + C\sum_{k\leqslant q}\left\|D_t^{k+1} u(0,\cdot)\right\|_{q+\frac{1}{3}-k}$$
$$+ C\sum_{k\leqslant q}\int_0^t \left\|D_t^k f(s,\cdot)\right\|_{q-k} s^{-\frac{1}{2}}\,ds\,. \tag{4.5}$$

Now in order to obtain the estimate (1.7) we must estimate higher derivatives of u at $t=0$ in the right hand side of (4.5). Thus we show that

$$\left\|D_t^{k+1} u(0,\cdot)\right\|_{q+\frac{1}{3}-k} \leqslant C\|v_0\|_{q+\frac{4}{3}} + C\|v_1\|_{q+\frac{1}{3}} + C\sum_{j\leqslant k-1}\left\|D_t^j f(0,\cdot)\right\|_{q-j} \tag{4.6}$$

for $k \geqslant 1$. Since for $k=1$

$$D_t^2 u\,(0,x) = -\big(2a(0,D_x)\,D_t + b(0,D_x) + P_1(0,x,D_t,D_x)\big)\,u + f(0,x)\,,$$

we see that

$$\left\|D_t^2 u(0,\cdot)\right\|_{q+\frac{1}{3}-1} \leqslant C\|v_0\|_{q+\frac{4}{3}} + C\|v_1\|_{q+\frac{1}{3}} + C\left\|f(0,\cdot)\right\|_{q+\frac{1}{3}-1}$$
$$\leqslant C\|v_0\|_{q+\frac{4}{3}} + C\|v_1\|_{q+\frac{1}{3}} + C\left\|f(0,\cdot)\right\|_q\,.$$

Now assume that the assertion holds for any $k' \leqslant k$. Then it follows that

$$\left\|D_t^{k+2} u(0,\cdot)\right\|_{q+\frac{1}{3}-(k+1)} \leqslant C\sum_{j=0}^{k+1}\left\|D_t^j u(0,\cdot)\right\|_{q+\frac{1}{3}-(j-1)} + C\left\|D^k f(0,\cdot)\right\|_{q+\frac{1}{3}-(k+1)}$$
$$\leqslant C\|v_0\|_{q+\frac{4}{3}} + C\|v_1\|_{q+\frac{4}{3}} + C\sum_{j\leqslant k}\left\|D_t^j f(0,\cdot)\right\|_{q-j}$$

since

$$D^{k+2}u(0, x)=D_t^k(-2aD_tu-bu-P_1u+f)\Big|_{t=0}.$$

Apply (4.6) to (4.5) then we obtain (1.6). This completes the proof of Theorem 1.2.

§ 5. Appendix; Construction of a phase function of a completely regularly hyperbolic operator whose principal symbol depends on time and space variables.

In this chapter we shall give a solution of the eikonal equation $p(x, \varphi_x)=0$, where $p(x, \xi)$ is a principal symbol of a completely regularly hyperbolic operator of second order.

Let $x\in R^n$, $x=(x_0, x')=(x_0, x_1, \cdots, x_n)$ and $\xi=(\xi_0, \xi')$ its dual variables. By a suitable canonical change of (x', ξ'), we may assume that

$$p(x, \xi)=\xi_0^2-\big(x_0+\zeta(x', \xi')\big)A(x, \xi'). \tag{5.1}$$

Here $\zeta(x', \xi')$ is homogeneouse degree 0, non-negative and $A(x, \xi')$ is a positive elliptic symbol of homogeneous degree 2 (see § 2). Suppose that $\zeta(0, \xi'^0)=0$ for some $\xi'^0\neq 0$. Then the characteristic equation $p(0, \xi_0, \xi'^0)=0$ with respect to ξ_0 has a double root 0.

Now under these assumption we shall prove the following

THEOREM 5.1. *There exist some conic neighborhood* $\Gamma\subset R_x^{n+1}\times R_{\xi'}^n\backslash 0$ *of* $(0, \xi'^0)$ *and real functions* $\theta(x, \xi')$, $\rho(x, \xi')\in C^\infty(R_x^{n+1}\times R_{\xi'}^n\backslash 0)$ *of homogeneous degree* 1 *and* $\frac{2}{3}$ *respectively such that*

$$\rho|_{x_0+\zeta(x', \xi')=0}=0, \qquad \rho_{x_0}(0, \xi'^0)\gneqq 0, \tag{5.2}$$

$$\det \theta_{x'\xi'}(0, \xi'^0)\neq 0 \tag{5.3}$$

and in the domain $\Gamma\cap\{(x, \xi');\ \rho\geqq 0\}$, $\varphi_\pm(x, \xi')=\theta(x, \xi')\pm\frac{2}{3}\rho^{\frac{3}{2}}(x, \xi')$ satisfies

$$p(x, \varphi_{\pm x})=0. \tag{5.4}$$

Let $X=R_x^{n+1}$, $T^*(X)=R_x^{n+1}\times R_\xi^{n+1}$,

$$V=\Big\{(x, \xi);\ p(x, \xi)=0\Big\},$$

$$M=\Big\{(x, 0, \xi');\ x_0=-\zeta(x', \xi')\Big\}\subset V$$

and

$$L_\nu=\left\{(x,0,\nu)\,;\ x_0=-\zeta(x',\nu)\right\}\subset M,$$

where $\nu\in S^{n-1}\cap R^n_{\xi'}$, $S^{n-1}=\{\nu\in R^n\,;\ |\nu|=1\}$. Here we regard ν as a parameter. Note that for any (x,ξ'), $x_0=-\zeta(x',\xi')$, the characteristic equation $p(x,\xi_0,\xi')=0$ has a double root $\xi_0=0$. Moreover M is a submanifold of $T^*(X)$ and a symplectic space with respect to the restriction of the symplectic 2-form $\sigma=\sum_{k=0}^{n}dx_k\wedge d\xi_k$. For fixed ν, L_ν is a Lagrangian submanifold of M with respect to $\sigma|_M$.

Let $\Phi^t(m)$ be the integral curve of the Hamiltonian vector field H_p which start from m at $t=0$. Then the orbits

(5.5) $$\Lambda_\nu=\left\{\Phi^t(m)\,;\ m\in L_\nu\right\}\subset V$$

is a Lagrangian submanifold of $T^*(X)$.

Indeed for any $m\in L_\nu$, H_p is trasversal to L_ν at m since $p_{x_0}\neq 0$. Thus we may take (t,x') as a local coordinate system of Λ_ν, hence $\dim\Lambda_\nu=n+1$. Let $(x(t,m),\ \xi(t,m))=\Phi^t(m)$. Then at $m=(x,0,\nu)\in L_\nu$, $t=0$,

$$\begin{cases} dx_0=\dfrac{\partial p}{\partial\xi_0}dt-\sum_{k=1}^{n}\zeta_{x_k}dx_k=-\sum_{k=1}^{n}\zeta_{x_k}dx_k\,, \\ dx_k=\dfrac{\partial p}{\partial\xi_k}dt+dx_k\,, \qquad k=1,\cdots,n\,, \\ d\xi_0=\dfrac{\partial p}{\partial x_0}dt=A(x,\nu)\,dt \\ d\xi_k=\dfrac{\partial p}{\partial x_k}dt=\zeta_{x_k}A(x,\nu)\,dt,\ \ k=1,\cdots,n\,, \end{cases}$$

thus $dx_0\wedge d\xi_0+\sum_{k=0}^{n}dx_k\wedge d\xi_k=0$ so $T^\sigma_m(\Lambda_\nu)=T_m(\Lambda_\nu)$. For any t, $T_{\Phi^t(m)}(\Lambda_\nu)=D\Phi^t(m)(T_m(\Lambda_\nu))$ and $D\Phi^t(m):\ T_m(\Lambda_\nu)\to T_{\Phi^t(m)}(\Lambda_\nu)$ is a symplectic map. This proves the assertion.

Let $\phi(x,\eta)\in C^\infty(X\times R^N)$ be a real function such that $\phi_\eta(x,\eta)$ is non-degenerate with respect to (x,η). Then $\Lambda_\phi=\{(x,\phi_x)\,;\ (x,\eta)\in C_\phi\}$ is a Lagragian submanifold of $T^*(X)$, where C_ϕ is a submanifold of $X\times R^N$ defined by the equation $\phi_\eta(x,\eta)=0$. Here we do not assume the homogeneity with respect to η.

The idea of the proof of Theorem 5.1 is that we shall construct a suitable phase function $\phi(x,\eta,\nu)$ containing ν as a smooth parameter such that $\Lambda_\nu=\Lambda_{\phi(\cdot,\cdot,\nu)}$ and decrease the fiber variable η to one dimensional parameter α from the fact that rank $\pi=n$ on L_ν, where π is the projection $\pi:\ \Lambda_\nu\to X$. Furthermore noting that the projection π is a simple fold, we shall deform $\phi(x,\alpha,\nu)$ to the form $\theta(x,\nu)+\rho(x,\nu)\alpha-\frac{\alpha^3}{3}$ by the Malgrange's preparation theorem. Then $\Lambda_\nu=\{(x,\theta_x+\sqrt{\rho}\,\rho_x)\}$, since $\phi_\alpha=\rho-\alpha^2$. Thus it is equal to $\{(x,\varphi_{\pm x})\}$, if $\varphi_\pm=\theta\pm\frac{2}{3}\rho^{3/2}$. If we extend $\theta(x,\nu)$, $\rho(x,\nu)$ homogemeously of order 1 and $\frac{2}{3}$ respectively, we obtain Theorem 5.1.

In order to construct the phase function $\phi(x,\eta,\nu)$, suppose that $\xi'^0/|\xi'^0|=\nu^0=(\nu_1^0,\cdots,\nu_n^0)$, $\nu_1^0\neq 0$, and make a change of variables

$$x_0=y_0,\ x_1=y_1+\frac{1}{2}(y_1^2+\cdots+y_n^2),\ x_2=y_2,\cdots,x_n=y_n\,. \tag{5.6}$$

Denoting that $T^*(X)=R_y^{n+1}\times R_\eta^{n+1}$, we see that

$$\eta=\frac{{}^t\partial x}{\partial y}\xi=\begin{pmatrix}\xi_0\\ \xi_1+y_1\xi_1\\ \vdots\\ \xi_n+y_n\xi_1\end{pmatrix}=\xi+\xi_1y'\,, \tag{5.7}$$

since

$$\frac{\partial x}{\partial y}=\begin{pmatrix}1 & 0 & 0 & \cdots & 0\\ 0 & 1+y_1 & y_2 & \cdots & y_n\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & & & \ddots & \vdots\\ 0 & \cdots & & \cdots & 1\end{pmatrix}$$

Thus at $(0,\nu^0)$ the tangent vector $(\delta x,\delta\xi)=(\delta x_0,\delta x',\delta\xi_0,\delta\xi')\in T_{(0,0,\nu^0)}(T^*(R_x^{n+1}))$ is changed as follows

$$\delta\eta=\delta\xi+y'\delta\xi_1+\xi_1\frac{\partial y'}{\partial x'}\delta x'\Big|_{\substack{x=0\\ \xi_0=0\\ \xi'=\nu^0}}=\delta\xi+\nu_1^0\delta x'\,, \tag{5.8}$$

where $\delta\eta$ is the corresponding fiber component of $(\delta y,\delta\eta)\in T_{(0,0,\nu^0)}(T^*(R_y^{n+1}))$. Note that the vectors $(\delta x,\delta\xi)=(0,\delta x',-p_{x_0}\delta t,0)$ generate the tangent space $T_{(0,0,\nu^0)}(\Lambda_{\nu^0})$ and for these vectors $\delta\eta=(-p_{x_0}\delta t,\nu_1^0\delta x')$. Because $-p_{x_0}$, $\nu_1^0\neq 0$, $\pi':\Lambda_{\nu^0}\to R_\eta^{n+1}$ is regular at $(0,0,\nu^0)$, where π' is the projection $R_y^{n+1}\times R_\eta^{n+1}\to R_\eta^{n+1}$. Thus there exist some open neighborhoods $U\subset R_\eta^{n+1}$ of $(0,\nu^0)$, $U'\subset S^{n-1}$ of ν^0 and C^∞-functions $Y(\eta,\nu)=(Y_0(\eta,\nu),\cdots,Y_n(\eta,\nu))\subset C^\infty(U\times U';R^{n+1})$ such that

$$\Lambda_\nu = \left\{ \left(Y(\eta, \nu), \eta \right) ;\ \eta \in U,\ \nu \in U' \right\} .$$

If $\omega = \sum_{k=0}^{n} Y_k d\eta_k$, then $d\omega = \sigma|_{\Lambda_\nu} = 0$ for fixed ν since Λ_ν is a Lagrangian, hence there exists a C^∞-function $H(\eta, \nu) \in C^\infty(U \times U')$ such that $dH = \omega$. Now we take

$$\tilde{\phi}(y, \eta, \nu) = \langle y, \eta \rangle - H(\eta, \nu) . \tag{5. 9}$$

Then $dH = \omega$ implies that $\Lambda_\nu = \Lambda_{\tilde{\phi}(\cdot, \cdot, \nu)}$. Substituting $\phi(x, \eta, \nu) = \tilde{\phi}(y(x), \eta, \nu)$ we have that $\Lambda_\nu = \Lambda_{\phi(\cdot, \cdot, \nu)}$, since the diagram

$$\begin{array}{ccc} C_{\phi(\cdot,\cdot,\nu)} & \xrightarrow{\ \kappa\ } & C_{\tilde{\phi}(\cdot,\cdot,\nu)} \\ & \searrow \quad \swarrow & \\ & \Lambda_\nu & \end{array}$$

commutes, where κ is a map $R_x^{n+1} \times R_\eta^{n+1} \ni (x, \eta) \mapsto (y(x), \eta) \in R_y^{n+1} \times R_\eta^{n+1}$.

The next step is to decrease the number of the fiber variables. From the above construction we may regard (t, x') and η as the local coordinates of Λ_ν respectively and from (5. 8) the mapping $x' \mapsto \eta'$ is regular at $(0, 0, \nu^0)$. Hence $\det \dfrac{\partial x'}{\partial \eta'}(Y') \neq 0$ so $\dfrac{\partial}{\partial \eta'} x'(H_{\eta'}) = \dfrac{\partial x'}{\partial y'} \dfrac{\partial H_{\eta'}}{\partial \eta'}$ is non-degenerate, whch implies that $\det \phi_{\eta'\eta'} = \det H_{\eta'\eta'} \neq 0$ at $\eta_0 = 0$, $\eta' = \nu^0$, $\nu = \nu^0$. By the implicit function theorem, there exists a C^∞-function $\eta'(x', \eta_0, \nu)$ which satisfies the equation

$$\phi_{\eta'}(x, \eta, \nu) = y' - H_{\eta'}(\eta, \nu) = 0 ,$$

where $y' = y'(x')$. Let $\alpha = \eta_0 = \xi_0$, ϕ_1 the C^∞-function

$$\phi_1(x, \alpha, \nu) = \phi\left(x, \alpha, \eta'(x', \alpha, \nu), \nu\right) . \tag{5. 10}$$

Then $\Lambda_\nu = \Lambda_{\phi_1(\cdot, \cdot, \nu)}$. In fact setting $\eta' = \eta'(x', \alpha, \nu)$ we have that $\phi_{1_\alpha}(x, \alpha, \nu) = 0$ if and only if $\phi_\eta(x, \alpha, \eta', \nu) = 0$, and $\phi_{1_x}(x, \alpha, \nu) = \phi_x(x, \alpha, \eta', \nu) + \phi_{\eta'}(x, \alpha, \eta', \nu) \dfrac{\partial \eta'}{\partial x'} = \phi_x(x, \alpha, \eta', \nu)$.

Now for fixed $\nu = \nu^0$, $x' = 0$, we shall examine the behavior of ϕ_1 with respect to (x_0, α). Put $N = \Lambda_{\nu^0} \cap \{(x_0, 0, \xi_0, 0)\}$. Since the (x_0, ξ_0)-components of the bicharacteristic strip of H_p satisfy the equations $\dot{x}_0 = 2\xi_0$, $\dot{\xi}_0 = -p_{x_0}$, there a C^∞-function $f(\xi_0)$ such that $N = \{(x_0, 0, \xi_0, 0) ;\ x_0 = f(\xi_0)\}$, where $f(0) = f'(0) = 0$, $f''(0) \gneqq 0$. On the other hand recalling that $\phi_{\eta'}(x_0, 0, \eta, \nu^0) = -H_{\eta'}(\eta, \nu^0)$, so $H_{\eta'}(\alpha, \eta'(0, \alpha, \nu^0), \nu^0) \equiv 0$, we obtain that

$$\phi_{1_\alpha}(x_0, 0, \alpha, \nu^0) = \frac{\partial}{\partial \alpha} \phi\left(x_0, 0, \alpha, \eta'(0, \alpha, \nu_0), \nu_0\right)$$

$$= x_0 - H_\alpha\left(\alpha, \eta'(0, \alpha, \nu^0), \nu^0\right) - H_{\eta'}\left(\alpha, \eta'(0, \alpha, \nu^0), \nu^0\right) \frac{\partial \eta'}{\partial \alpha}$$
$$= x_0 - H_\alpha\left(\alpha, \eta'(0, \alpha, \nu^0), \nu^0\right).$$

In view of (5.6), (5.9) and (5.10), $\phi_{1_{x_0}} = \alpha = \xi_0$. Hence $N = \{(x_0, 0, \phi_{1_{x_0}}, 0) = (x_0, 0, \xi_0, 0)\,;\ x_0 = H_\alpha(\xi_0, \eta'(0, \xi_0, \nu^0), \nu^0)\}$ which means that

$$\phi_{1_\alpha}(0, 0, \xi^0) = \phi_{1_{\alpha\alpha}}(0, 0, \xi^0) = 0, \ \phi_{1_{\alpha\alpha\alpha}}(0, 0, \xi^0) \gtrless 0.$$

By the Taylor expansion with respect to α we see that

$$\phi_1(x, \alpha, \nu) = \theta_1(x, \nu) + \rho_1(x, \nu)\,\alpha + \mu(x, \nu)\,\alpha^2 - \sigma(x, \alpha, \nu)\frac{\alpha^3}{3},$$

where $\rho_1(0, \nu^0) = 0$, $\rho_{1_{x_0}}(0, \nu^0) \gtrless 0$, $\mu(0, \nu^0) = 0$ and $\sigma(0, 0, \nu^0) \gtrless 0$.

Taking $\sigma^{\frac{1}{3}}(x, \alpha, \nu)\,\alpha$ as a new independant variable, which is also denoted by α, we have

LEMMA 5.2. *There exist open neighborhoods* $U \times U' \subset R_x^{n+1} \times S^{n-1}$ *of* $(0, \nu^0)$, $U \times I \times U' \subset R_x^{n+1} \times R_\alpha \times S^{n-1}$ *of* $(0, 0, \nu^0)$ *and* C^∞*-real functions* $\theta_1(x, \nu)$, $\rho_1(x, \nu)$, $\mu(x, \nu) \in C^\infty(U \times U')$ *and* $\tilde\alpha(x, \alpha, \nu) \in C^\infty(U \times I \times U')$ *such that, if* ϕ_1 *is defined by*

$$\phi_1(x, \alpha, \nu) = \theta_1(x, \nu) + \rho_1(x, \nu)\,\tilde\alpha(x, \alpha, \nu) + \mu(x, \nu)\,\tilde\alpha^2(x, \alpha, \nu) - \frac{\alpha^3}{3},$$

then

$$(5.11) \qquad \Lambda_\nu = \Lambda_{\phi_1(\cdot, \cdot, \nu)}.$$

Moreover $\rho_1(0, \nu^0) = 0$, $\rho_{1_{x_0}}(0, \nu^0) \gtrless 0$, $\mu(0, \nu^0) = 0$, $\tilde\alpha(x, 0, \nu) = 0$ *and* $\frac{\partial \tilde\alpha}{\partial \alpha}(x, 0, \nu) = \sigma^{-\frac{1}{3}}(x, 0, \nu) \gtrless 0$.

Finally we shall prove

PROPOSITION 5.3. *Let* ϕ_2 *be a* C^∞*-function*

$$\phi_2(x, \alpha, \nu) = \theta_0 + c x_0 \alpha - \frac{\alpha^3}{3},$$

where $c = \rho_{1_{x_0}}(0, \nu^0)\,\sigma^{-1/3}(0, 0, \nu^0)$, $\theta_0 = \theta_1(0, \nu^0)$.
Then there exist open neighborhoods $U \times U' \subset R^{n+1} \times S^{n-1}$ *of* $(0, \nu^0)$, $U \times I \times U' \subset R_x^{n+1} \times R_\alpha \times S^{n-1}$ *of* $(0, 0, \nu^0)$ *and* C^∞*-functions* $X(x, \nu)$, $\Theta(x, \nu) \in C^\infty(U \times U')$ *and* $A(x, \alpha, \nu) \in C^\infty(U \times I \times U')$ *such that*

$$(5.12) \qquad X(0, \nu^0) = 0, \ \det \frac{\partial X}{\partial x}(x, \nu) \neq 0,$$

$$(5.13) \qquad A(0, \alpha, \nu^0) = \alpha, \ \frac{\partial A}{\partial \alpha}(x, \alpha, \nu) \neq 0$$

and

$$\phi_1\Big(X(x,\nu),\, A(x,\alpha,\nu),\, \nu\Big) = \phi_2(x,\alpha,\nu) + \Theta(x,\nu) \tag{5.14}$$
$$= \theta_0 + \Theta(x,\nu) + c x_0 \alpha - \frac{\alpha^3}{3}.$$

Note that both ϕ_1 and ϕ_2 are $(n+1)+(n-1)$ dimensional unfoldings of the right 3-determined function $\theta_0 - \frac{\alpha^3}{3}$ and 3-transversal ([13]). If we apply general theory of the stability of unfoldings, we can construct C^∞-diffeomorphisms $(X(x,\nu), N(x,\nu))$, $A(x,\alpha,\nu)$, which satisfy that $\phi_1(X(x,\nu), A(x,\alpha,\nu), N(x,\nu)) = \phi_2(x,\alpha,\nu) + \Theta(x,\nu)$. However it is not clear that $X(x,\nu)$ is a C^∞-diffeomorphism with respect to x for a fixed ν (See (5.12)). Thus to prove Proposition 5.3, using the Malgrange's preparation theorem we show the following

LEMMA 5.4. *Let E be a C^∞-function*

$$E(x,\alpha,\nu,t) = t\phi_1(x,\alpha,\nu) + (1-t)\,\phi_2(x,\alpha,\nu). \tag{5.15}$$

Then there exist a C^∞-function $h(x,\alpha,\nu,t)$ defined in an open neighborhood of $(0,0,\nu^0,0) \in R_x^{n+1} \times R_\alpha \times S^{n-1} \times R_t$ and C^∞-functions $k(x,\nu,t)$, $l(x,\nu,t)$ defined in an open neighborhood of $(0,\nu^0,0) \in R_x^{n+1} \times S^{n-1} \times R_t$ such that

$$\frac{\partial E}{\partial t}(x,\alpha,\nu,t) = -h(x,\alpha,\nu,t)\frac{\partial E}{\partial \alpha} - k(x,\nu,t)\frac{\partial E}{\partial x_0} + l(x,\nu.t), \tag{5.16}$$

where

$$h(0,\alpha,\nu^0,t) \equiv 0 \ \textit{and}\ k(0,\nu^0,t) = l(0,\nu^0,0) \equiv 0. \tag{5.17}$$

PROOF. Remarking the facts that $\rho_1(0,\nu^0) = \mu(0,\nu^0) = 0$ and $\tilde{\alpha}(x,0,\nu) = 0$, we see that

$$\phi_1(x,\alpha,\nu) = \theta_1(x,\nu) + c x_0 \alpha + f(x,\alpha,\nu) - \frac{\alpha^3}{3}, \tag{5.18}$$

where f satisfies

$$\frac{\partial^2 f}{\partial\alpha\partial x_0}(0,0,\nu^0) = 0,\ \frac{\partial^j f}{\partial \alpha^j}(0,\alpha,\nu^0) \equiv 0, \qquad j = 1,2,3. \tag{5.19}$$

Setting $F = t(\theta_1 - \theta_0 + f)$ we obtain, from (5.15),

$$\frac{\partial E}{\partial \alpha} = c x_0 - \alpha^2 + \frac{\partial E}{\partial \alpha}, \tag{5.20}$$

$$\frac{\partial E}{\partial x_0} = c\alpha + \frac{\partial F}{\partial x_0}. \tag{5.21}$$

Note that $\partial E/\partial\alpha$ has regularity of order 2 at $(0, 0, \nu^0, 0)$ with respect to α. By the Malgrange's preparation theorm, we have

$$\frac{\partial E}{\partial t}=q(x, \alpha, \nu, t)\frac{\partial E}{\partial\alpha}+r_1(x, \nu, t)\,\alpha+r_0(x, \nu, t)\,, \tag{5.22}$$

$$\frac{\partial F}{\partial x_0}=q'(x, \alpha, v, t)\frac{\partial E}{\partial\alpha}+s_1(x, \nu, t)\,\alpha+s_0(x, \nu, t)\,. \tag{5.23}$$

Here q, q' are C^∞-functions defined in an open neighborhood of $(0, 0, \nu^0, 0)$ and r_1, r_0, s_1, s_0 are C^∞-functions defined in an open neighborhood of $(0, \nu^0, 0)$. Moreover from (5.19), (5.23) implies that $s_1(0, \nu^0, 0)=0$.

Here we remark that

$$\frac{\partial E}{\partial x_0}=(c+s_1)\,\alpha+q'\frac{\partial E}{\partial\alpha}+s_0\,. \tag{5.24}$$

Let

$$k(x, \nu, t)=\frac{-r_1(x, \nu, t)}{c+s_1(x, \nu, t)}\,,$$

$$h(x, \alpha, \nu, t)=-q(x, \alpha, \nu, t)-k(x, \nu, t)\,q'(x, \alpha, \nu, t) \quad \text{and}$$

$$l(x, \nu, t)=r_0(x, \nu, t)+k(x, \nu, t)\,s_0(x, \nu, t)\,,$$

then from (5.22), (5.24) we obtain (5.16). Since $\partial E/\partial t(0, \alpha, \nu^0, t)=\phi_1(0, \alpha, \nu^0)-\phi_2(0, \alpha, \nu^0)\equiv 0$, (5.19) implies that $r_0(0, \nu^0, t)=r_1(0, \nu^0, t)\equiv 0$ and $q(0, \alpha, \nu^0, t)\equiv 0$, which proves the lemma.

PROOF of PROPOSITION 5.3. Now we define C^∞-mapping $\tilde{E}$; $R_x^{n+1}\times R_\alpha\times S^{n-1}\times R_t\to R_x^{n+1}\times R_y\times S^{n-1}\times R_t$ as follows

$$\tilde{E}(x, \alpha, \nu, t)=\big(x, E(x, \alpha, \nu, t), \nu, t\big)\,. \tag{5.25}$$

Let X, Y be C^∞-vector fields on $R_x^{n+1}\times R_\alpha\times S^{n-1}\times R_t$, $R_x^{n+1}\times R_y\times S^{n-1}\times R_t$ respectively such that

$$\begin{cases} X=k(x, \nu, t)\dfrac{\partial}{\partial x_0}+h(x, \alpha, \nu, t)\dfrac{\partial}{\partial\alpha}+\dfrac{\partial}{\partial t}\,, \\ Y=k(x, \nu, t)\dfrac{\partial}{\partial x_0}+l(x, \nu, t)\dfrac{\partial}{\partial y}+\dfrac{\partial}{\partial t}\,. \end{cases} \tag{5.26}$$

From (5.16), (5.25) and (5.26),

$$D\tilde{E}(X)=Y\,, \tag{5.27}$$

where $D\tilde{E}$ is the differential of $\tilde{E}$.

Let $\varphi^s(x, \alpha, \nu, t)$ be the integral curve of X such that $\varphi^0(x, \alpha, \nu, t)=$

(x, α, ν, t), $\psi^s(x, y, \nu, t)$ the integral curve of Y such that $\psi^0(x, y, \nu, t)=(x, y, \nu, t)$. The coefficients of $\frac{\partial}{\partial x_0}$ and $\frac{\partial}{\partial y}$ of Y do not depend on y, and those of $\frac{\partial}{\partial x'}$ and $\frac{\partial}{\partial \nu}$ of Y are zero. Therefore $\psi^t(x, y, \nu, 0)$ is written in the form

$$\psi^t(x, y, \nu, 0) = \left(X_0(x, \nu, t), x', y+\Theta(x, \nu, t), \nu, t\right). \tag{5.28}$$

Noting that the coefficients of $\partial/\partial x_0$ of X and Y are the same we can write $\varphi^t(x, \alpha, \nu, 0)$ in the following form:

$$\varphi^t(x, \alpha, \nu, 0) = \left(X_0(x, \nu, t), x', A(x, \alpha, \nu, t), \nu, t\right). \tag{5.29}$$

From (5.27), (5.28) and (5.29) we see

$$\begin{aligned}
&\left(X_0(x, \nu, t), x', E(x, \alpha, \nu, 0)+\Theta(x, \nu, t), \nu, t\right)\\
&\quad= \psi^t\left(x, E(x, \alpha, \nu, 0), \nu, 0\right)\\
&\quad= \tilde{E}\left(X_0(x, \nu, t), x', A(x, \alpha, \nu, t), \nu, t\right)\\
&\quad= \left(X_0(x, \nu, t), x', E\left(X_0(x, \nu, t), x', A(x, \nu, t), \nu, t\right), \nu, t\right).
\end{aligned} \tag{5.30}$$

Let $\phi^t(x, \alpha, \nu)=E(x, \alpha, \nu, t)=t\phi_1+(1-t)\phi_2$, $A^t(x, \alpha, \nu)=A(x, \alpha, \nu, t)$, $\Theta^t(x, \nu)=\Theta(x, \nu, t)$ and $X^t(x, \nu)=(X_0(x, \nu, t), x')$. Then $\phi^0=\phi_2$ and from (5.30)

$$\phi_2(x, \alpha, \nu)+\Theta^t(x, \nu) = \phi^t\left(X^t(x, \nu), A^t(x, \alpha, \nu), \nu\right).$$

If $t>0$ is sufficiently small, X^t and A^t are C^∞-diffeomorphisms of x and α respectively. By the same argument, substituting ϕ^{t_1}, ϕ^{t_2} for ϕ_1, ϕ_2 respectively, we can prove Lemma 5.4 for any $0\leqq t_1<t_2\leqq 1$, and so construct C^∞-diffeomorphisms which are joining ϕ^{t_1} and ϕ^{t_2} if t_2-t_1 is sufficiently small. Since the interval $[0, 1]$ is compact, composing such the finite C^∞-diffeomorphisms we obtain (5, 14).

Proof of Theorem 5.1. Let $\tilde{X}(x, \nu)=(\tilde{X}_0(x, \nu), x')$ be the inverse mapping of $X(x, \nu)$, ϕ the function

$$\begin{aligned}
\phi(x, \alpha, \nu) &= \theta_0+\Theta\left(\tilde{X}(x, \nu), \nu\right)+c\tilde{X}_0(x, \nu)\,\alpha-\frac{\alpha^3}{3}\\
&= \theta(x, \nu)+\rho(x, \nu)\,\alpha-\frac{\alpha^3}{3},
\end{aligned}$$

where $\theta(x, \nu)=\theta_0+\Theta(\tilde{X}(x, \nu), \nu)$, $\rho(x, \nu)=c\tilde{X}_0(x, \nu)$. Then $\Lambda_\nu=\Lambda_{\phi_1}=\Lambda_{\phi_2+\Theta}=\Lambda_\phi$. Furthermore if $\phi_\alpha=\rho-\alpha^2=0$, $\phi_x=\theta_x\pm\sqrt{\rho}\,\rho_x$, hence setting $\varphi_\pm=\theta\pm\frac{2}{3}\rho^{\frac{3}{2}}$ we see that $\Lambda_\nu=\{(x, \varphi_{\pm x})\}\subset V$. If $\theta(x, \xi')=|\xi'|\theta(x, \xi'/|\xi'|)$, $\rho(x, \xi')=|\xi'|^{\frac{2}{3}}\rho(x, \xi'/|\xi'|)$, *then* $\varphi_\pm(x, \xi')=\theta(x, \xi')\pm\frac{2}{3}\rho^{\frac{3}{2}}(x, \xi')$ solves the equation (5. 4). Finally note that if $x_0+\zeta(x', \nu)=0$, $(x, \varphi_{\pm x})\in L_\nu$. Since $\rho_{x_0}(0, \nu^0)\neq 0$, it follows that $\rho=0$ and $\theta_{x'}=\nu$ at $x_0=\zeta(x', \nu)$, $\xi'=\nu$, which proves (5. 2). (5. 3). This completes the proof.

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