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On Amitsur cohomology of rings of algebraic integers

Dedicated to Professor G. Azumaya on his 60th birthday

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In connection with the study of Azumaya algebras over rings, we introduced in [4] certain Amitsur-type cohomology groups $\boldsymbol{H}^{q}(S/R)$ for an extension S/R of commutative rings. The present article is a supplement to that paper, and deals with special features of groups $\boldsymbol{H}^{q}(S/R)$ in arithmetical context.

As is the case for the groups $\boldsymbol{H}^{q}(S, G)$ of group cohomology-type [3], [6], we can apply the device of mapping cones to the construction of groups $\boldsymbol{H}^{q}(S/R)$, thus dispensing with the intermediary of the whole category of invertible modules. This is done in § 1, based upon the general foundations in [6] § 1. In § 2 we deal with local fields, and in § 3 global fields, where we proceed almost parallel to [3] § 6. Parallel though they are, the results are not the same since, roughly speaking, $\boldsymbol{H}^{q}(S/R)$ almost ignores the ramification, while $\boldsymbol{H}^{q}(S, G)$ is essentially involved with it. The relationship between these two series of cohomology groups is studied to some extent in [5], but remains to be further clarified. As an example, we show in the final § 4 that for the integer rings of imaginary quadratic fields the unit-valued Amitsur cohomology vanishes in every dimension.

§ 1. Groups $\boldsymbol{H}^{q}(S/R)$ via mapping cone

1.1. Let R be a commutative ring (with unity). Let F be a covariant functor from the category of commutative R-algebras to the category of abelian groups. Denote the Amitsur's complex of F concerning an R-algebra S by $\mathrm{Am}(S/R, F)$, and its cohomology groups by $H^{q}(S/R, F)$. (F need not be meaningful on the whole category of R-algebras. It is only required that R is defined in a subcategory sufficient to work with Amitsur cohomology.) A morphism of functors $f: F\rightarrow F'$ yields a complex morphism $\mathrm{Am}(S/R, F)\rightarrow \mathrm{Am}(S/R, F')$. We denote the *mapping cone* of this morphism by $\mathrm{Am}(S/R, f)$. Hence we have an exact sequence

$$0\longrightarrow \mathrm{Am}(S/R, F')\longrightarrow \mathrm{Am}(S/R, f)\longrightarrow \mathrm{Am}(S/R, F)_{\#}\longrightarrow 0$$

where $C_{\#}$ for a cochain complex C is defined by $C_{\#}^{q}=C^{q+1}$, $d_{\#}^{q}=-d^{q+1}$.

General constructions about mapping cones are carried out in § 1 of [6], and are applied in § 2 to the group cohomology. This time, we adapt it to Amitsur cohomology and will quickly summarize the main facts in the following lines.

We denote the cohomology of the complex $\mathrm{Am}(S/R,f)$ by $H^q(S/R,f)$. Then the above exact sequence of complexes leads to the *first exact sequence* associated to f:

$$\cdots\longrightarrow H^q(S/R,F)\longrightarrow H^q(S/R,F') \longrightarrow H^q(S/R,f)\longrightarrow H^{q+1}(S/R,F)\longrightarrow\cdots \tag{1.1}$$

Associated to $f: F\to F'$, we have two functors $\ker f$ and $\operatorname{coker} f$, together constituting an exact sequence of functors:

$$0\longrightarrow \ker f\longrightarrow F\overset{f}{\longrightarrow} F'\longrightarrow \operatorname{coker} f\longrightarrow 0 \tag{1.2}$$

This gives rise to another exact sequence due to MacLane [8]:

$$\cdots\longrightarrow H^q(S/R,\ker f)\longrightarrow H^{q-1}(S/R,f) \longrightarrow H^{q-1}(S/R,\operatorname{coker} f)\longrightarrow H^{q+1}(S/R,\ker f)\longrightarrow\cdots \tag{1.3}$$

which we call the *second exact sequence* associated to f.

Let

$$\begin{array}{ccc} F & \xrightarrow{f} & F' \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi'} \\ G & \xrightarrow{g} & G' \end{array}$$

be a commutative square of functor morphisms. It gives rise to a commutative square of their Amitsur complexes

$$\begin{array}{ccc} \mathrm{Am}(S/R,F) & \longrightarrow & \mathrm{Am}(S/R,F') \\ \downarrow & & \downarrow \\ \mathrm{Am}(S/R,G) & \longrightarrow & \mathrm{Am}(S/R,G') \end{array} \tag{1.4}$$

Following § 1. 2 of [6], we obtain the first exact sequence associated to $\{\varphi,\varphi'\}$:

$$\cdots\longrightarrow H^q(Y)\longrightarrow H^q(S/R,f)\longrightarrow H^q(S/R,g)\longrightarrow H^{q+1}(Y)\longrightarrow\cdots \tag{1.5}$$

under suitable conditions, where Y is the *center* of the square (1. 4).

Let φ: $S/R\to S'/R'$ be a morphism of algebras ([4], § 6), and assume that the following square

$$
\begin{array}{ccc}
\mathrm{Am}\,(S/R, F) & \longrightarrow & \mathrm{Am}\,(S/R, F') \\
\downarrow & & \downarrow \\
\mathrm{Am}\,(S'/R', F) & \longrightarrow & \mathrm{Am}\,(S'/R', F')
\end{array} \tag{1.6}
$$

is meaningful and commutative, then we have the following exact sequence for change of rings:

$$
\cdots \longrightarrow H^q(Y) \longrightarrow H^q(S/R, f) \longrightarrow H^q(S'/R', f) \longrightarrow H^{q+1}(Y) \longrightarrow \cdots \tag{1.7}
$$

under suitable conditions, where Y is the center of the square (1. 6).

1. 2. Now we assume that R is an integral domain with the field of quotients k, and restrict our attention to the category of R-faithfully flat R-orders Λ in finite dimensional commutative separable k-algebras A. The q-fold tensor product $\Lambda^q = \Lambda \otimes_R \cdots \otimes_R \Lambda$ is an order in the separable algebra $A^q = A \otimes_k \cdots \otimes_k A$, and the machinary of the Amitsur cohomology can be set up in this category. Let U be the functor of units: $\Lambda \to U(\Lambda)$, while U_k be the functor $\Lambda \to U(\Lambda \otimes k) = U(A) = A^*$. Let further $I(\Lambda)$ be the group of invertible Λ-ideals in $A = \Lambda \otimes k$. Then the assignment $a \in A^* \to (a) = a\Lambda \in I(\Lambda)$ defines a morphism of functors pr : $U_k \to I$, with the kernel U and the cokernel denoted Pic, so that we have an exact sequence

$$
0 \longrightarrow U \longrightarrow U_k \xrightarrow{\mathrm{pr}} I \longrightarrow \mathrm{Pic} \longrightarrow 0 \tag{1.8}
$$

We shall apply the construction of § 1. 1 to (1. 8), and obtain the following facts, which are parallel to the case of group cohomology of [6] § 3.

We introduce the notation

$$
\boldsymbol{H}^q(\Lambda/R) = H^{q-1}(\Lambda/R, \mathrm{pr})
$$

Then we have the *first exact sequence*

$$
\begin{aligned}
\cdots \longrightarrow H^q(A/k, U) \longrightarrow H^q(\Lambda/R, I) \longrightarrow \boldsymbol{H}^{q+1}(\Lambda/R) \\
\longrightarrow H^{q+1}(A/k, U) \longrightarrow \cdots
\end{aligned} \tag{1.9}
$$

and the *second exact sequence*

$$
\begin{aligned}
\cdots \longrightarrow H^q(\Lambda/R, U) \longrightarrow \boldsymbol{H}^q(\Lambda/R) \longrightarrow H^{q-1}(\Lambda/R, \mathrm{Pic}) \\
\longrightarrow H^{q+1}(\Lambda/R, U) \longrightarrow \cdots
\end{aligned} \tag{1.10}
$$

The latter is the exact sequence of [4] Theorem 1. 1.

Noticing that we are assuming that Λ is R-faithfully flat, we easily observe that

$$
\boldsymbol{H}^0(\Lambda/R) \simeq U(R), \tag{1.11}
$$

(which is [4] Proposition 2. 1). For $q=1$, we have

(1. 12) $\quad \boldsymbol{H}^1(\Lambda/R) \simeq \mathrm{Pic}\,(R)$

([4] Theorem 2. 2). This follows from the first exact sequence:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \boldsymbol{H}^0(\Lambda/R) & \longrightarrow & H^0(A/k, U) & \longrightarrow & H^0(\Lambda/R, I) & \longrightarrow & \boldsymbol{H}^1(\Lambda/R) & \longrightarrow & H^1(A/k, U) \\ & & \wr\wr & & \wr\wr & & \wr\wr & & & & \| \\ 0 & \longrightarrow & U(R) & \longrightarrow & k^* & \longrightarrow & I(R) & & & & 0 \end{array}$$

We quote the following isomorphism without proof ([4] Theorem 5. 2):

(1. 13) $\quad \boldsymbol{H}^2(\Lambda/R) \simeq \mathrm{Br}\,(\Lambda/R)\,,$

which holds when Λ is R-faithfully projective (cf. also [14]).

Let $\varphi : \Lambda/R \rightarrow \Lambda'/R'$ be a morphism of algebras, and assume the U-injectivity and Pic-surjectivity for every $\Lambda^r \rightarrow \Lambda'^r$ $(r=1, 2, \cdots)$ (cf. [4] § 6). Then the exact sequence for the change of rings (1. 7) reads in the present context as follows:

(1. 14) $\quad \cdots \longrightarrow H^{q-1}(Y) \longrightarrow \boldsymbol{H}^q(\Lambda/R) \longrightarrow \boldsymbol{H}^q(\Lambda'/R') \longrightarrow H^q(Y) \longrightarrow \cdots$

([4] Theorem 6. 1). Here, Y is the center of the square

$$\begin{array}{ccc} \mathrm{Am}\,(A/k, U) & \longrightarrow & \mathrm{Am}\,(\Lambda/R, I) \\ \downarrow & & \downarrow \\ \mathrm{Am}\,(A'/k', U) & \longrightarrow & \mathrm{Am}(\Lambda'/R', I) \end{array}$$

where k' is the field of quotients of R', and $A'=\Lambda'\otimes k'$. Explicitly, Y^q consists of $[P, \alpha']$ $(P \in I(\Lambda^{q+1}),\ \alpha' \in U(A'^{q+1}))$ such that $P \otimes_{\Lambda^{q+1}} \Lambda'^{q+1} = \alpha' \Lambda'^{q+1}$, where $[P, \alpha']$ and $[Q, \beta']$ are identified if their 'difference' equals to $[\alpha \Lambda^{q+1}, \varphi(\alpha)]$ with some $\alpha \in A^{q+1}$. The boundary is defined by $d[P, \alpha'] = [dP, d\alpha']$. This agrees with the formulation in [4] (where Y is denoted as $\mathrm{Am}\,(\varphi, \mathrm{Pic})$).

§ 2. Local fields

We begin with

PROPOSITION 2. 1. *If F is a C_1-field, then we have $H^q(A/F, U)=0$ $(q \geq 1)$ for any finite dimensional primary algebra A.*

PROOF. Let A' be the residue class algebra of A by the radical. Then we have $H^q(A/F, U) \simeq H^q(A'/F, U)$ by [12] Proposition 3. 3. So we assume that A is an extension field of F. If K/F is the maximal purely inseparable subextension of A/F, we have the following exact sequence

$$(*) \qquad \begin{array}{l} \cdots \longrightarrow H^{q-1}(A/K, U) \longrightarrow H^q(K/F, U) \longrightarrow H^q(A/F, U) \\ \qquad\qquad\qquad\qquad\qquad\qquad \longrightarrow H^q(A/K, U) \longrightarrow \cdots \end{array}$$

([12] Theorem 4. 3). Now, $H^q(K/F, U)=0$ for $q\neq 2$ by Berkson's theorem, and $H^2(K/F, U)\simeq \mathrm{Br}(K/F)$ also vanishes by the C_1-assumption. Hence $H^q(K/F, U)$ vanishes for every $q\geq 1$. Next we consider the separable extension A/K. Let L be a finite Galois extension of K, containing A. Let $G=\mathrm{Gal}(L/K)$ and $H=\mathrm{Gal}(L/A)$. Then $H^q(A/K, U)$ is isomorphic to the relative Galois cohomology group $H^q([G:H], L^*)$ by [11] Theorem 1. Now we have $H^q(D, L^*)=0$ for every subgroup D of G and $q\geq 1$, since the fixed subfield of D, being a finite extension of F, is likewise a C_1-field (cf. e. g. [13] Chap. IV § 3). In this case, there is an exact sequence [1]:

$$0\longrightarrow H^q\big([G:H], L^*\big)\longrightarrow H^q(G, L^*)\longrightarrow H^q(H, L^*)$$

from which follows that $H^q([G:H], L^*)=0$, *i.e.* $H^q(A/K, U)=0$ $(q\geq 1)$. Applying these facts to the above exact sequence (*), we obtain $H^q(A/F, U)=0$ $(q\geq 1)$.

Conjecture. The triviality of $H^q(A/F, U)$ will hold for all A without the assumption of primary-ness.

We apply the above Proposition to prove the following result.

PROPOSITION 2. 2. *Let $\mathfrak{o}$ be a complete discrete valuation ring, k its field of quotients. Let K/k be a finite extension, and $\mathfrak{O}$ the integral closure of $\mathfrak{o}$ in K. Assume that the residue field $F=\mathfrak{o}/\mathfrak{p}$ is a C_1-field· Then the Amitsur cohomology $H^q(\mathfrak{O}/\mathfrak{o}, U)$ vanishes for $q\geq 1$.*

PROOF. Put $A=\mathfrak{O}/\mathfrak{p}\mathfrak{O}$, which is a finite dimensional primary algebra over F. Let t be a uniformizing parameter in k: $\mathfrak{p}=(t)$, and introduce a filtration of $U(\mathfrak{O})$ by

$$V^{(j)}(\mathfrak{O})=\big\{u\in U(\mathfrak{O})\big|u\equiv 1 \bmod t^j\mathfrak{O}\big\}, \qquad j=0, 1, 2, \cdots$$

Putting $W^{(j)}(\mathfrak{O})=V^{(j)}(\mathfrak{O})/V^{(j+1)}(\mathfrak{O})$, we have

$$W^{(j)}(\mathfrak{O})\simeq\begin{cases} U(A) & (j=0)\\ A \text{ (additive group)} & (j>0)\end{cases}$$

Apply a similar construction to $\mathfrak{O}^q$ for every $q\geq 1$, and we have a series of subgroups of $U(\mathfrak{O}^q)$:

$$V^{(j)}(\mathfrak{O}^q)=\big\{u\in U(\mathfrak{O}^q)\big|u\equiv 1^q \bmod t^j\mathfrak{O}^q\big\}$$

(where 1^q is the identity of $\mathfrak{O}^q$), and isomorphisms

$$W^{(j)}(\mathfrak{O}^q)=V^{(j)}(\mathfrak{O}^q)/V^{(j+1)}(\mathfrak{O}^q)\simeq\begin{cases} U(A^q) & (j=0)\\ A^q & (j\geq 0)\end{cases}$$

One immediately verifies that these yield isomorphisms of Amitsur's complexes:

$$\mathrm{Am}\,(\mathfrak{O}/\mathfrak{o}, W^{(j)}) \simeq \begin{cases} \mathrm{Am}\,(A/F, U) & (j=0) \\ \mathrm{Am}\,(A/F, \text{additive}) & (j>0) \end{cases}$$

Now the preceding proposition shows that $H^q(A/F, U)=0$ for $q\geq 1$, while $H^q(A/F, \text{additive})=0$ as is well known. It follows that if $u\in U(\mathfrak{O}^{q+1})$ is such that $du=1^{q+2}$, we can successively find $v_j\in V^j(\mathfrak{O}^{q+1})$, $j=0, 1, 2, \cdots$, so that $u\equiv d(v_0\cdots v_j) \bmod V^{(j+1)}$. Clearly $\{v_0, v_0 v_1, \cdots\}$ is a Cauchy sequence, and the limit $v=\lim v_0\cdots v_j$ satisfies $u=dv$, q. e. d.

§ 3. Global fileds

3.1. Let k be an algebraic number field, and R the ring of integers. Let A be a finite dimensional commutative separable algebra over k, and Λ an R-order in A. $k_{\mathfrak{p}}$ denotes the completion of k at a prime $\mathfrak{p}$, $R_{\mathfrak{p}}$ the closure of R in $k_{\mathfrak{p}}$, $A_{\mathfrak{p}}=A\otimes_k k_{\mathfrak{p}}$ and $\Lambda_{\mathfrak{p}}=\Lambda\otimes_R R_{\mathfrak{p}}$ which is an $R_{\mathfrak{p}}$-order in $A_{\mathfrak{p}}$. (For an archimedean $\mathfrak{p}$, we put $R_{\mathfrak{p}}=k_{\mathfrak{p}}$, and $\Lambda_{\mathfrak{p}}=A_{\mathfrak{p}}$.) Let $J(A)$ be the idele group of A, which is defined as the restricted direct product of $U(A_{\mathfrak{p}})$ with respect to $U(\Lambda_{\mathfrak{p}})$, the local unit groups. $J(A)$ is independent of the specified order Λ.

An idele $\boldsymbol{a}=(\cdots, a_{\mathfrak{p}}, \cdots)$ determines an invertible Λ-ideal P of A such that $P_{\mathfrak{p}}=a_{\mathfrak{p}}\Lambda_{\mathfrak{p}}$ for every $\mathfrak{p}$. Following Fröhlich we denote this ideal P as $\boldsymbol{a}\Lambda$. The map $\boldsymbol{a}\mapsto\boldsymbol{a}\Lambda$ defines an epimorphism from $J(A)$ to the group $I(\Lambda)$ of invertible Λ-ideals, and the kernel is the group of unit ideles $UJ(\Lambda)=\prod_{\mathfrak{p}} U(\Lambda_{\mathfrak{p}})$. Thus we have an exact sequence

$$0\longrightarrow UJ(\Lambda)\longrightarrow J(A)\xrightarrow{w} I(\Lambda)\longrightarrow 0 \tag{3.1}$$

(cf. [2] Theorem 1).

Denote by A^q (resp. Λ^q) the q-fold tensor product $A\otimes_k\cdots\otimes_k A$ (resp. $\Lambda\otimes_R\cdots\otimes_R\Lambda$). Λ^q is an R-order in the separable algebra A^q, and (3.1) gives rise to a series of exact sequences

$$0\longrightarrow UJ(\Lambda^q)\longrightarrow J(A^q)\longrightarrow I(\Lambda^q)\longrightarrow 0 \qquad (q\geq 1)\,. \tag{3.2}$$

As is easily observed, the aggregate of these sequences may be interpreted as an exact sequence of Amitsur's complexes. It yields a long exact sequence

$$\begin{aligned}\cdots\longrightarrow H^q(\Lambda/R, UJ)\longrightarrow H^q(A/k, J)\xrightarrow{w^q} H^q(\Lambda/R, I)\\ \longrightarrow H^{q+1}(\Lambda/R, UJ)\longrightarrow\cdots\end{aligned} \tag{3.3}$$

PROPOSITION 3.1. *Let K be a finite extension field of k, and denote by r the number of infinite primes of k which ramify in the extension K/k. For an R-order S in K, we have*

$$H^1(S/R, UJ)=0$$

$$H^2(S/R, UJ)\simeq(\boldsymbol{Z}/2\boldsymbol{Z})^r$$

PROOF. For a finite prime $\mathfrak{p}$, the $\mathfrak{p}$-component $H^q(S_\mathfrak{p}/R_\mathfrak{p}, U)$ of $H^q(S/R, UJ)$ vanishes for $q=1$, 2. This is clear for $q=1$, since $H^1(S_\mathfrak{p}/R_\mathfrak{p}, U)\simeq \mathrm{Pic}\,(S_\mathfrak{p}/R_\mathfrak{p})$ [7] and $R_\mathfrak{p}$ is local. For $q=2$, we argue as follows. Put $F=R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ and $\mathfrak{A}=S_\mathfrak{p}/\mathfrak{p}S_\mathfrak{p}$. Then F is a finite field and $\mathfrak{A}$ is a finite dimensional F-algebra. Hence we have $H^2(\mathfrak{A}/F, U)\simeq \mathrm{Br}\,(\mathfrak{A}/F)$ [11], and this vanishes clearly. Then, applying the standard arguments as in the proof of Proposition 2.2, we obtain $H^2(S_\mathfrak{p}/R_\mathfrak{p}, U)=0$. Therefore we have $H^q(S/R, UJ)\simeq \prod_{\mathfrak{p}\ \text{infinite}} H^q(S_\mathfrak{p}/R_\mathfrak{p}, U)$ for $q=1, 2$. Since $H^1(\boldsymbol{C}/\boldsymbol{R}, U)=0$ and $H^2(\boldsymbol{C}/\boldsymbol{R}, U)\simeq \mathrm{Br}\,(\boldsymbol{C}/\boldsymbol{R})\simeq \boldsymbol{Z}/2\boldsymbol{Z}$, we have the results as described in the Proposition.

PROPOSITION 3.2. $H^1(S/R, I)=0$.

PROOF. We make use of the exactness of (3.3). First, we have $H^1(K/k, J)=0$. This follows immediately from the local triviality $H^1(K_\mathfrak{p}/k_\mathfrak{p}, U)=0$, $H^1(S_\mathfrak{p}/R_\mathfrak{p}, U)=0$. Next, any $P\in I(S^2)$ can be represented as $\boldsymbol{a}S^2$ with $\boldsymbol{a}$ such that $a_\mathfrak{p}=1$ for every infinite prime $\mathfrak{p}$. Then $d\boldsymbol{a}$ satisfies the same condition. Since $H^2(S_\mathfrak{p}/R_\mathfrak{p}, U)=0$ for finite primes, this means that the map $H^1(S/R, I)\to H^2(S/R, UJ)$ in (3.3) is a 0-map. It follows that $H^1(S/R, I)=0$.

3.2. Let $C(A)$ be the idele class group $J(A)/U(A)$. Then we have the following *basic diagram*, analogous to the one in [6] § 4.

PROPOSITION 3.3. *There are homomorphisms* ω^q: $H^q(A/k, C)\to \boldsymbol{H}^{q+1}(\Lambda/R)$ $(q=0, 1, 2, \cdots)$ *such that the following diagram is commutative*:

$$\begin{array}{ccccccccc}
\cdots\longrightarrow & H^q(A/k, U) & \longrightarrow & H^q(A/k, J) & \longrightarrow & H^q(A/k, C) & \longrightarrow & H^{q+1}(A/k, U) & \longrightarrow\cdots \\
 & \| & & \downarrow w^q & & \downarrow \omega^q & & \| & \\
\cdots\longrightarrow & H^q(A/k, U) & \longrightarrow & H^q(\Lambda/R, I) & \longrightarrow & \boldsymbol{H}^{q+1}(\Lambda/R) & \longrightarrow & H^{q+1}(A/k, U) & \longrightarrow\cdots
\end{array}$$

where the upper exact sequence is derived from the exact sequence $0\to U\to J\to C\to 0$, and the lower one is the first exact sequence.

PROOF. This is an immediate consequence of the naturality of the first exact sequence applied to the commutative diagram

$$\begin{array}{ccc}
U(A^q) & \longrightarrow & J(A^q) \\
\| & & \downarrow \\
U(\Lambda^q\otimes k) & \xrightarrow{\ \mathrm{pr}\ } & I(\Lambda^q)
\end{array}$$

Consider the most important case $q=1$, again assuming that $A=K$ is an extension field of k and $\Lambda=S$ is an R-order in K. Since $H^2(K/k, U)\simeq \mathrm{Br}(K/k)$ by Amitsur, and $\boldsymbol{H}^2(S/R)\simeq \mathrm{Br}(S/R)$ by [14], [4], we have the following diagram:

$$\begin{array}{ccccccccc} & & 0 & \longrightarrow & \mathrm{Br}(K/k) & \longrightarrow & H^2(K/k, J) & \xrightarrow{\ j\ } & H^2(K/k, C) \\ & & \downarrow & & \| & & \downarrow{\scriptstyle w^2} & & \downarrow{\scriptstyle \omega^2} \\ 0 & \longrightarrow & \mathrm{Br}(S/R) & \longrightarrow & \mathrm{Br}(K/k) & \longrightarrow & H^2(S/R, I) & \longrightarrow & \boldsymbol{H}^3(S/R) \end{array}$$

(The left-most 0 is due to Proposition 3. 2.) The situation is quite similar as in [3] § 6, and $\mathrm{Br}(S/R)$ is isomorphic to a subgroup of $H^2(K/k, J)$ which is the simultaneous kernel of maps denoted j and w^2 respectively in the above diagram. For convenience, we assume that the extension K/k is normal. Now, $\ker w^2$ is isomorphic to $(\boldsymbol{Z}/2\boldsymbol{Z})^r$ by the considerations of § 3. 1, and each element of it is described in terms of a system of Hasse invariants taking value in $\boldsymbol{Z}/2\boldsymbol{Z}$ at real infinite primes of k. Since the map j is given by the sum of local invariants, we have the following result.

PROPOSITION 3. 4. *Let K/k be a finite Galois extension, and r the number of infinite primes of k which ramify in the extension K/k.*

1) *For an R-order S in K, the map $\mathrm{Br}(S/R)\to\mathrm{Br}(K/k)$ is injective, and*

2) $\mathrm{Br}(S/R)\begin{cases} =0 & (r=0) \\ \simeq (Z/2\boldsymbol{Z})^{r-1} & (r\geq 1)\end{cases}$

It follows that, if K is totally imaginary, $\mathrm{Br}(S/R)$ is the whole $\mathrm{Br}(R)$, and we obtain the structure theorem for $\mathrm{Br}(R)$ as is described in [10] Theorem 6. 36.

§ 4. Imaginary quadratic fields.

PROPOSITION 4. 1. *For the integer ring R of an imaginary quadratic field k, $H^q(R/\boldsymbol{Z}, U)$ vanishes for every $q\geq 1$.*

COROLLARY 4. 2. *Under the same assumption, we have $\boldsymbol{H}^q(R/\boldsymbol{Z})\simeq H^{q-1}(R/\boldsymbol{Z}, \mathrm{Pic})$ for every $q\geq 1$.*

This is an immediate consequence of the exactness of (1. 10). For $q=1, 2$, these groups are trivial. If the conjecture of § 2 is confirmed, we will obtain the triviality of these groups for all $q\geq 1$.

PROOF. Let $k=\boldsymbol{Q}(\sqrt{m})$, where m is a square-free negative integer. As is well known, we have

$$U(R)=\begin{cases}\{\pm 1\} & (m\neq -1,\ -3)\\ \{\pm 1,\ \pm i\} & (m=-1)\\ \{\pm 1,\ \pm\omega,\ \pm\omega^2\} & (m=-3)\end{cases}$$

where ω is a cubic root of unity $(\neq 1)$.

LEMMA. $U(R^q)=\{e_1\otimes\cdots\otimes e_q | e_i\in U(R)\}$.

PROOF OF LEMMA. First, let $q=2$. There is an embedding $R^2\to R\times R$, denoted $\alpha\mapsto(\alpha_1, \alpha_2)$, defined by the map $a_1\otimes a_2\mapsto(a_1 a_2, a_1\bar{a}_2)$, where $\bar{a}$ denotes the complex conjugate of a. We consider the case $m=-3$. Then $\{1, \omega\}$ is an integral basis of R. For $\alpha\in U(R^2)$, $\varepsilon=\alpha_2\alpha_1^{-1}$ must satisfy $1-\varepsilon\equiv 0$ (mod $\mathfrak{d}$), where $\mathfrak{d}$ is the different, generated by $\omega-\omega^2$. Hence ε is one of 1, ω, ω^2. If $\alpha=a\otimes 1+b\otimes\omega\in U(R^2)$, then $\alpha_1=a+b\omega$ $\alpha_2=a+b\omega^2$, and we easily verify that

if $\varepsilon=1$, then $\alpha=a\otimes 1$, $a\in U(R)$,

if $\varepsilon=\omega$, then $\alpha=b\otimes\omega$, $b\in U(R)$, and

if $\varepsilon=\omega^2$, then $\alpha=a\otimes 1+a\otimes\omega=(-a)\otimes\omega^2$, $-a\in U(R)$.

This proves the assertion for $m=-3$. Other cases are easier and we omit the verification of these cases. Now we proceed by induction on q. Let

$$\varphi_q:\ R^q\longrightarrow R^{q-1}\times R^{q-1}\ ;\ \alpha\longmapsto(\alpha_1, \alpha_2)$$

be defined by $a_1\otimes a_2\otimes\beta\mapsto(a_1 a_2\otimes\beta,\ a_1\bar{a}_2\otimes\beta)$ (where $q\geq 3$, and a_1, $a_2\in R$, $\beta\in R^{q-2}$). We observe that

(4.1) $\varphi_{q+1}(\alpha\otimes a)=(\alpha_1\otimes a, \alpha_2\otimes a)$ (where $q\geq 2$, and $\alpha\in R^q$, $a\in R$)

Again we consider the case $m=-3$. Then, our task is to show the following fact:

(4.2) If $\varepsilon=\alpha\otimes 1+\beta\otimes\omega\in U(R^{q+1})$, then $\beta=0$, or $\alpha=0$, or $\alpha=\beta$.

By (4.1), we have

$$\varphi_{q+1}(\varepsilon)=(\alpha_1\otimes 1+\beta_1\otimes\omega,\ \alpha_2\otimes 1+\beta_2\otimes\omega)$$

We may assume that (4.2) holds for these two units of R^q in the right hand side. If $\beta_1=0$, then β_2 is a non-unit since $\beta_1-\beta_2\in\mathfrak{d}\otimes R^{q-1}$. Hence $\alpha_2\neq 0$. The case $\alpha_2=\beta_2$ is also excluded, since this should imply that β_2 is a unit. Hence, only the case $\beta_2=0$ remains. This means that $\beta=0$. Similarly, if $\alpha_1=0$, then we have $\alpha=0$. Further, if $\alpha_1=\alpha_2$ we have $\alpha_2=\beta_2$, which means $\alpha=\beta$. Thus (4.2) is verified, and the Lemma is true for $m=-3$. Other values of m can be dealt with more easily.

Now return to the proof of Proposition. Write $u \in U(R^q)$ as $u = \pm e_1 \otimes \cdots \otimes e_q$, $e_i \in U'(R)$, where $U'(R) = \{1\}$, $\{1, i\}$, $\{1, \omega, \omega^2\}$ according as $m \neq -1, -3$, $m=-1$, $m=-3$. Then

$$du = \begin{cases} \pm 1 \otimes e_1 e_2 \otimes 1 \otimes e_3 e_4 \otimes \cdots \otimes e_{q-1} e_q \otimes 1 & (q \text{ even}) \\ e_1^{-1} \otimes e_1 \otimes e_3^{-1} \otimes e_3 \otimes \cdots \otimes e_q^{-1} \otimes e_q & (q \text{ odd}) \end{cases}$$

Noticing that $e_1 \otimes \cdots \otimes e_q = e_1' \otimes \cdots \otimes e_q'$ for $e_i, e_i' \in U'(R)$ implies $e_i = e_i'$ for every i, we immediately observe that any u such that $du=1$ can be expressed as $u=dv$, q. e. d.

Remark. Morris [9] shows that $H^2(R/\boldsymbol{Z}, U)=0$ even when R is the integer ring of a real quadratic field. We can not yet determine the structure of $H^q(R/\boldsymbol{Z}, U)$ for general q in this case. This is due to the fact that $U(R^q)$ is no longer finite, and the analogy to the above Lemma fails in this case.

Example. $R = \boldsymbol{Z}[\sqrt{2}]$, where $\varepsilon = 1+\sqrt{2}$ is a fundamental unit. $U(R^3)$ has rank 4 and contains units which can not be expressed as $\pm \varepsilon^i \otimes \varepsilon^j \otimes \varepsilon^k$. *E.g.*

$$u = 1 \otimes 1 \otimes 17(17+12\sqrt{2}) + \sqrt{2} \otimes \sqrt{2} \otimes 6(24+17\sqrt{2}).$$

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