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Double integral theorem of Haar measures

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On a group G we consider only those uniformities $\boldsymbol{U}$ for which the right transformation group R_G is equi-continuous, i. e., for any $U\in\boldsymbol{U}$ there is $V\in\boldsymbol{U}$ such that $xVy\subset xyU$ for every $x, y\in G$. A set $A\subset G$ is said to be *totally bounded* for $\boldsymbol{U}$ if for any $U\in\boldsymbol{U}$ we can find a finite system $x_\nu\in G$ $(\nu=1,2,\cdots,n)$ for which we have $A\subset\bigcup_{\nu=1}^{n} x_\nu U$. The linear lattice Φ of all uniformly continuous functions φ on G for which $\{x: \varphi(x)\neq 0\}$ are totally bounded for $\boldsymbol{U}$ is called the *trunk* of $\boldsymbol{U}$. A positive linear functional μ on Φ is called a *measure* on Φ and its value is denoted by $\int\varphi(x)\,\mu(dx)$ for $\varphi\in\Phi$.

For a transformation T on G, if both T and T^{-1} are uniformly continuous for $\boldsymbol{U}$, then for any $\varphi\in\Phi$, setting $\psi(x)=\varphi(xT)$ for $x\in G$, we obtain $\psi\in\Phi$. A measure μ on Φ is called a *Haar measure* of G for $\boldsymbol{U}$ if $\mu\neq 0$ and μ is invariant by R_G, i. e.,

$$\int\varphi(xy)\,\mu(dx)=\int\varphi(x)\,\mu(dx) \qquad \text{for} \quad \varphi\in\Phi \quad \text{and} \quad y\in G.$$

A uniformity $\boldsymbol{U}$ on G is said to be *locally totally bounded* if there is $U\in\boldsymbol{U}$ such that xU is totally bounded for every $x\in G$. According to the Theorem of Existence in [3], if $\boldsymbol{U}$ is locally totally bounded, then there is a Haar measure of G for $\boldsymbol{U}$. If every left transformation $L_x(X\in G)$ is uniformly continuous for $\boldsymbol{U}$ in addition, then we can apply the Theorem of Uniqueness in [3], and we have that the Haar measures are uniquely determined except for constant multiplication, i. e., for any two Haar measures μ and ρ there is a positive number α such that

$$\int\varphi(x)\,\mu(dx)=\alpha\int\varphi(x)\,\rho(dx) \qquad \text{for every} \quad \varphi\in\Phi.$$

For a topological group G we defined the proper uniformity on G in [6]. For the proper uniformity the right transformation group R_G is equi-continuous and every left transformation $L_x(x\in G)$ is uniformly continuous. Therefore for a locally compact topological group G there exists a Haar

* This paper was completed by the auther about 1971.

measure that is uniquely determined except for constant multiplication, as well known.

For a set S of a group G, the relative uniformity of $\boldsymbol{U}$ on S is denoted by $\boldsymbol{U}^S$, i. e., $\boldsymbol{U}^S=\{U^S: U\in\boldsymbol{U}\}$ where $xU^S=xU\cap S$ for $x\in S$. If S is a subgroup of G, then it is clear by definition that the right transformation group R_S is equi-continuous on S for $\boldsymbol{U}^S$.

For two subgroups S and H of G, if $G=SH$ and $S\cap H=\{e\}$, then we can consider G the product space of S and H because $ux=vy$ for $u, v\in S$ and $x, y\in H$ implies $u=v$ and $x=y$. In [6] we proved the

PRODUCT MEASURE THEOREM *For two subgroups S and H of a group G, if $G=SH$, $S\cap H=\{e\}$, $\boldsymbol{U}=\boldsymbol{U}^S\times\boldsymbol{U}^H$ for the relative uniformities $\boldsymbol{U}^S$ and $\boldsymbol{U}^H$, and every L_x $(x\in G)$ is uniformly continuous for $\boldsymbol{U}$, then for Haar measures μ_S and μ_H of S and H respectively, the product measure $\mu_S\times\mu_H$ is a Haar measure of G for $\boldsymbol{U}$; i. e., for the trunk Φ and Φ_H of $\boldsymbol{U}$ and $\boldsymbol{U}^H$ respectively, setting*

$$\psi(x)=\int\varphi(ux)\,\mu_S(du) \qquad \text{for} \quad \varphi\in\Phi \quad \text{and} \quad x\in H,$$

we have $\psi\in\Phi_H$, and setting

$$\int\varphi(x)\,\mu_S\times\mu_H(dx)=\int\psi(x)\,\mu_H(dx)$$

we obtain a Haar measure $\mu_S\times\mu_H$ of G for $\boldsymbol{U}$.

Let S be a subgroup of a group G. Considering each coast Sx $(x\in G)$ an element, we obtain a space. This space is called the *coset space* of S and is denoted by S_G, i. e., $S_G=\{Sx;\ x\in G\}$. Setting $xM_S=Sx$ for $x\in G$, we obtain a mapping M_S from G onto S_G. This mapping M_S is called the *coset mapping*. We define the *coset uniformity* $\boldsymbol{U}_S$ on S_G by the strongest uniformity for which M_S is uniformly continuous.

Setting $(Sx)C_y=Sxy$ for $x, y\in G$, we obtain a transformation group C_G on S_G. This transformation group C_G is called the *coset transformation group* of S. If the coset mapping M_S is uniformly open, then C_G is equi-continuous. If $\boldsymbol{U}$ is locally totally bounded in addition, then the coset uniformity $\boldsymbol{U}_S$ also is locally totally bounded, and there exists an invariant measure μ_S by C_G. If the system L_x $(x\in S)$ is equi-continuous on G for $\boldsymbol{U}$, then M_S is uniformly open. In this paper we will prove the

DOUBLE INTEGRAL THEOREM *Let S be a subgroup of a group G. For a uniformity $\boldsymbol{U}$ on G we suppose that every left transformation L_x $(x\in G)$ is uniformly continuous, the system L_x $(x\in S)$ is equi-continuous, A^{-1} is totally bounded for any totally bounded set $A\subset G$ (this condition is satisfied* *if U is complete), and U is locally totally bounded. For an invariant measure μ_0 by the coset transformation group C_G for the coset uniformity U_S on S_G and a Haar measure μ_S of S for the relative uniformity U^S and for the trunks Φ and Φ_0 of U and U_S respectively, setting*

$$\psi(Sx) = \int \varphi(ux)\, \mu_S(du) \qquad \text{for} \quad \varphi \in \Phi \quad \text{and} \quad x \in G,$$

we have $\psi \in \Phi_0$, and setting

$$\int \varphi(x)\, \mu(dx) = \int \psi(Sx)\, \mu_0(dSx),$$

we obtain a Haar measure μ of G for U.

For two subgroups S and H of a group G such that $G=SH$ and $S\cap H=\{e\}$, setting $xP_H=Sx$ for $x\in H$, we obtain a one-to-one mapping P_H from H onto S_G, and for the representation T_G of G on H defined in [5], we have $xP_HC_z=xT_zP_H$ for $x\in H$ and $z\in G$. Setting $uxM_H=x$ for $u\in S$ and $x\in H$, we obtain a mapping M_H from G onto H. If M_H is uniformly continuous for the relative uniformity U^H of U on H, then P_H is a unimorphism from H with U^H to S_G with the coset uniformity U_S. For the trunk Φ_H of U^H, setting $\varphi(Sx)=\varphi(x)$ for $x\in H$, we can consider Φ_H the trunk of U_S. Since $Tx=R_x$ for $x\in H$ by definition, every invariant measure μ_H by T_G is a Haar measure of H for U^H. Conversely, because of the uniqueness of Haar measures, a Haar measure μ_H of H for U^H is invariant by T_G. Thus, considering μ_H on the trunk Φ_0 of U_S, μ_H is invariant by C_G. Therefore, applying the Double Integral Theorem, we obtain another product measure theorem.

If S is an invariant subgroup of a group G, then the coset space S_G forms the quotient group G/S, and $C_x=R_{Sx}$ for every $x\in G$. Therefore, every invariant measure μ_S by C_G is a Haar measure of G/S for U_S.

We already proved the Double Integral Theorem generally for a transitive transformation group G on a space S in [3]. If we set $S=G$ and $G=R_G$, then as a special case we obtain a double integral theorem for an invariant subgroup, but under the stronger condition that the left transformation group L_G is equi-continuous for U.

In this paper we construct an algebraic theory of coset transformation groups and develop it with uniformities in order to establish the Double Integral Theorem. Many papers are listed in refereces for those who are interested in this field.

1. Coset Transformation Groups Let G be a group. For any subgroup $S\subset G$, the space of all cosets Sx ($x\in G$) is called the *coset space* of S and is denoted by S_G; i. e., $S_G=\{Sx: x\in G\}$.

For any $x\in G$, setting $(Su)C_x=Sux$ for every $u\in G$, we obtain a transformation C_x on S_G, and we have

(1. 1) $C_xC_y=C_{xy}$ and $C_x^{-1}=C_{x^{-1}}$ for $x, y\in G$.

Thus C_x $(x\in G)$ form a transformation group on S_G that is called the *coset transformation group* for a subgroup $S\subset G$ and is denoted by C_G or $C^S{}_G$ if we need to indicate S.

(1. 2) $C^S{}_x$ $(x\in G)$ is a homomorphism from G to the coset transformation group $C^S{}_G$, and $\bigcap_{u\in G} u^{-1}$ Su is its kernel, i. e.,

$$\bigcap_{u\in G} u^{-1}Su = \{x: C^S{}_x = E\}.$$

Proof It is clear by (1. 1) that $C^S{}_x$ $(x\in G)$ is a homomorphism from G to $C^S{}_G$. By definition $C^S{}_x=E$ is equivalent to $Sux=Su$ for every $u\in G$; i. e., $Suxu^{-1}=S$ for every $u\in G$. This is equivalent to $uxu^{-1}\in S$, i. e., $x\in u^{-1}Su$ for every $u\in G$.

We say that a subgroup $S\subset G$ is *simple* if $\bigcap_{u\in G} u^{-1}Su=\{e\}$, as defined in [5]. By (1. 2) we have

(1. 3) $C^S{}_x$ $(x\in G)$ is an isomorphism if and only if S is simple.

For any subgroup $S\subset G$, setting $S_0=\bigcap_{u\in G} u^{-1}Su$, we obtain an invariant subgroup S_0 of G, and we have

(1. 4) S/S_0 is a simple subgroup of the quotient group G/S_0.

Proof By definition the quotient group G/S_0 consists of cosets S_0x $(x\in G)$, and for any $u\in G$ we have

$$(S_0u)^{-1}(S/S_0)(S_0u) = (S_0u^{-1})(S/S_0)(S_0u) = \{S_0u^{-1}xu: x\in S\}$$

because $uS_0=S_0u$ for every $u\in G$. If $S_0y\in\{S_0u^{-1}xu: x\in S\}$ for every $u\in G$, then $y\in S_0u^{-1}Su$ for every $u\in G$. On the other hand, we have

$$S_0u^{-1}Su = u^{-1}S_0Su = u^{-1}Su$$

because $S_0\subset S$ and $S_0S=S$. Therefore $y\in\bigcap_{u\in G} u^{-1}Su=S_0$, and we conclude that S/S_0 is a simple subgroup of G/S_0.

2. Congruence Let M be a transformation from a space S_1 to another space S_2; i. e., M is a one-to-one mapping from S_1 onto S_2. For a transformation T on S_2, MTM^{-1} is a transformation on S_1. We say that a transformation group T_1 on S_1 is *congruent* to a transformation group T_2 on S_2 by M if $T_1=MT_2M^{-1}$, i. e., $T_1=\{MTM^{-1}: T\in T_2\}$.

We also say that T_1 is *congruent* to T_2 if there is a transformation

M from S_1 to S_2 such that T_1 is congruent to T_2 by M. It is clear that if T_1 is congruent to T_2, then T_1 is isomorphic to T_2. We can easily prove that congruence is symmetric, i.e., if T_1 is congruent to T_2, then T_2 is congruent to T_1, and congruence is transitive, i.e., if T_1 is congruent to T_2 and T_2 is congruent to T_3, then T_1 is congruent to T_3.

Let P be a homomorphism from a group G to another group H. For a subgroup $S\subset G$ we have that

(2.1) Setting $(Sx)M=(SP)(xP)$ for $x\in G$, we obtain a transformation M from the coset space S_G of S in G to the coset space $(SP)_H$ of SP in H if and only if $SPP^{-1}=S$.

PROOF We have $(SP)(xP)=(SP)(yP)$ if and only if $xy^{-1}P=(xP)(yP)^{-1}\in SP$. If $SPP^{-1}=S$, then $xy^{-1}\in SPP^{-1}=S$, and $Sx=Sy$ whenever $(SP)(xP)=(SP)(yP)$. Thus M is one-to-one. Conversely, if M is one-to-one, then $xP\in SP$ implies $(Sx)M=(SP)(xP)=SP$. On the other hand, we have $(Se)M=(SP)(eP)=SP$ because P is a homomorphism. Thus $Sx=Se$, i.e., $x\in S$. Therefore $SPP^{-1}\subset S$, and we have $SPP^{-1}=S$ because we always have $SPP^{-1}\supset S$.

(2.2) $SPP^{-1}=S$ if and only if S includes the kernel of P.

PROOF Let K be the kernel of P. Since $xP=e\in H$ for every $x\in K$, we have $xP\in SP$ for every $x\in K$. Thus, if $SPP^{-1}=S$, then $K\subset S$. Conversely, we suppose that $K\subset S$. For any $x\in G$, if $xP\in SP$, then there is $y\in S$ such that $xP=yP$, and $Kx=Ky$ because K is the kernel of P. Then we have $x\in Ky\subset KS=S$ by assumption. Therefore $SPP^{-1}\subset S$, and we have $SPP^{-1}=S$.

(2.3) If $SPP^{-1}=S$, then the coset transformation group $C^S{}_G$ is congruent to $C^{SP}{}_H$ by M in (2.1).

PROOF For any $x, y\in G$ we have

$$\begin{aligned}(Sy)\, MC^{SP}{}_{xP}M^{-1} &= (SP)(yP)(xP)\, M^{-1} = (SP)(yxP)\, M^{-1}\\ &= Syx = (Sy)\, C^S{}_x\,.\end{aligned}$$

Therefore $C^S{}_G=MC^{SP}{}_HM^{-1}$.

For an invariant subgroup K of a group G, setting $xP=Kx$ for every $x\in G$, we obtain a homomorphism P from G to the quotient group G/K, and K is the kernel of P. Therefore, by (2.1), (2.2), and (2.3) we have

(2.4) For a subgroup S and an invariant subgroup K of a group G, setting $(Sx)\, M=(S/K)(Kx)$ we obtain a transformation M from S_G to $(S/K)_{G/K}$ if and only if $K\subset S$, and then the coset transformation group $C^S{}_G$ is congruent to $C^{S/K}{}_{G/K}$ by M.

CONGRUENCE THEOREM 2. 5. *For simple subgroups S and K of groups G and H respectively, the coset transformation group $C^S{}_G$ is congruent to $C^K{}_H$ if and only if there is an isomorphism P from G to H such that $SP=K$.*

PROOF If $C^S{}_G$ is congruent to $C^K{}_H$ by a transformation M from S_G to K_H, then setting

$$C^S{}_x = MC^K{}_{xP} M^{-1} \qquad for \quad x \in G,$$

we obtain an isomorphism P from G to H by (1. 3) because both S and K are simple by assumption and by (1. 1) we have

$$C^S{}_{xy} = C^S{}_x C^S{}_y = MC^K{}_{xP} C^K{}_{yP} M^{-1} = MC^K{}_{(xP)(yP)} M^{-1} \text{ and}$$
$$C^S{}_{x^{-1}} = (C^S{}_x)^{-1} = M(C^K{}_{xP})^{-1} M^{-1} = M(C^K{}_{(xP)^{-1}}) M^{-1}.$$

Since $(Se)\, C^S{}_x = Se$ if and only if $x \in S$, we have

$$SP = \left\{xP:\ (Se)\, C^S{}_x = Se\right\} = \left\{u:\ (Se)\, MC^K{}_u = (Se)\, M\right\}.$$

Since M is a transformation from S_G to K_H, there is $u_0 \in H$ such that $(Se)\, M = Ku_0$, and we have

$$SP = \left\{u:\ Ku_0 u = Ku_0\right\} = \left\{u:\ u_0 u u_0^{-1} \in K\right\} = u_0^{-1} K u_0.$$

Setting $xQ = u_0 (xP)\, u_0^{-1}$ for $x \in G$, we obtain an isomorphism Q from G to H, and $SQ = u_0 (SP)\, u_0^{-1} = K$.

Conversely, if there is an isomorphism Q from G to H such that $SQ=K$, then setting $(Sx)\, M = K(xQ)$ for $x \in G$, we obtain a transformation M from S_G to K_H, and for any $x, y \in G$ we have

$$(Sx)\, MC^K{}_{yQ} M^{-1} = K(xQ)\, C^K{}_{yQ} M^{-1} = K\big((xy)\, Q\big)\, M^{-1}$$
$$= Sxy = (Sx)\, C^S{}_y.$$

Therefore $C^S{}_y = MC^K{}_{yQ} M^{-1}$ for every $y \in G$.

Referring to (1. 3) and (2. 4), by this theorem we obtain

CONGRUENCE THEOREM 2. 6. *$C^S{}_G$ is congruent to $C^K{}_H$ if and only if for*

$$S_0 = \bigcap_{x \in G} x^{-1} S x \quad and \quad K_0 = \bigcap_{u \in H} u^{-1} K u$$

there is an isomorphism P from G/S_0 to H/K_0 such that $(S/S_0)\, P = K/K_0$.

3. Representations on Subgroups Let H be an *adjoint* of a subgroup S in a group G; i. e., H is a subgroup of G, $S \cap H = \{e\}$, and $G = SH$, as defined in [5]. It is clear by definition that $S_G = \{Sx:\ x \in H\}$, and setting $xP_H = Sx$ for $x \in H$, we obtain a transformation P_H from H to the coset space S_G.

Since S is an adjoint of H too, we defined the representation T_G of G on H for S in [5], and we have

$$T_{xu}=R_x D_u,\ xu=(uS_x)(xD_u),\ uS_x\in S, \quad \text{and} \quad xD_u\in H$$

for $x\in H$ and $u\in S$. For x, $y\in H$ and $u\in S$ we have

$$(Sy)\,C^S{}_{xu}=Syxu=S(uS_{yx})\Big((yx)\,D_u\Big)S(yR_xD_u)=S(yT_{xu})\,.$$

Thus, $yT_{xu}=yP_H C^S{}_{xu}P_H^{-1}$ for every $y\in H$, and we obtain $T_z=P_H C^S{}_z P_H^{-1}$ for $z\in G$. Therefore we can state

REPRESENTATION THEOREM 3.1. *For an adjoint H of S in G, setting $xP_H=Sx$ for $x\in H$, we obtain a transformation P_H from H to the coset space S_G. The representation T_G of G on H for S is congruent to the coset transformation group $C^S{}_G$ by P_H, and*

$$(Sx)\,C^S{}_G=S(xT_y) \qquad \text{for} \quad x\in H \quad \text{and} \quad y\in G\,.$$

As an immediate consequence of this theorem, we have

REPRESENTATION THEOREM 3.2. *For two adjoints H and K of a subgroup S in a group G, the representation of G on H for S is congruent to that of G on K for S by the transformation M from H to K defined by $Sx=S(xM)$ for $x\in H$.*

Referring to Congruence Theorem 2.6, by Representation Theorem 3.2 we obtain

REPRESENTATION THEOREM 3.3. *The representation of a group G on a subgroup S for its adjoint S_1 is congruent to the representation of a group H on a subgroup K for its adjoint K_1 if and only if for $S_0=\bigcap_{x\in G} x^{-1}S_1x$ and $K_0=\bigcap_{u\in H} u^{-1}K_1u$, there is an isomorphism P from G/S_0 to H/K_0 such that $(S_1/S_0)\,P=K_1/K_0$.*

4. Skew-Coset Transformation Groups For a subgroup S of a group G the space of all skew-cosets uS $(u\in G)$ is called the *skew-coset space* of S and is denoted by ${}_GS$; i.e., ${}_GS=\{uS:\ u\in G\}$.

For any $x\in G$, *setting* $(uS)\,\tilde{C}_x=xuS$ for every $u\in G$, we obtain a transformation $\tilde{C}_x$ on ${}_GS$, and we have

$$(4.1) \qquad \tilde{C}_x\tilde{C}_y=\tilde{C}_{yx} \quad \text{and} \quad \tilde{C}_x^{-1}=\tilde{C}_{x^{-1}} \quad \text{for} \quad x,\ y\in G\,.$$

The transformations $\tilde{C}_x$ $(x\in G)$ form a transformation group on ${}_GS$ that is called the *skew-coset transformation group* for a subgroup $S\subset G$ and is denoted by $\tilde{C}_G$ or by $\tilde{C}^S{}_G$ if we need to indicate S.

A mapping P from a group G onto a group H is called a *skew-homo-*

morphism if $(xy)P=(yP)(xP)$ for $x, y\in G$. For a skew-homomorphism P from G to H, setting $S=\{x: xP=e\}$, we obtain an invariant subgroup S of G that is called the *kernel* of P.

By (4. 1) we have

(4. 2) $\tilde{C}^S{}_x$ $(x\in G)$ is a skew-homomorphism from G to $\tilde{C}^S{}_G$, and $\bigcap_{u\in G} u^{-1}Su$ is its kernel.

A skew-homomorphism is called a *skew-isomorphism* if it is one-to-one. By (4. 2) we have

(4. 3) $\tilde{C}^S{}_x$ $(x\in G)$ is a skew-isomorphism if and only if S is simple.

Congruence Theorem 4. 4. *Setting $(Sx)P_S=x^{-1}S$ for $x\in G$, we obtain a transformation P_S from S_G to ${}_GS$ such that $C^S{}_G$ is congruent to $\tilde{C}^S{}_G$ by P_S, and $P_S\tilde{C}^S{}_xP_S^{-1}=C^S{}_{x^{-1}}$ for every $x\in G$.*

Proof We have $Sx=Sy$ if and only if $xy^{-1}\in S$, and we have $x^{-1}S=y^{-1}S$ if and only if $xy^{-1}\in S$. Thus P_S is a transformation. Furthermore, for $x, y\in G$ we have

$$(Sx)P_S\tilde{C}^S{}_yP_S^{-1}=(yx^{-1}S)P_S^{-1}=Sxy^{-1}=(Sx)C^S{}_{y^{-1}}\,.$$

Thus, $P_S\tilde{C}^S{}_GP_S^{-1}=C^S{}_G$.

Let S be an invariant subgroup of G; i. e., $x^{-1}Sx=S$ for every $x\in G$. Since $xS=Sx$ for every $x\in G$, we have ${}_GS=S_G$. Furthermore, S_G forms the quotient group G/S, and $(Sx)(Sy)=Sxy$ for $x, y\in G$. We also have

$$(Sx)C_y=Sxy=(Sx)(Sy)=(Sx)R_{Sy} \quad \text{and}$$
$$(Sx)\tilde{C}_y=ySx=Syx=(Sy)(Sx)=(Sx)L_{Sy}$$

for $x, y\in G$ where R_{Sy} is the right transformation on G/S and L_{Sy} is the left transformation on G/S, as defined in [3].

Now we can state

Quotient Group Theorem 4. 5. *For an invariant subgroup S of a group G, the coset space S_G forms the quotient group G/S and we have ${}_GS=S_G$, $xS=Sx$ for every $x\in G$, and*

$C_x=R_{Sx}$ and $\tilde{C}_x=L_{Sx}$ for $x\in G$ on the quotient group G/S.

5. Strong Uniformities Let $M_\lambda(\lambda\in\Lambda)$ be a mapping from a space S_λ to a space R for each $\lambda\in\Lambda$, and a uniformity $\boldsymbol{U}_\lambda$ is defined on S_λ for each $\lambda\in\Lambda$. For the trivial uniformity on R that consists of only one connector every M_λ is uniformly continuous. Let $\boldsymbol{V}_\gamma(\gamma\in\Gamma)$ be the system of all uniformities on R for which M_λ is uniformly continuous for every $\lambda\in\Lambda$. For the weakest stronger uniformity $\bigvee_{\gamma\in\Gamma} \boldsymbol{V}_\gamma$ on R every M_λ is uniformly continuous by Theorem 21. 1 in [3]. Therefore there exists the strongest among the uniformities on R for which M_λ is uniformly continuous for every $\lambda\in\Lambda$. This strongest uniformity on R is called the *strong uniformity* on R by $M_\lambda(\lambda\in\Lambda)$ for $\boldsymbol{U}_\lambda(\lambda\in\Lambda)$.

Let M be a *full* mapping from S to R; i. e., M is a mapping from S onto R. For uniformities $\boldsymbol{U}$ and $\boldsymbol{V}$ on S and R respectively, M is said to be uniformly open if for any $U\in\boldsymbol{U}$ there is $V\in\boldsymbol{V}$ such that $xUM\supset xMV$ for every $x\in S$, as defined in [3].

(5. 1) If M is uniformly continuous and uniformly open, then $\boldsymbol{V}$ is the strong uniformity by M for $\boldsymbol{U}$.

Proof If M is uniformly continuous for another uniformity $\boldsymbol{V}_0$ on R, then for any $V\in\boldsymbol{V}_0$ we can find $U\in\boldsymbol{U}$ by definition such that $xUM\subset xMV$ for every $x\in S$. Since M is uniformly open for $\boldsymbol{V}$ by assumption, for this U we can find $W\in\boldsymbol{V}$ by definition such that $xUM\supset xMW$ for every $x\in S$. Since M is full, we obtain $V\geqq W$, and we conclude that $\boldsymbol{V}_0\subset\boldsymbol{V}$. Therefore $\boldsymbol{V}$ is the strong uniformity by definition.

Let N be a mapping from R to a space K with a uniformity $\boldsymbol{W}$. If both M and N are uniformly continuous, then the composed mapping MN also is uniformly continuous by Theorem 14.3 in [3]. If M is uniformly open and MN is uniformly continuous, then N is uniformly continuous by Theorem 14. 4 in [3]. Thus, if M is uniformly continuous and uniformly open, then N is uniformly continuous if and only if MN is uniformly continuous. Therefore we have

(5. 2) If a full mapping M from S to R is uniformly continuous and uniformly open, then for a mapping N from R to a space K, the strong uniformity on K by N is the strong uniformity on K by the composed mapping MN.

Since M is a full mapping from S to R, for any connector U on S we can define a connector UM^+ on R by

(5. 3) $$uUM^+ = \bigcup_{xM=u} xUM \quad \text{for } u\in R,$$

and UM^- on R by

(5. 4) $$uUM^- = \bigcap_{xM=u} xUM \quad \text{for } u\in R.$$

For any connector V on R we obviously have

(5. 5) $$MVM^{-1}M^+ = MVM^{-1}M^- = V.$$

We also have

(5.6) $xM(UM^{-})\subset xUM\subset xM(UM^{+})$ for $x\in S$.

We can easily prove

(5.7) $V\leqq U$ implies $VM^{+}\leqq UM^{+}$ and $VM^{-}\leqq UM^{-}$.

As an immediate consequence of (5.7), we have

(5.8) $(U\cap V)M^{+}\leqq UM^{+}\cap VM^{+}$ and $(U\cap V)M^{-}\leqq UM^{-}\cap VM^{-}$.

We will prove

(5.9) $U^{-1}M^{+}=(UM^{+})^{-1}$.

PROOF For $u, v\in R$ we have $u\in v(U^{-1}M^{+})$ if and only if we can find $x, y\in S$ such that $xM=u$, $yM=v$, and $x\in yU^{-1}$ because $v(U^{-1}M^{+})=\bigcup_{yM=v} yU^{-1}M$ by definition. Likewise, we have $v\in u(UM^{+})$ if and only if we can find $x, y\in S$ such that $xM=u$, $yM=v$, and $y\in xU$. Since we have $x\in yU^{-1}$ if and only if $y\in xU$ by definition, we have $u\in v(U^{-1}M^{+})$ if and only $v\in u(UM^{+})$, and we obtain (5.9) by definition.

(5.10) $(UM^{-})(VM^{-})\leqq(UV)M^{-}$.

PROOF We suppose that $u\in v(UM^{-})(VM^{-})$. By definition we can find $w\in v(UM^{-})$ such that $u\in w(VM^{-})$, and we have $w\in yUM$ and $u\in zVM$ for any $y, z\in S$ with $yM=v$ and $zM=w$. We can find $z\in S$ such that $zM=w$ and $z\in yU$, and for such z we have $zV\subset yUV$. Thus $u\in yUVM$ for any $y\in S$ with $yM=v$, and we have $u\in v((UV)M^{-})$ by definition.

For uniformities $\boldsymbol{U}$ and $\boldsymbol{V}$ on S and R respectively, M is uniformly continuous by definition if and only if for any $V\in\boldsymbol{V}$ we can find $U\in\boldsymbol{U}$ such that $xUM\subset xMV$ for every $x\in S$. On the other hand, we have $xUM\subset xMV$ for every $x\in S$ if and only if $u(UM^{+})\subset uV$ for every $u\in R$ by definition. Therefore we can state

(5.11) M is uniformly continuous if and only if for any $V\in\boldsymbol{V}$ we can find $U\in\boldsymbol{U}$ such that $UM^{+}\leqq V$.

M is uniformly open by definition if and only if for any $U\in\boldsymbol{U}$ we can find $V\in\boldsymbol{V}$ such that $xUM\supset xMV$ for every $x\in S$. On the other hand, we have $xUM\supset xMV$ for every $x\in S$ if and only if $u(UM^{-})\supset uV$ for every $u\in R$ by definition. Therefore we have

(5.12) M is uniformly open if and only if for any $U\in\boldsymbol{U}$ we can find $V\in\boldsymbol{V}$ such that $UM^{-}\geqq V$.

It is clear by definition that

(5.13) $UM^+ \leqq VM^-$ if and only if $xM=yM$ implies $xUM \subset yVM$.

STRONG UNIFORMITY THEOREM 5.14. *For the strong uniformity $\boldsymbol{V}$ on R by a full mapping M from S with a uniformity $\boldsymbol{U}$ to R, M is uniformly open if and only if for any $U \in \boldsymbol{U}$ we can find $V \in \boldsymbol{U}$ such that $VM^+ \leqq UM^-$, and then $UM^+ (U \in \boldsymbol{U})$ form a basis of $\boldsymbol{V}$.*

PROOF If M is uniformly open for $\boldsymbol{V}$ on R, then $UM^- \in \boldsymbol{V}$ for every $U \in \boldsymbol{U}$ by (5.12). Since M is uniformly continuous by definition, for any $U \in \boldsymbol{U}$ we can find $V \in \boldsymbol{U}$ by (5.11) such that $VM^+ \leqq UM^-$.

Conversely, we suppose that for any $U \in \boldsymbol{U}$ there is $V \in \boldsymbol{U}$ such that $VM^+ \leqq UM^-$. For any $U \in \boldsymbol{U}$ we can find $V \in \boldsymbol{U}$ by definition such that $VV \leqq U$. Referring to (5.7), for such V we can find a symmetric $W \in \boldsymbol{U}$ by assumption such that $WM^+ \leqq VM^-$. Then, by (5.9), (5.10), (5.7), and (5.6) we have

$$(WM^+)(WM^+)^{-1} = (WM^+)(WM^+) \leqq (VM^-)(VM^-) \\ \leqq (VV)M^- \leqq UM^- \leqq UM^+ .$$

Thus, by (5.8) we conclude that there exists a unique uniformity $\boldsymbol{V}_0$ on R such that UM^+ $(U \in \boldsymbol{U})$ form a basis of $\boldsymbol{V}_0$. For this uniformity $\boldsymbol{V}_0$, M is uniformly continuous by (5.11) and uniformly open by (5.12). Therefore $\boldsymbol{V}_0$ is the strong uniformity on R by M by (5.1).

We say that a uniformity $\boldsymbol{U}$ on a space S is *unimorphic* to a uniformity $\boldsymbol{V}$ on a space R if there is a transformation P from S to R such that both P and the inverse P^{-1} are uniformly continuous, and then P is called a *unimorphism*, as defined in [3]. It is clear by definition that the inverse P^{-1} is uniformly continuous if and only if P is uniformly open. Thus a unimorphism P is a transformation that is uniformly continuous and uniformly open simultaneously.

For a transformation P from S to R we have $UP^+ = UP^-$ for any connector U on S by definitions (5.3) and (5.4). Thus by Strong Uniformity Theorem 5.14 we have

UNIMORPHISM THEOREM 5.15. *A transformation P from a space S with a uniformity $\boldsymbol{U}$ to a space R is a unimorphism for the strong uniformity on R by P for $\boldsymbol{U}$.*

A transformation group G on a space S with a uniformity $\boldsymbol{U}$ is said to be *equivalent* to a transformation group H on a space R with a uniformity $\boldsymbol{V}$ if G is congruent to H by some unimorphism from S to R, as defined in [3].

A transformation group G on S with a uniformity $\boldsymbol{U}$ is said to be *equi-continuous* if for any $U\in\boldsymbol{U}$ there is $V\in\boldsymbol{U}$ such that $V\leqq TUT^{-1}$ for every $T\in G$. With this definition, we can easily prove

EQUIVALENCE THEOREM 5.16. *When a transformation group G is equivalent to a transformation group H, G is equi-continuous if and only if H is equi-continuous.*

6. Coset Uniformities Let G be a group. For $e\in n\subset G$ we define a connector $U(n)$ on G by

(6.1) $$xU(n)=nx \quad \text{for} \quad x\in G.$$

As proved in [3], we have

(6.2) $$AU(n)=nA \quad \text{for} \quad \emptyset\neq A\subset G,$$

(6.3) $$\bigcap_{\lambda\in\Lambda} U(n_\lambda)=U(\bigcap_{\lambda\in\Lambda} n_\lambda),$$

(6.4) $$U(n)\leqq U(m) \text{ if and only if } n\subset m,$$

(6.5) $$U(m)\,U(n)=U(nm), \quad \text{and}$$

(6.6) $$U(n)^{-1}=U(n^{-1}).$$

A set class N of G is called a *neighborhood* on G if $e\in n$ for every $n\in N$; $N\ni n\subset m$ implies $N\ni m$; $N\ni n$, m implies $N\ni n\cap m$; and for any $n\in N$ there is $m\in N$ such that $m^{-1}m\subset n$. For any neighborhood N on G, (6.3)-(6.6) shows that there exists a unique uniformity on G such that $U(n)$ $(n\in N)$ form a basis. This uniformity on G is called the *induced uniformity* on G by N and is denoted by $\boldsymbol{U}(N)$.

For the induced uniformity $\boldsymbol{U}(N)$ the right transformation group R_G is equi-continuous because $xR_yU(n)=nxy=xU(n)\,R_y$ for any $x,y\in G$ and $n\in N$. Conversely, if for a uniformity $\boldsymbol{U}$ on G the right transformation group R_G is equi-continuous, then there exists a unique neighborhood N such that $\boldsymbol{U}=\boldsymbol{U}(N)$, as proved in [3]. On a group G we consider only those uniformities for which the right transformation group R_G is equi-continuous.

For a subgroup $S\subset G$ we defined the coset space S_G. For a set $\emptyset\neq A\subset G$ we make use of the notation $S_A=\{Sx: x\in A\}$ as a set of S_G in distinction from SA that is a set of G. However we will use both S_x and Sx as a coset.

We can easily prove

(6.7) $$S_{SA}=S_A \quad \text{for} \quad \emptyset\neq A\subset G,$$

(6.8) $$S_A\subset S_B \text{ if and only if } A\subset SB,$$

(6.9) $$S_A\subset S_B \text{ implies } S_{Ax}\subset S_{Bx}, \quad \text{and}$$

(6. 10) $\bigcup_{\lambda\in\Lambda} S_{A_\lambda} = S_{\bigcup_{\lambda\in\Lambda} A_\lambda}$.

We will prove

(6. 11) For $\emptyset \neq A \subset G$, setting $S_x U = S_{Ax}$ for $x \in G$, we obtain a connector U on S_G if and only if $AS \subset SA$ and $A \cap S \neq \emptyset$.

PROOF For any $z \in A \cap S$ we have $z \in A$ and $z^{-1} \in S$, and hence $e = z^{-1}z \in SA$. Conversely, if $e \in SA$, then we can find $x \in A$ and $u \in S$ such that $e = ux$, and $x = u^{-1} \in S$. Therefore we have $A \cap S \neq \emptyset$ if and only if $e \in SA$.

If U is a connector on S_G, then $S_e \in S_{Ae} = S_{Au}$ for every $u \in S$. Thus $e \in SA$ and $SA \supset Au$ for every $u \in S$ by (6. 8). Then $SA \supset AS$ and $A \cap S \neq \emptyset$. Conversely, if $SA \supset AS$ and $A \cap S \neq \emptyset$, then $SA \supset Au$ and $SAu \supset A$ for $u \in S$, and $S_A = S_{Au}$ for every $u \in S$ by (6. 8). Since $e \in SA$, we have $S_x \in S_{Ax} = S_{Aux}$ for $u \in S$ and $x \in G$ by (6. 9). Therefore U is a connector on S_G.

For $\emptyset \neq A \subset G$ we have $ASS = AS \subset SAS$. Thus, by (6. 11) we have

(6. 12) For $e \in n \subset G$, setting $S_x U_n = S_{nSx}$ for $x \in G$, we obtain a connector U_n on S_G.

Such a connector U_n is called a *proper* connector on S_G. A uniformity $\boldsymbol{U}$ on S_G is said to be *proper* if $\boldsymbol{U}$ has a basis that consists of proper connectors.

For a transformation group G on a space S, a connector U on S is said to be *invariant* by G if $XUX^{-1} = U$ for every $X \in G$, i. e., $xXU = xUX$ for all $x \in S$ and $X \in G$. A transformation group G on a space S with a uniformity $\boldsymbol{U}$ is equi-continuous if and only if $\boldsymbol{U}$ has a basis that consists of invariant connectors. This is Theorem 32.5 in [3].

PROPER CONNECTOR THEOREM 6. 13. *A connector U on S_G is invariant by the coset transformation group C_G if and only if U is a proper connector.*

PROOF For any proper connector U_n on S_G we have

$$S_x C_y U_n = S_{nSxy} = S_{nSx} C_y = S_x U_n C_y$$

for every $x, y \in G$. Therefore U_n is invariant by C_G.

Conversely, if a connector U on S_G is invariant by C_G, then setting $n = \{x : S_x \in S_e U\}$, we have $e \in n \subset G$, and for any $x \in G$ we have

$$S_x U = S_e C_x U = S_e U C_x = S_n C_x .$$

Since $S_u = S_e$ for every $u \in S$, we have $S_n = S_u U = S_n C_u = S_{nu}$ for every $u \in S$, and $S_n = S_{nS}$ by (6. 10). Therefore, we have

$$S_x U = S_{nSx} = S_x U_n \qquad \text{for every} \quad x \in G ,$$

i. e., U is a proper connector.

PROPER UNIFORMITY THEOREM 6.14. *The coset transformation group C_G is equi-continuous on S_G for a uniformity $\boldsymbol{U}$ on S_G if and only if $\boldsymbol{U}$ is proper.*

PROOF If $\boldsymbol{U}$ is proper, then for any $U\in\boldsymbol{U}$ there is a proper connector $U_n\in\boldsymbol{U}$ by definition such that $U_n\leqq U$. Then by Proper Connector Theorem 6.13 we have

$$U_n = C_x U_n C_x^{-1} \leqq C_x U C_x^{-1} \quad \text{for every} \quad x\in G.$$

Therefore C_G is equi-continuous for $\boldsymbol{U}$.

Conversely, if C_G is equi-continuous for $\boldsymbol{U}$, then for any $U\in\boldsymbol{U}$ there is $V\in\boldsymbol{U}$ such that $V\leqq C_x U C_x^{-1}$ for every $x\in G$. Setting

$$U^c = \bigcap_{x\in G} C_x U C_x^{-1}$$

we obtain a connector $U^c\in\boldsymbol{U}$, and $C_x U^c C_x^{-1} = U^c$ for every $x\in G$. Thus U^c is a proper connector by Proper Connector Theorem 6.13. Since $U^c\leqq U$ by definition, U^c $(U\in\boldsymbol{U})$ form a basis of $\boldsymbol{U}$. Therefore $\boldsymbol{U}$ is proper by definition.

Setting $xM_S = S_x$ for $x\in G$, we obtain a full mapping M_S from G to the coset space S_G. This mapping M_S is called the *coset mapping* for a subgroup $S\subset G$.

For $e\in n\subset G$ the connector $U(n)$ defined by (6.1) is invariant by the right transformation group R_G because

$$xR_y U(n) R_y^{-1} = nxyy^{-1} = nx = xU(n)$$

for every $x, y\in G$. For $U(n)$ and the coset mapping M_S for a subgroup $S\subset G$ we have

$$S_x\big(U(n) M_S^{+}\big) = \bigcup_{yM_S=S_x} nyM_S = \bigcup_{u\in S} nu\ xM_S = \big(\bigcup_{u\in S} nu\big)\, xM_S$$
$$= nS\ xM_S = S_{nSx} = S_x U_n$$

for every $x\in G$. So we have

(6.15) $$U(n) M_S^{+} = U_n \quad \text{for} \quad e\in n\subset G.$$

For the coset mapping M_S we also have

(6.16) $$U(n) M_S^{+} \leqq U(m) M^{-} \text{ if and only if } nS\subset Sm.$$

PROOF For any $x\in G$, by definition (5.4) we have

$$S_x\big(U(m) M_S^{-}\big) = \bigcap_{yM_S=S_x} yU(m) M_S = \bigcap_{u\in S} muxM_S = \bigcap_{u\in S} S_{mux}.$$

If $U(n)\,M_S^+ \leqq U(m)\,M_S^-$, then by (6.15) we have

$$S_{nS} = S_e U_n \subset S_{mu} \qquad \text{for every} \quad u \in S.$$

Referring to (6.8), we obtain $nS \subset Smu$, and $nS = nSu^{-1} \subset Sm$.

Conversely, if $nS \subset Sm$, then $nSx = nSux \subset Smux$ for $u \in S$ and $x \in G$. Thus we have $S_{nSx} \subset S_{mux}$ for $u \in S$ and $x \in G$ by (6.8), and for any $x \in G$ we have

$$S_x\big(U(n)\,M^+\big) = S_{nSx} \subset \bigcap_{u \in S} S_{mux} = S_x\big(U(m)\,M^-\big).$$

Let N be a neighborhood on a group G. For a subgroup $S \subset G$, the strong uniformity $\boldsymbol{U}_S$ on the coset space S_G by the coset mapping M_S for the induced uniformity $\boldsymbol{U}(N)$ is called the *coset uniformity* on S_G for N.

Referring to Strong Uniformity Theorem 5.4, by (6.16) we have

Coset Uniformity Theorem 6.17. *For the coset uniformity $\boldsymbol{U}_S$ on S_G for a neighborhood N on G, the coset mapping M_S is uniformly open if and only if for any $m \in N$ there is $n \in N$ such that $nS \subset Sm$, and then $\boldsymbol{U}_S$ is proper.*

If left transformations L_u $(u \in S)$ are equi-continuous on G for $\boldsymbol{U}(N)$, then for any $m \in N$ there is $n \in N$ such that $U(n) \leqq L_u U(m)\,L_u^{-1}$ for every $u \in S$. Thus we have

$$n = eU(n) \subset eL_u U(m)\,L_u^{-1} = muL_{u^{-1}} = u^{-1}mu \qquad \text{for} \quad u \in S,$$

and $nu^{-1} \subset u^{-1}m \subset Sm$ for every $u \in S$. Therefore $nS \subset Sm$, and by Coset Uniformity Theorem 6.17 we have

Coset Uniformity Theorem 6.18. *If left transformations L_u $(u \in S)$ are equi-continuous on G for N, then the coset uniformity $\boldsymbol{U}_S$ on S_G is uniformly open and proper.*

7. Skew-Coset Uniformities For a subgroup S of a group G we defined the skew-coset space ${}_GS$. For $\emptyset \neq A \subset G$ we make use of the notation

$${}_AS = \{xS:\ x \in A\}$$

as a set of ${}_GS$. However, we will use both ${}_xS$ and xS as a skew-coset.

We can easily prove

(7.1) $\quad {}_{AS}S = {}_AS,$

(7.2) $\quad {}_AS \subset {}_BS$ if and only if $A \subset BS,$

(7.3) $\quad {}_AS \subset {}_BS$ implies ${}_{XA}S \subset {}_{XB}S,$

(7.4) $\quad \bigcup_{\lambda \in \Lambda} {}_{A_\lambda}S = {}_{\bigcup_{\lambda \in \Lambda} A_\lambda}S,$

(7. 5) For $\emptyset \neq A \subset G$, setting ${}_{x}S\tilde{U} = {}_{xA}S$ for $x \in G$, we obtain a connector $\tilde{U}$ on ${}_{G}S$ if and only if $SA \subset AS$ and $A \cap S \neq \emptyset$, and

(7. 6) For $e \in n \subset G$, setting ${}_{x}S\tilde{U}_n = {}_{xSn}S$ for $x \in G$, we obtain a connector $\tilde{U}_n$ on ${}_{G}S$.

Such a connector $\tilde{U}_n$ is called a *skew-proper* connector on ${}_{G}S$. A uniformity $\boldsymbol{U}$ on ${}_{G}S$ is said to be *skew-proper* if $\boldsymbol{U}$ has a basis that consists of skew-proper connectors.

We defined a transformation P_S from S_G to ${}_{G}S$ by $S_x P_S = {}_{x^{-1}}S$ for $x \in G$ in Congruence Theorem 4. 4. For this transformation P_S we have

(7. 7) $P_S \tilde{U}_n P_S^{-1} = U_{n^{-1}}$ and $P_S^{-1} U_n P_S = \tilde{U}_{n^{-1}}$.

PROOF For any $x \in G$ we have

$$S_x P_s \tilde{U}_n P_S^{-1} = {}_{x^{-1}}S\tilde{U}_n P_S^{-1} = {}_{x^{-1}Sn}SP_S^{-1} = S_{n^{-1}Sx} = S_x U_{n^{-1}}$$

by (6. 12) and (7. 6).

By Congruence Theorem 4. 4 we also have

(7. 8) $P_S \tilde{C}_x P_S^{-1} = C_{x^{-1}}$ and $P_S^{-1} C_x P_S = \tilde{C}_{x^{-1}}$.

It is clear by definition that a connector U on S_G is invariant by C_x if and only if $P_S^{-1} U P_S$ is invariant by $P_S^{-1} C_x P_s$. Therefore, referring to Proper Connector Theorem 6. 13, by (7. 7) and (7. 8) we have

SKEW-PROPER CONNECTOR THEOREM 7. 9. *A connector U on ${}_{G}S$ is invariant by the skew-coset transformation group $\tilde{C}_G$ if and only if U is a skew-proper connector.*

For a uniformity $\tilde{\boldsymbol{U}}$ on ${}_{G}S$ we have a uniformity

$$P_S \tilde{\boldsymbol{U}} P_S^{-1} = \{P_S U P_S^{-1} : U \in \tilde{\boldsymbol{U}}\}$$

on S_G, and it is clear by definition that P_S is a unimorphism from S_G to ${}_{G}S$ for those uniformities. Referring to (7. 7), we can easily prove

(7. 10) A uniformity $\tilde{\boldsymbol{U}}$ on ${}_{G}S$ is skew-proper if and only if $P_S \tilde{\boldsymbol{U}} P_S^{-1}$ is proper on S_G.

Referring to proper Uniformity Theorem 6. 14, by (7. 10) we obtain

SKEW-PROPER UNIFORMITY THEOREM 7. 11. *The skew-coset transformation group $\tilde{C}_G$ is equi-continuous for a uniformity $\tilde{\boldsymbol{U}}$ on ${}_{G}S$ if and only if $\tilde{\boldsymbol{U}}$ is skew-proper.*

A full mapping $\tilde{M}_S$ from G to ${}_{G}S$ defined by $x\tilde{M}_S = {}_{x^{-1}}S$ for $x \in G$ is called the *skew-coset mapping* for a subgroup $S \subset G$. With this definition, we obviously have

(7.11) $\widetilde{M}_S = M_S P_S$ and $M_S = \widetilde{M}_S P_S^{-1}$.

We will prove that

(7.12) $P_S\bigl(U(n)\,\widetilde{M}_S^+\bigr)\,P_S^{-1} = U(n)\,M_S^+$ for $e \in n \subset G$.

PROOF If $y\widetilde{M}_S = {}_{x^{-1}}S$, then $yM_S = y\widetilde{M}_S P_S^{-1} = {}_{x^{-1}}SP_S^{-1} = S_x$ by (7.11); and if $yM_S = S_x$, then $y\widetilde{M}_S = yM_S P_S = S_x P_S = {}_{x^{-1}}S$. Thus, we have $y\widetilde{M}_S = {}_{x^{-1}}S$ if and only if $yM_S = S_x$.

For any $x \in G$, by definition (5.4) we have

$$S_x P_S\bigl(U(n)\,\widetilde{M}_S^+\bigr)\,P_S^{-1} = {}_{x^{-1}}S\bigl(U(n)\,\widetilde{M}_S^+\bigr)\,P_S^{-1} = \bigcup_{y\widetilde{M}_S = {}_{x^{-1}}S} yU(n)\,\widetilde{M}_S P_S^{-1}$$
$$= \bigcup_{yM_S = S_x} yU(n)\,M_S = S_x\bigl(U(n)\,M_S^+\bigr).$$

Furthermore, we have

$$S_x P_S\bigl(U(n)\,\widetilde{M}_S^-\bigr)\,P_S^{-1} = {}_{x^{-1}}S\bigl(U(n)\,\widetilde{M}_S^-\bigr)\,P_S^{-1} = \bigcap_{y\widetilde{M}_S = {}_{x^{-1}}S} yU(n)\,\widetilde{M}_S P_S^{-1}$$
$$= \bigcap_{yM_S = S_x} yU(n)\,M_S = S_x\bigl(U(n)\,M_S^-\bigr)$$

for every $x \in G$. Therefore we also have

(7.13) $P_S\bigl(U(n)\,\widetilde{M}_S^-\bigr)\,P_S^{-1} = U(n)\,M_S^-$ for $e \in n \subset G$.

Let N be a neighborhood on a group G. For a subgroup $S \subset G$ the strong uniformity $\widetilde{\boldsymbol{U}}_S$ on the skew-coset space ${}_GS$ by the skew-coset mapping $\widetilde{M}_S$ for the induced uniformity $\boldsymbol{U}(N)$ is called the *skew-coset uniformity* on ${}_GS$ for N.

For the coset uniformity $\boldsymbol{U}_S$ on S_G we have

(7.14) $P_S \widetilde{\boldsymbol{U}}_S P_S^{-1} = \boldsymbol{U}_S$.

PROOF For $\boldsymbol{U}_S$ on S_G and $P_S^{-1}\boldsymbol{U}_S P_S$ on ${}_GS$, since P_S is uniformly continuous and $\widetilde{M}_S = M_S P_S$ by (7.11), $\widetilde{M}_S$ is uniformly continuous, and we have $P_S^{-1}\boldsymbol{U}_S P_S \subset \widetilde{\boldsymbol{U}}_S$ because $\widetilde{\boldsymbol{U}}_S$ is the strong uniformity on ${}_GS$ by $\widetilde{M}_S$. Thus we have $\boldsymbol{U}_S \subset P_S \widetilde{\boldsymbol{U}}_S P_S^{-1}$. For $\widetilde{\boldsymbol{U}}_S$ on ${}_GS$ and $P_S^{-1}\widetilde{\boldsymbol{U}}_S P_S$ on S_G, since P_S^{-1} is uniformly continuous and $M_S = \widetilde{M}_S P_S^{-1}$ by (7.11), M_S is uniformly continuous, and we have $P_S \widetilde{\boldsymbol{U}}_S P_S^{-1} \subset \boldsymbol{U}_S$ because $\boldsymbol{U}_S$ is the strong uniformity on S_G for M_S. Therefore we obtain (7.14).

Referring to Strong Uniformity Theorem 5.14, by (7.12) and (7.13) we obtain

(7.15) The skew-coset Mapping $\widetilde{M}_S$ is uniformly open for the skew-coset uniformity $\widetilde{\boldsymbol{U}}_S$ if and only if the coset mapping M_S is uniformly open for the coset uniformity $\boldsymbol{U}_S$.

Referring to Coset Uniformity Theorem 6.17, by (7.10) and (7.14) we have

(7.16) If the skew-coset mapping $\tilde{M}_S$ is uniformly open for the skew-coset uniformity $\tilde{\boldsymbol{U}}_S$, then $\tilde{\boldsymbol{U}}_S$ is skew-proper.

Now we suppose that S is an invariant subgroup of G. Since $xS=Sx$ for every $x\in G$ by definition, the coset space S_G forms the quotient group G/S, and the coset mapping M_S is a homomorphism from G to G/S whose kernel is S. Since for $\emptyset\neq n\subset G$ we have $Sn=nS$, by Coset Uniformity Theorem 6.17 we conclude that for any neighborhood N on G the homomorphism M_S is uniformly open for the coset uniformity $\boldsymbol{U}_S$.

For $\emptyset\neq n\subset G$ we make use of the notation $n/S=S_n=nM_S$ and $N/S=NM_S$ for a set class N of G. Then we have

$$(7.17)\qquad U_n=U(n/S)\qquad \text{for}\quad e\in n\subset G\,.$$

because $S_xU_n=S_{nSx}=(n/S)\,S_x=S_xU(n/S)$ for every $x\in G$. Furthermore, we can easily prove that N/S forms a neighborhood on G/S for any neighborhood N on G. Therefore, by (7.17) we conclude that the coset uniformity $\boldsymbol{U}_S$ is the induced uniformity by N/S, and we can state

Invariant Subgroup Theorem 7.18. *If S is an invariant subgroup of a group G, then the coset mapping M_S is a homomorphism from G to the quotient group G/S; for any neighborhood N on G, N/S forms a neighborhood on G/S; the coset uniformity $\boldsymbol{U}_S$ is the induced uniformity $\boldsymbol{U}(N/S)$; and M_S is uniformly open for $\boldsymbol{U}_S$.*

If S is an invariant subgroup of G, then ${}_GS=S_G$, and P_S is a transformation on G/S. Furthermore, P_S is the inversion on G/S because

$$S_xP_S=x^{-1}S=Sx^{-1}=S_x^{-1}\qquad \text{for every}\quad x\in G\,.$$

Thus we have $P_S^{-1}=P_S$.

For the skew-coset uniformity $\tilde{\boldsymbol{U}}_S$ we have

(7.19) P_S is uniformly continuous as a transformation on G/S for $\boldsymbol{U}_S$ if and only if $\tilde{\boldsymbol{U}}_S=\boldsymbol{U}_S$.

Proof If $\tilde{\boldsymbol{U}}_S=\boldsymbol{U}_S$, then $P_S\boldsymbol{U}_SP_S^{-1}=\boldsymbol{U}_S$ by (7.14), and P_S is uniformly continuous as a transformation on G/S by definition. Conversely, if P_S is uniformly continuous as a transformation on G/S for $\boldsymbol{U}_S$, then $P_S\boldsymbol{U}_SP_S^{-1}\subset\boldsymbol{U}_S$ by definition that implies $\boldsymbol{U}_S\subset P_S^{-1}\boldsymbol{U}_SP_S$. Since $P_S^{-1}=P_S$, we obtain $\boldsymbol{U}_S=P_S\boldsymbol{U}_SP_S^{-1}=\tilde{\boldsymbol{U}}_S$ by (7.14).

(7.20) P_S is uniformly continuous as a transformation on G/S for $\boldsymbol{U}_S$

if and only if the left transformation group $L_{G/S}$ is equi-continuous for $\boldsymbol{U}_S$.

Generally we will prove

Inversion Theorem 7.21. *The inversion Iv on a group G is uniformly continuous for an induced uniformity $\boldsymbol{U}(N)$ if and only if the left transformation group L_G is equi-continuous for $\boldsymbol{U}(N)$.*

Proof The inversion Iv is uniformly continuous for $\boldsymbol{U}(N)$ by definition if and only if for any $m\in N$ there is $n\in N$ such that $(nx)\,Iv\subset m(xIv)$ for $x\in G$; i.e., $x^{-1}n^{-1}\subset mx^{-1}$ for $x\in G$. The left transformation group L_G is equi-continuous for $\boldsymbol{U}(N)$ by definition if and only if for any $m\in N$ there is $n\in N$ such that $(nx)\,L_y\subset m(xL_y)$ for $x, y\in G$; i.e., $yn\subset my$ for $y\in G$. Since $n\in N$ implies $n^{-1}\in N$, we obtain Inversion Theorem 17.20.

As an immediate consequence of Invariant Subgroup Theorem 7.17, we have

Homomorphism Theorem 7.22. *Let M be a homomorphism from a group G to another group H. For a neighblrhood N on G, NM forms a neighborhood on H, and the induced uniformity $\boldsymbol{U}(NM)$ on H is the strong uniformity on H by M for the induced uniformity $\boldsymbol{U}(N)$ on G, and M is uniformly open.*

8. Relative Uniformities on Adjoints Let H be an adjoint of a subgroup S in a group G; i.e., $S\cap H=\{e\}$ and $G=SH$. Since for $u, v\in S$ and $x, y\in H$ we have $ux=vy$ if and only if $u=v$ and $x=y$, setting $(ux)\,M_H=x$ for $u\in S$ and $x\in H$, we obtain a full mapping M_H from G to H. This mapping M_H is called the *adjoint mapping* for an adjoint H of S.

Concerning the adjoint mapping M_H, we can easily prove

(8.1) $\quad (AB)\,M_H=BM_H \quad$ for $\emptyset\neq A\subset S$ and $\emptyset\neq B\subset G$,

(8.2) $\quad (AX)\,M_H=X \quad$ for $\emptyset\neq A\subset S$ and $\emptyset\neq X\subset H$,

(8.3) $\quad (AX)\,M_H=AM_HX \quad$ for $\emptyset\neq A\subset G$ and $\emptyset\neq X\subset H$,

(8.4) $\quad XM_H^{-1}=SX \quad$ for $\emptyset\neq X\subset H$, and

(8.5) $\quad AM_HM_H^{-1}=SA \quad$ for $\emptyset\neq A\subset G$.

Let N be a neighborhood on a group G. Setting

$$N^H=\{n^H\colon n\in N\} \qquad \text{where} \quad n^H=n\cap H \quad \text{for} \quad n\in N,$$

we obtain a neighborhood N^H on H, and we can easily prove that the induced uniformity $\boldsymbol{U}(N^H)$ on H is the relative uniformity of the induced uniformity $\boldsymbol{U}(N)$ on G. The neighborhood N^H on H is called the *relative neighborhood* of N on H.

ADJOINT MAPPING THEOREM 8.6. *The adjoint mapping M_H is uniformly continuous for the relative neighborhood N^H if and only if for any $m\in N$ there is $n\in N$ such that $nS\subset Sm^H$, and then M_H is uniformly open.*

PROOF M_H is uniformly continuous if and only if for any $m\in N$ there is $n\in N$ by definition such that $(nux)M_H\subset m^H((ux)M_H)$ for $u\in S$ and $x\in H$; i. e., $(nu)M_H\subset m^H$ for every $u\in S$ because $(nux)M_H=(nu)M_Hx$ by (8.3).

If $(nu)M_H\subset m^H$ for every $u\in S$, then $nu\subset Sm^H$ for every $u\in S$ by (8.4), and we obtain $nS\subset Sm^H$. Conversely, if $nS\subset Sm^H$, then $nu\subset Sm^H$ for every $u\in S$, and $(nu)M_H\subset(Sm^H)M_H=m^H$ by (8.2). Therefore, M_H is uniformly continuous if and only if for any $m\in N$ there is $n\in N$ such that $nS\subset Sm^H$

Furthermore, since $n^H\subset n$ and $m^H\subset m$, $nS\subset Sm^H$ implies $n^HS\subset S_m$, and $n^H\subset Smu$ for every $u\in S$. Thus, referring to definition (5.4), by (8.2) and (8.3) we have

$$n^Hx\subset\bigcap_{u\in S}(mux)M_H=x\left(U(m)M_H^{-}\right)$$

for every $x\in H$. Therefore M_H is uniformly open by (5.12).

We defined a transformation P_H from H to S_G by $xP_H=S_x$ for $x\in H$ in Representation Theorem 3.1. For the coset mapping M_S from G to S_G, by definition we have

(8.7) $$M_S=M_HP_H.$$

Since $m^H\subset m$, we have that $nS\subset Sm^H$ implies $nS\subset Sm$. Thus, by Adjoint Mapping Theorem 8.6 and Coset Uniformity Theorem 6.17 we conclude that if M_H is uniformly continuous for N^H, then the coset mapping M_S is uniformly open for the coset uniformity $\boldsymbol{U}_S$ on S_G, and P_H is a unimorphism by (5.2) and Unimorphism Theorem 5.15. Therefore we have

ADJOINT MAPPING THEOREM 8.8. *If the adjoint mapping M_H is uniformly continuous for the relative neighborhood N^H, then the coset mapping M_S is uniformly open for the coset uniformity $\boldsymbol{U}_S$ on S_G for which P_H is a unimorphism from H to S_G.*

According to Representation Theorem 3.1, the representation T_G of a group G on its subgroup H is congruent to the coset transformation group C_G on S_G. Referring to Equivalence Theorem 5.16, Coset Uniformity Theorem 6.17, and Proper Uniformity Theorem 6.14, we obtain

REPRESENTATION THEOREM 8.9. *If the adjoint mapping M_H is uniformly continuous for the relative neighborhood N^H, then the representation T_G of a group G on its subgroup H is equi-continuous for N^H.*

Since $G=SH$ and $S\cap H=\{e\}$, we can consider G the product space of S and H, and then M_H is the projection of G on H. Thus, if $\boldsymbol{U}=$

$\boldsymbol{U}^S \times \boldsymbol{U}^H$ for the relative uniformities of $\boldsymbol{U}$ on S and H respectively, then M_H is uniformly continuous by Theorem 24.2 in [3]. Therefore, Representation Theorem 8.9 is a generalization of the Deviation Theorem in [6].

9. Double Integral Theorem Let N be a neighborhood on a group G. If N is regular, i.e., each left transformation L_x $(x \in G)$ is uniformly continuous for the induced uniformity $\boldsymbol{U}(N)$, then the inversion Iv is continuous by Theorem 41.11 in [3]. If N is complete in addition, i.e., the induced uniformity $\boldsymbol{U}(N)$ is complete, then for any totally bounded set $A \subset G$ for $\boldsymbol{U}(N)$, the closure A^- is compact, and $A^- Iv$ also is compact because Iv is continuous. Since $AIv \subset A^- Iv$, AIv is totally bounded and $A^{-1} = AIv$ by definition. Therefore we have

(9.1) If N is regular and complete, then for any totally bounded set $A \subset G$ for $\boldsymbol{U}(N)$, A^{-1} also is totally bounded for $\boldsymbol{U}(N)$.

A set $A \subset G$ is said to be right totally bounded for N if for any $n \in N$ we can find a finite system $x_\nu \in G$ $(\nu = 1, 2, \cdots, n)$ such that $A \subset \bigcup_{\nu=1}^{n} n x_\nu$.

It is clear by definition that a set $A \subset G$ is right totally bounded if and only if A is totally bounded for the induced uniformity $\boldsymbol{U}(N)$.

A neighborhood N is said to be right totally bounded if there is a set $n \in N$ that is right totally bounded for N. According to Theorem 40.3 in [3], N is right totally bounded if and only if the induced uniformity $\boldsymbol{U}(N)$ is locally totally bounded; i.e., there is $U_0 \in \boldsymbol{U}(N)$ such that xU_0 is totally bounded for $\boldsymbol{U}(N)$ for every $x \in G$.

If N is locally uniformly regular, i.e., N is regular and there is $n \in N$ such that the left transformations L_x $(x \in n)$ are equi-continuous for $\boldsymbol{U}(N)$, then by Theorem 4.11 in [3], Iv is locally uniformly continuous for $\boldsymbol{U}(N)$; i.e., there is $n \in N$ such that Iv is uniformly continuous on $xU(n)$ for every $x \in G$. Thus, for any totally bounded set $A \subset G$ for $\boldsymbol{U}(N)$, Iv is uniformly continuous on A, and A^{-1} also is totally bounded for $\boldsymbol{U}(N)$. Therefore we have

(9.2) If N is locally uniformly regular, then for any totally bounded set $A \subset G$ for $\boldsymbol{U}(N)$, A^{-1} also is totally bounded for $\boldsymbol{U}(N)$.

Now we suppose that for any right totally bounded set $A \subset G$, A^{-1} also is right totally bounded for N, and N is regular and right totally bounded. Let S be a subgroup of G such that left transformations L_u $(u \in S)$ are equi-continuous for the induced uniformity $\boldsymbol{U}(N)$; i.e., for any $m \in N$ there is $n \in N$ such that $nu \subset um$ for every $u \in S$. Since $nu \subset um$ for every $u \in S$ implies $nS \subset Sm$, the coset mapping M_S is uniformly open for the coset uniformity $\boldsymbol{U}_S$ on S_G by Coset Uniformity Theorem 6.17, and the coset transformation group C_G is equi-continuous on S_G for $\boldsymbol{U}_S$ by Proper Uniformity Theorem 6.14. Since N is right totally bounded by assumption, there is $m_0 \in N$ by definition such that m_0 is right totally bounded for N. Since M_S is uniformly open, there is $U_0 \in \boldsymbol{U}_S$ such that $xM_S U_0 \subset xU(m_0) M_S$ for every $x \in G$. Since $xU(m_0) = m_0 s$ by definition and $m_0 x$ is totally bounded for $\boldsymbol{U}(N)$, $xU(m_0) M_S$ is totally bounded for $\boldsymbol{U}_S$ because M_S is uniformly continuous. Therefore $xM_S U_0$ is totally bounded for $\boldsymbol{U}_S$ for every $x \in G$, and hence $\boldsymbol{U}_S$ is locally totally bounded by definition.

According to the Theorem of Existence in [3], there exists a measure μ_0 on S_G that is invariant by C_G; i.e., for the trunk Φ_0 of U_S we have

$$\int \varphi(S_x C_y)\, \mu_0(dS_x) = \int \varphi(S_x)\, \mu_0(dS_x) \qquad \text{for} \quad \varphi \in \Phi_0 \quad \text{and} \quad y \in G\,.$$

The relative neighborhood N^S also is right totally bounded by definition, and the induced uniformity $\boldsymbol{U}(N^S)$ on S is regular and locally totally bounded. Thus, there exists a Haar measure μ_S of S; i.e., for the trunk Φ_S of $\boldsymbol{U}(N^S)$ we have

$$\int \varphi(xy)\, \mu_S(dx) = \int \varphi(x)\, \mu_S(dx) \qquad \text{for} \quad \varphi \in \Phi_S \quad \text{and} \quad y \in S\,.$$

Let Φ be the trunk of $\boldsymbol{U}(N)$. We can consider $\Phi \subset \Phi_S$ by definition. For any $\varphi \in \Phi$ we have

$$\int \varphi(uvx)\, \mu_S(du) = \int \varphi(ux)\, \mu_S(du) \qquad \text{for} \quad v \in S \quad \text{and} \quad x \in G\,.$$

Thus, setting

$$\psi(S_x) = \int \varphi(ux)\, \mu_S(du) \qquad \text{for} \quad x \in G\,,$$

we obtain a function ψ on S_G. We will prove $\psi \in \Phi_0$.

The set $A_0 = \{x : \varphi(x) \neq 0\}$ is right totally bounded for N by definition, and $\varphi(x) = 0$ for every $x \overline{\in} A_0$. Since $m_0 \in N$ is right totally bounded by assumption, setting $A = m_0 A_0$ and $B = S \cap AA^{-1}$, we obtain right totally bounded sets A and B by Theorem 40.6 in [3], and $B \overline{\ni} u \in S$ implies $uA \cap A = \emptyset$ because for any $x \in uA \cap A$ where $u \in S$ we have $u^{-1}x \in A$, $x \in A$, and

$$u = x(u^{-1}x)^{-1} \in S \cap AA^{-1} = B\,.$$

Therefore $\varphi(ux) = 0$ for $B \overline{\ni} u \in S$ and $x \in A$ because $A \supset A_0$.

According to Theorem 19.1 in [3], there is a uniformly continuous function φ_0 on G for $U(N)$ such that $\varphi_0(u)=1$ for $u\in B$, $1\geqq\varphi_0\geqq 0$, and $\varphi_0(x)=0$ for $x\overline{\in} m_0 B=BU(m_0)$. Since $m_0 B$ is right totally bounded for N by Theorem 40.6 in [3], we have $\varphi_0\in\Phi$.

Since φ is uniformly continuous on G for $U(N)$, for any $\varepsilon>0$ there is $m\in N$ such that $mm\subset m_0$ and

$$\left|\varphi(x)-\varphi(y)\right|<\varepsilon \qquad \text{for} \quad x\in my\,.$$

For such $m\in N$ there is $n\in N$ such that $un\subset mu$ for every $u\in S$ because L_u $(u\in S)$ are equi-continuous for $U(N)$ by assumption. Then, $x\in ny$ implies $ux\in uny\subset muy$ for every $u\in S$, and we have

$$\left|\varphi(ux)-\varphi(uy)\right|<\varepsilon \qquad \text{for} \quad u\in S \quad \text{and} \quad x\in ny\,.$$

Since $\varphi(ux)=\varphi(uy)=0$ for $B\overline{\ni} u\in S$ and $x,y\in A$, we have

$$\left|\varphi(ux)-\varphi(uy)\right|<\varepsilon\varphi_0(u) \qquad \text{for} \quad u\in S,\ x\in ny, \quad \text{and} \quad x,y\in A\,,$$

and we obtain

$$\left|\psi(S_x)-\psi(S_n)\right|\leqq\varepsilon\int\varphi_0(u)\,\mu_S(du) \qquad \text{for} \quad x\in ny \quad \text{and} \quad x,y\in A\,.$$

Since M_S is uniformly open, there is a symmetric $U\in\boldsymbol{U}_S$ such that $xM_S U\subset nxM_S$ for every $x\in G$, and we have $S_x U\subset S_{nx}$ for every $x\in G$. If $S_x\in S_y U$ and $S_y\in S_{nA_0}$, then we can find $y_0\in G$ such that $S_y=S_{y_0}$ and $y_0\in nA_0$, and $x_0\in G$ such that $S_x=S_{x_0}$ and $x_0\in ny_0$. Since $nn\subset m_0$, we have $y_0\in m_0 A_0=A$ and $x_0\in nnA_0\subset m_0 A_0=A$. Thus we have

$$\left|\psi(S_x)-\psi(S_y)\right|\leqq\varepsilon\int\varphi_0(u)\,\mu_S(du) \qquad \text{for} \quad S_x\in S_y U \text{ and } S_y\in S_{nA_0}\,.$$

If $S_y\overline{\in} S_{nA_0}$, they $y\overline{\in} SnA_0$ by (6.8), and $uy\overline{\in} SnA_0\supset A_0$ for every $u\in S$. Therefore $\varphi(uy)=0$ for every $u\in S$, and $\psi(S_y)=0$. If $S_y\overline{\in} S_{nA_0}$ and $S_x\in S_y U$, then, since $S_{nA_0}\cap S_{A_0}U$, we have $S_{A_0}\cap S_y U^{-1}=\emptyset$. Since U is symmetric by assumption, we obtain $S_x\overline{\in} S_{A_0}$, and $ux\overline{\in} A_0$ for every $u\in S$. Thus we have $\psi(S_x)=0$ for $S_x\in S_y U$ and $S_y\overline{\in} S_{nA_0}$. Therefore ψ is uniformly continuous. Since M_S is uniformly continuous and nA_0 is totally bounded for $U(N)$, S_{nA_0} is totally bounded for $\boldsymbol{U}_S$. Therefore $\psi\in\Phi_0$ by definition.

Setting $\int\varphi(x)\,\mu(dx)=\int\psi(S_x)\,\mu_0(dS_x)$ for $\varphi\in\Phi$ we obtain a measure μ on Φ, and for any $z\in G$ we have

$$\psi(S_{xz})=\int\varphi(uxz)\,\mu_S(du),\ \psi(S_{xz})=\psi(S_x C_z)\,, \quad \text{and}$$

$$\int\psi(S_x C_z)\,\mu_0(dS_x)=\int\psi(S_x)\,\mu_0(dS_x)\,.$$

Thus we have $\int\varphi(xz)\,\mu(dx)=\int\varphi(x)\,\mu(dx)$ for $\varphi\in\Phi$; i. e., μ is a Haar measure of G for $\boldsymbol{U}(N)$.

Now we can state

DOUBLE INTEGRAL THEOREM 9. 3. *Let N be a neighborhood on a group G such that for the induced uniformity $\boldsymbol{U}(N)$ every left transformation L_x $(x\in G)$ is uniformly continuous, A^{-1} is totally bounded for any totally bounded set $A\subset G$, and $\boldsymbol{U}(N)$ is locally totally bounded. Let S be a subgroup of G such that left transformations L_x $(x\in S)$ are equi-continuous for $\boldsymbol{U}(N)$. Then, for the coset uniformity $\boldsymbol{U}_S$ on the coset space S_G the coset mapping M_S is uniformly open and the coset transformation group C_G is equi-continuous. Let Φ be the trunk of $\boldsymbol{U}(N)$ on G and Φ_0 the trunk of $\boldsymbol{U}_S$ on S_G. For a Haar measure μ_S of S for the relative neighborhood N^S and an invariant measure μ_0 by C_G on S_G, setting*

$$\psi(S_x)=\int\varphi(ux)\,\mu_S(du)\qquad \text{for}\quad x\in G\quad \text{and}\quad \varphi\in\Phi\,,$$

we have $\psi\in\Phi_0$, and setting

$$\int\varphi(x)\,\mu(dx)=\int\psi(S_x)\,\mu_0(dS_x)\,,$$

we obtain a Haar measure μ of G for $\boldsymbol{U}(N)$.

10. Skew-Double Integral Theorem If the coset mapping M_S is uniformly open for the coset uniformity $\boldsymbol{U}_S$ on S_G, then the skew-coset mapping $\tilde{M}_S$ is uniformly open for the skew-coset uniformity $\tilde{\boldsymbol{U}}_S$ on ${}_GS$ by (7.15), P_S is a unimorphism from S_G to ${}_GS$ by (7. 14), and $\tilde{C}_G$ is equi-continuous by Skew-Proper Uniformity Theorem 7. 11 and (7. 16).

Let $\tilde{\Phi}_0$ be the trunk of $\tilde{\boldsymbol{U}}_S$ on ${}_GS$. For functions φ and $\tilde{\varphi}$ on S_G and ${}_GS$ respectively, if $\tilde{\varphi}({}_xS)=\varphi(S_{x^{-1}})$ for every $x\in G$, then we have $\tilde{\varphi}\in\tilde{\Phi}_0$ if and only if $\varphi\in\Phi_0$ because P_S is a unimorphism. If $\boldsymbol{U}_S$ is locally totally bounded, then so is $\tilde{\boldsymbol{U}}_S$, and for any invariant measure $\tilde{\mu}_0$ by $\tilde{C}_G$ on $\tilde{\Phi}_0$, setting

$$\int\varphi(S_x)\,\mu_0(dS_x)=\int\tilde{\varphi}({}_xS)\,\tilde{\mu}_0(d_xS)$$

for $\tilde{\varphi}({}_xS)=\varphi(S_{x^{-1}})$ for $x\in G$, we obtain an invarinat measure μ_0 by C_G on Φ_0 because

$$\varphi(S_{x^{-1}}C_y)=\varphi(S_{x^{-1}y})=\tilde{\varphi}({}_{y^{-1}x}S)=\tilde{\varphi}({}_xS\tilde{C}_{y^{-1}})$$

for every $x\in G$ and

$$\int\varphi(S_xC_y)\,\mu_0(dS_x)=\int\tilde{\varphi}({}_xS\tilde{C}_{y^{-1}})\,\tilde{\mu}_0(d_xS)=\int\tilde{\varphi}({}_xS)\,\tilde{\mu}_0(d_xS)=\int\varphi(S_x)\,\mu_0(dS_x)\,.$$

Furthermore, for any $\varphi \in \Phi$, setting

$$\tilde{\phi}({}_xS) = \int \varphi(ux^{-1})\,\mu_S(du) \quad \text{and} \quad \phi(S_x) = \int \varphi(ux)\,\mu_S(du)$$

for $x \in G$, we have $\tilde{\phi}({}_xS) = \phi(S_{x^{-1}})$ for every $x \in G$, and

$$\int \tilde{\phi}({}_xS)\,\tilde{\mu}_0(d_xS) = \int \phi(S_x)\,\mu_0(dS_x)\,.$$

Therefore by Double Integral Theorem 9.3 we have

SKEW-DOUBLE INTEGRAL THEOREM 10.1. *Under the same assumption as Double Integral Theorem* 9.3, *for the skew-coset uniformity* $\tilde{\boldsymbol{U}}_S$ *on* ${}_GS$, *the skew-coset mapping* $\widetilde{M}_S$ *is uniformly open and the skew-coset transformation group* $\tilde{C}_G$ *is equi-continuous. Let* Φ *be the trunk of* $\boldsymbol{U}(N)$ *on* G *and* $\tilde{\Phi}_0$ *the trunk of* $\tilde{\boldsymbol{U}}_S$ *on* ${}_GS$. *For a Haar measure* μ_S *of* S *for the relative neighborhood* N^S *and an invariant measure* $\tilde{\mu}_0$ *by* $\tilde{C}_G$ *on* ${}_GS$, *setting*

$$\tilde{\phi}({}_xS) = \int \varphi(ux^{-1})\,\mu_S(du) \qquad \textit{for} \quad x \in G \quad \textit{and} \quad \varphi \in \Phi\,,$$

we have $\tilde{\phi} \in \tilde{\Phi}_0$, *and setting*

$$\int \varphi(x)\,\mu(dx) = \int \tilde{\phi}({}_xS)\,\tilde{\mu}_0(d_xS)\,,$$

we obtain a Haar measure μ *of* G *for* $\boldsymbol{U}(N)$.

If S is an invariant subgroup of G, then S_G is the quotient group G/S and the coset transformation group C_G is the right transformation group $R_{G/S}$ by Quotient Group Theorem 4.5. Furthermore, the coset uniformity $\boldsymbol{U}_S$ is the induced uniformity $\boldsymbol{U}(N/S)$ by the neighborhood N/S on G/S. Therefore, every invariant measure μ_0 by C_G on G/S for $\boldsymbol{U}_S$ is a Haar measure of G/S for $\boldsymbol{U}(N/S)$. Consequently, we can state

INVARIANT SUBGROUP THEOREM 10.2. *In Double Integral Theorem* 9.3, *if* S *is an invariant subgroup of* G, *then* μ_0 *is a Haar measure of the quotient group* G/S *for the induced uniformity* $\boldsymbol{U}(N/S)$.

Now we suppose that S has an adjoint H in G and the adjoint mapping M_H is uniformly continuous for the relative neighborhood N^H on H. According to Adjoint Mapping Theorem 8.8, P_H is a unimorphism from H to S_G, and the representation T_G of G on H is congruent to C_G by P_H by Representation Theorem 3.1. For any function ϕ on H, setting $\phi(xP_H) = \phi(x)$ for $x \in H$, we can consider ϕ a function on S_G. Since P_H is a unimorphism, the trunk Φ_H of $\boldsymbol{U}(N^H)$ coincides with Φ_0, and we have

$$\phi(S_xC_z) = \phi(xT_zP_H) = \phi(xT_z) \qquad \text{for} \quad x \in H \quad \text{and} \quad z \in G\,.$$

Since $\Phi_0=\Phi_H$, any measure on Φ_H is a measure on Φ_0. If a measure μ_H on Φ_H is invariant by T_G, then μ_H is invariant by C_G. On the other hand, if μ_H is invariant by T_G, then μ_H is a Haar measure of H for the induced uniformity $\boldsymbol{U}(N^H)$ because $T_x=R_x$ for $x\in H$ by definition. Since every left transformation L_z $(z\in G)$ is uniformly continuous by assumption, we can easily prove that every L_x $(x\in H)$ is uniformly continuous for $\boldsymbol{U}(N^H)$. Thus, we can apply the Theorem of Uniqueness in [3], and we conclude that the Haar measures of H for $\boldsymbol{U}(N^H)$ are uniquely determined except for constant multiplication. Therefore, any Haar measure μ_H of H for $\boldsymbol{U}(N^H)$ is invariant by T_G.

Now we have another

PRODUCT MEASURE THEOREM 10.3. *Under the same assumption as Double Integral Theorem* 9.3, *if S has an adjoint H in G such that the adjoint mapping M_H is uniformly continuous for the relative neighborhood N^H on H, then for the trunk Φ_H of $\boldsymbol{U}(N^H)$ and a Haar measure μ_H of H for $\boldsymbol{U}(N^H)$, setting*

$$\psi(x)=\int\varphi(ux)\,\mu_S(du) \qquad \text{for} \quad \varphi\in\Phi,$$

we have $\psi\in\Phi_H$, and setting

$$\int\varphi(x)\,\mu(dx)=\int\psi(x)\,\mu_H(dx),$$

we obtain a Haar measure μ of G for $\boldsymbol{U}(N)$.

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