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Bimodule structure of certain Jordan algebras relative to subalgebras with one generator

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Dedicated to Goro Azumaya on his sixtieth birthday

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Throughout this paper "algebra" will mean finite dimensional algebra with unit over a field F and, unless otherwise indicated, "algebra" without modifier will mean associative algebra. An algebra is called a *Frobenius algebra* if there exists a non-degenerate associative bilinear form $f(x, y)$ on A, where associativity means that

$$f(ab, c)=f(a, bc) \tag{0.1}$$

for a, b, $c\in A$. This condition is readily seen to be equivalent to: A contains a hyperplane that contains no non-zero one sided ideal.

A number of years ago we proved the following result on generation of central simple algebras.

0.1. THEOREM. *Let A be a central simple algebra of degree n, C a commutative Frobenius subalgebra of n dimensions. Then A contains an element b such that $A=CbC$ (Jacobson* [1]).

It is well known that an algebra with a single generator is Frobenius (see e.g. Jacobson [1], p. 219). Hence we have the following consequence of this theorem.

0.2. COROLLARY. *Let A be a central simple algebra of degree n, a an element of A such that $[F[a]:F]=n$. Then A contains an element b such that $A=F[a]bF[a]$.*

The proof of Theorem 0.1 is based on the following facts:

1. The tensor product of Frobenius algebras is Frobenius. 2. If C is a commutative Frobenius algebra then any faithful representation of C contains the regular representation as a direct component. 3. If B is a subalgebra of a central simple algebra then A regarded as a bimodule for B in the natural way can be regarded as a faithful module for $B\otimes B^{op}$. This follows from the fact that A is faithful as $A\otimes A^{op}$ module which in turn follows from the simplicity of $A\otimes A^{op}$.

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In this note we propose to investigate to what extent the analogue of the above Corollary is valid for special Jordan algebras.

1. We recall that a subspace H of an algebra A is a special Jordan algebra if H contains 1 and H is closed under the composition aba and hence under $abc+cba$. If the characteristic is $\neq 2$ this is equivalent to closure under $a.b=\frac{1}{2}(ab+ba)$. Associated with H we have a special universal envelope $S(H)$, which is analogous to the universal enveloping algebra of a Lie algebra. We recall the definition. First, we define an associative specialization σ of H into the algebra B as a linear map such that $\sigma(1)=1$, $\sigma(aba)=\sigma(a)\sigma(b)\sigma(a)$. Then a special universal envelope is a pair $(S(H), \sigma_u)$ where $S(H)$ is an algebra and σ_u is an associative specialization of H into $S(H)$ such that if σ is any associative specialization of H into B then we have a unique (associative algebra) homomorphism η of $S(H)$ into B such that

$$\begin{tikzcd} H \arrow[r, "\sigma_u"] \arrow[d, "\sigma"'] & S(H) \arrow[dl, "\eta"] \\ B & \end{tikzcd}$$

is commutative. The pair $(S(H), \sigma_u)$ is unique in the usual strong sense of uniqueness of universals in category theory. The existence of $S(H), \sigma)$ is easily proved. We refer the reader to Jacobson [2] or [3] for this and proofs of other properties which we shall state. Among these we note that $S(H)$ is generated by $\sigma_u(H)$ and $S(H)$ has a (unique) in- volution ι such that $\iota\sigma_u(a)=\sigma_u(a)$, $a\in H$. Unlike the situation for universal enveloping algebras of Lie algebras, $S(H)$ is finite dimensional for finite dimensional H.

Let H be a subalgebra of a special Jordan algebra K with ambient associative algebra B. If $a, b\in H$ we write $U_K(a)$ for the linear map $x\mapsto axa$ in K and $U_K(a, b)=U_K(a+b)-U_K(a)-U_K(b)$. This is $x\mapsto axb+bxa$. If $K=H$ we drop the subscripts K and write $U(a)$ and $U(a, b)$. We now define the *squared special Jordan algebra* $S''(H)$ to be the subalgebra of $S(H)\otimes_F S(H)$ generated by the elements $\sigma(a)\otimes\sigma(a)$, $a\in H$. We shall now show that K has a left $S''(H)$*-module* structure in which

$$(1.1) \qquad \bigl(\sigma(a)\otimes\sigma(a)\bigr)\, x=U_K(a)\, x, \qquad a\in H,\ x\in K.$$

First, we have the homomorphism of $S(H)$ into B such that $\sigma(a)\mapsto a$, $a\in H$. Combining this with $b\mapsto b_L$ where b_L is $x\mapsto bx$ we obtain the homomorphism of $S(H)$ into $\mathrm{End}_F\, B$ such that $\sigma(a)\mapsto a_L$. Similarly, since we have the involution ι of $S(H)$ such that $\iota\sigma_u(a)=\sigma_u(a)$, we have the homomorphism of $S(H)$ into $\mathrm{End}_F B$ such that $\sigma(a)\mapsto a_R$. Since left multiplications commute with right multiplications we obtain a homomorphism of $S(H)\otimes S(H)$ into $\mathrm{End}_F B$ such that $\sigma(a)\otimes\sigma(b)\mapsto a_L b_R$. Then B is a $S(H)\otimes S(H)$ module in which $(\sigma(a)\otimes(b))\,x=axb,\ x\in B$. Hence B is an $S''(H)$-module in which $(\sigma(a)\otimes\sigma(a))\,x=axa$. Since H is a subalgebra of K, K is a submodule of B as $S''(H)$-module. Then K is an $S''(H)$-module in which we have (2).

It is clear from the definition that $S''(H)$ is contained in the subalgebra of $S(H)\otimes S(H)$ of elements fixed under the exchange automorphism such that $c\otimes d\mapsto d\otimes c,\ c,\ d\in S(H)$. In important cases $S''(H)$ coincides with this subalgebra. This is the case if H is the subalgebra of K generated by a single element a. For, it follows from the power formulas $U(a)\,1=a^2$, $U(a)\,a^k=a^{k+2}$ that the subalgebra generated by a is $F[a]$. It is readily seen that the special universal envelope of $F[a]$ is $F[a]$ together with the identity map, so $F[a]''$ is the subalgebra of $F[a]\otimes F[a]$ generated by the elements $b\otimes b,\ b\in F[a]$. It is readily seen also that this is the subalgebra of fixed points under the exchange automorphism. If the dimensionality $[F[a]:F]=m$ then $(1, a, \cdots, a^{m-1})$ is a base for $F[a]$ and $(a^i\otimes a^i,\ a^i\otimes a^j+a^j\otimes a^i,\ 0\leq i<j\leq m-1)$ is a base for $F[a]''$.

We recall that an element e of H is idempotent if $e^2=e$ and e and f are orthogonal idempotents if $e\circ f\equiv ef+fe=U_e f=U_f e=0$. These Jordan relations are equivalent to the associative relations $ef=0=fe$ in the ambient associativealgebra A. Let $\{e_1, \cdots, e_k\}$ be a set of non-zero orthogonal idempotents that are supplementary in the sense that $\Sigma e_i=1$. Then the operations $U(e_i)$, $U(e_i, e_j)$, $i<j$, are orthogonal idempotent endomorphisms such that $\Sigma U(e_i)+\Sigma U(e_i, e_j)=1$. Hence these give a Peirce decomposition of H as

$$H=\bigoplus_{i\leq j} H_{ij} \tag{1.2}$$

where $H_{ii}=U(e_i)\,H$, $H_{ij}=U(e_i, e_j)\,H$ for $i<j$. (Jacobson [4], p. 2. 1 ff.). Since the e_i form a su form a supplementary set of orthogonal idempotents in A we have the two-sided Peirce decomposition $A=\bigoplus A_{ij}$ where $A_{ij}=e_i A e_j$. Then

$$H_{ii}=H\cap A_{ii}\,,\qquad H_{ij}=H\cap(A_{ij}+A_{ji})\,,\qquad i<j \tag{1.4}$$

Evidently H_{ii} is a special Jordan algebra with unit e_i.

We shall now derive two results on Peirce decompositions of H that are valid in the abstract case (cf. Jacobson [2], p. 105 or McCrimmon [1], p. 294).

1.5. PROPOSITION. *We have a module action of* $S(H_{ii})\otimes S(H_{jj})$ *on* H_{ij}, $i<j$, *in which*

(1.6) $$\left(\sigma_u(h_{ii})\otimes\sigma_u(h_{jj})\right) h_{ij} = U(h_{ii}, h_{jj})\, h_{ij}\,.$$

PROOF. Let $h_{ii}=e_i a e_i$, $h_{ij}=e_i x e_j+e_j x e_i$, $a, x\in H$. Then

$$U(h_{ii}, e_j)\, h_{ij} = e_i a e_i\, (e_i x e_j+e_j x e_i)\, e_j+$$
$$e_j(e_i x e_j+e_j x e_i)\, e_i a e_i = e_i a e_i x e_j+e_j x e_i a e_i\,.$$

It follows that if h'_{ii} is a second element of H_{ii} then

$$U(h_{ii}, e_j)\, U(h'_{ii}, e_j)\, U(h_{ii}, e_j)\, h_{ij}$$
$$= U\left(U(h_{ii})\, h'_{ii}, e_j\right) h_{ij}\,.$$

Also $U(e_i, e_j) h_{ij}=h_{ij}$. Hence we have a homomorphism of $S(H_i)$ into End_F H_{ij} such that $\sigma_u(h_{ii})\mapsto U(h_{ii}, e_j)|H_{ij}$. Similarly, we have a homomorphism of $S(H_j)$ into $\mathrm{End}_F\, H_{ij}$ such that $\sigma_u(h_{jj})\mapsto U(e_i, h_{jj})|H_{ij}$. Next we can verify that

$$U(h_{ii}, e_j)\, U(e_i, h_{jj})\, h_{ij} = U(h_{ii}, h_{jj})\, h_{ij}$$
$$= U(e_i, h_{jj})\, U(h_{ii}, e_j)\, h_{ij}\,.$$

It follows that we have a homomorphism of $S(H_{ii})\otimes S(H_{jj})$ into $\mathrm{End}_F\, H_{ij}$ such that $\sigma_u(h_{ii})\otimes\sigma_u(h_{jj})\mapsto U(h_{ii}, h_{jj})|H_{ij}$. Hence we have a module action of $S(H_i)\otimes S(H_j)$ on H_{ij} for which (1.6) holds. □

We now suppose that H is a subalgebra of the Jordan algebra K with ambient algebra B containing A as a subalgebra. Suppose $H=H_1\oplus H_2 \oplus\cdots\oplus H_s$ where H_i is an ideal in H (H_i is a subspace and $U(h_i)\, h$ and $U(h)\, h_i\in H_i$ for $h_i\in H_i$, $h\in H$). We have $1=\Sigma 1_i$, $1_i\in H_i$ and the 1_i are orthogonal idempotents. The decomposition $H=\oplus H_i$ is the Peirce decomposition of H relative to this set of idempotents. Let E be the ring of endomorphisms in K generated by the $U_K(h)$, $h\in H$. Then E contains the supplementary set $\Sigma=\{U_K(1_i), U_K(1_i, 1_j), i<1)\}$ of orthogonal idempotent operators. If $h_i\in H_i$ then $U_K(1_i)\, U_K(h_i)\, U_K(1_i)=U_K(U_K(1_i)\, h_i)=U_K(h_i)$. Hence $U_K(1_i)U_K(h_i)=U_K(h_i)=U_K(h_i)\, U_K(1_i)$ and $U_K(1_j)\, U_K(h_i)=0=U_K(h_i)\, U_K(1_j)$ for $j\neq i$, $U_K(1_j, 1_k)\, U_K(h_i)=0=U_K(h_i)\, U_K(1_j, 1_k)$ for $j<k$. Direct verification shows also that $U_K(1_i, 1_j)\, U_K(h_i, h_j) = U_K(h_i, h_j) = U_K(h_i, h_j)\, U_K(1_i, 1_j)$ for $h_i\in H_i$, $h_j\in H_j$, $i<j$. It now follows that multiplication of $U_K(h_i, h_j)$ on either side by any idempotent operator $\neq U_K(1_i, 1_j)$ in the set Σ gives 0. We use these results to prove

1.7. PROPOSITION. *The idempotents $U_K(1_i)$, $U_K(1_i, 1_j)$ are central in E so $E=\bigoplus_{i\le j} E_{ij}$ where $E_{ii}=U_K(1_i)\, E$, $E_{ij} = U_K(1_i, 1_j)\, E$, $i<j$, are ideals. Moreover, $U_K(h'_i)\in E_{ii}$ and $U_K(h_i, h_j)\in E_{ij}$ if $h_i\in H_i$, $h_j\in H_j$ and $i<j$.*

PROOF. Any $h \in H$ can be written as Σh_i, $h_i \in H_i$. Since $U_K(h)$ is a quadratic function of h we have $U_K(h) = \Sigma U_K(h_i) + \sum_{i<j} U_K(h_i, h_j)$. Thus E is generated by the $U_K(h_i)$ and the $U_K(h_i, h_j)$. Since these commute with the idempotents $U_K(1_i)$, $U_K(1_i, 1_j)$, these idempotents are central and we have $E = \bigoplus_{i<j} E_{ij}$ when the E_{ij} are as defined in the statement of the proposition. These are ideals and since $U_K(h_i) = U_K(1_i)\, U_K(h_i)$, $U_K(h_i) \in E_{ii}$. Similarly $U_K(h_i, h_j) \in E_{ij}$. □

2. The special Jordan algebras we shall consider in this paper are the following: 1. A^+ where A is central simple assocative and A^+ is obtained from A by replacing the associative product by the composition $U(a)\, b = aba$. 2. The subalgebra $H(A, j)$ of A^+ of j-symmetric elements ($j(h) = h$) of A^+ where A is central simple with involution j. 3. $H(A, j)$ where A is simple with center a separable quadratic extension of the base field and j is an involution of second kind. All of these Jordan algebras are central simple. Under extension of the base field F to its algebraic closure $\bar{F}$ the Jordan algebras in 1. and 3. become $M_n(\bar{F})^+$ and the algebras in 2. become either $H(M_n)$ the Jordan algebra of $n \times n$ symmetric matrices or $H(M_n, s)$ the Jordan algebra of $n \times n$ (n even) symplectic symmetric matrices. In the first case j is said to be of *orthogonal type* and in the second of *symplectic type*.

If a is an element of a Jordan algebra H we call the degree of the minimum polynomial of a the *degree* of a. The maximum degree of the elements of H is called the *degree* of H. For the algebras we have listed the degrees in all cases are n except in the case $H(A, j)$ where j is of symplectic type, in which case the degree is $\nu = n/2$.

From now on H will denote one of the central simple Jordan algebras we have listed. In this section we shall show that H is a faithful $F[a]''$-module for any a that is *separable* in the sense that its minimum polynomial has distinct roots and the same result holds for all a if char $F \neq 2$. The statement that H is a faithful $F[a]''$-module is equivalent to the following: If $[F[a] : F] = m$ so $(1, a, \cdots, a^{m-1})$ is a base for $F[a]/F$ then the $m(m+1)/2$ linear transformations $U(a^i)$, $U(a^i, a^j)$, $0 \leq i < j \leq m-1$, are linearly independent. Since we can extend F to its algebraic closure $\bar{F}$ and replace H by $H_{\bar{F}}$, it suffices to prove this for F algebraically closed.

2.1. LEMMA. *If e and f are non-zero orthogonal idempotents in H then $U(e, f) \neq 0$.*

PROOF. Since $e+f$ is idempotent and $U_e H$ is a Jordan algebra of the same type as H we may assume $e+f=1$. Also we may assume F is algebraically closed. Then we can we can write $e=\sum_{1}^{m} e_i$, $f=\sum_{m+1}^{n} e_j$ where $\{e_k|1\leq k\leq n\}$ is a supplementary set of orthogonal idempotents $\neq 0$. Then we have the Peirce decomposition $\oplus H_{kl}$, relative to $\{e_k\}$. Since F is algebraically closed $H_{kk}=Fe_k$ $(\neq 0)$. Since H is simple every $H_{kl}\neq 0$ (Jacobson [4], p. 3.26). Then the orthogonal idempotent operators $U(e_k)$ and $U(e_k, e_l)$ are $\neq 0$ and so these are linearly independent. Hence $U(e,f)=\sum_{i\leq m, j>m} U(e_i, e_j)\neq 0$. □

2.2. THEOREM. *Let H be one of the central simple Jordan algebras listed above and let a be a separable element of H. Then H is faithful as $F[a]''$-module.*

PROOF. We may assume F algebraically closed. Then $F[a]=Fe_1\oplus\cdots\oplus Fe_m$ where the e_i are orthogonal idempotents and $a=\Sigma\alpha_i e_i$ where the α_i are distinct. Then every $U(e_i)$, $U(e_i, e_j)$ is non-zero so these are linearly independent for $1\leq i<j\leq m$. Hence the $U(a^i)$, $U(a^i, a^j)$ for $0\leq i<j\leq m-1$ are linearly independent. □

We now assume char $F\neq 2$ and we shall prove

2.3. THEOREM. *Let H be as before and assume char $F\neq 2$. Then H is a faithful $F[a]''$-module for every $a\in H$.*

PROOF. We assume first that $a=z$ is nilpotent with minimum polynomial λ^m. Then it is known (Jacobson [5]) that z can be imbedded in a subalgebra that is a direct sum of ideals H_i where H_i can be identified with the Jordan algebra of $m_i\times m_i$ matrices that are j_i-symmetric relative to the involution

$$j_i : x\mapsto M_i({}^t x)\, M_i^{-1} \tag{2.4}$$

where

$$M_i=\sum_{k+l=m_i+1} e_{kl}=M_i^{-1} \tag{2.5}$$

(e_{kl} the matrix with 1 in the (k, l)-position, 0's elsewhere). Moreover, we may suppose $m_1=m$ and the component z_i of z in H_i is

$$z_i=\sum_{1}^{m_i-1} e_{k,\,k+1} \tag{2.6}$$

To prove the operators $U(z^i, z^j)$, $0\leq i\leq j\leq m-1$, $(U(z^j, z^i)=2U(z^i))$ are linearly independent it suffiices to show this is the case for their restrictions to H_1. Hence we may assume H is the Jordan algebra of $m\times m$ matrices h such that $M({}^t h)\, M=h$ where $M=\sum_{k+l=m+1} e_{kl}$ and $z=\sum_{1}^{m-1} e_{k,k+1}$. Now H contains

(2.7) $$h_{kl}=e_{kl}+Me_{lk}M=e_{kl}+e_{m+1-l,m+1-k}$$

and

(2.8) $$(h_{kl}\mid 1\leq k,\ l\leq m,\ k+1\leq m+1)$$

is a base for H. We have

(2.9) $$U(z^i, z^j)\, h_{kl}=h_{k-i,l+j}+h_{k-j,l+i}$$

where $h_{u,v}=0$ unless the (u, v) satisfy the inequalities for (k, l) given in (2.8).

These formulas show that every $U(z^i, z^j)$, $0\leq i\leq j\leq m-1$, $\neq 0$ and that no two of the matrices of these linear transformations relative to the base (2.8) (taken in some order) have non-zero entries in the same position. Hence the matrices are linearly independent and the operators $U(z^i, z^j)$ are linearly independent.

Now let a be arbitrary and assume F is algebraically closed. Then $F[a]$ contains a supplementary set of orthogonal idempotents $\{e_i|1\leq i\leq s\}$ such that $a_i=U(e_i)\,a=\alpha_i e_i+z_i$ where $\alpha_i\in F$ and $z_i\in U(e_i)\,F[a]$ has minimum polynomial λ^{m_i}. Then

(2.10) $$(e_1, z_1, \cdots, z_1^{m_1-1};\ e_2, z_2, \cdots, z_2^{m_2-1}, \cdots;\ \cdots z_s^{m_s-1})$$

is a base for $F[a]$. It auffices to prove the linear independence of the set $\{U(x, y)\}$ where x and y are chosen in (2.10) and $x\leq y$ in the order displayed in (2.10). By Proposition 1.7 and the result just proved on nilpotent elements, it suffices to show that for every $i\neq j$ the operators $U(z_i^k, z_j^l)$, $0\leq k\leq m_i-1$, $0\leq l\leq m_j-1$ are linearly independent. By replacing H by $U(e_i+e_j)\,H$ we may assume $s=2$ and write $e=e_i$, $f=e_j$, $z=z_i$, $w=z_j$. If we apply the imbedding theorem for nilpotent elements to the algebras $U(e)\,H$ and $U(f)\,H$ we see that there exists a subalgebra of H of the form $\bigoplus_1^t H_k$ where H_k is an ideal in this subalgebra that can be identified with the algebra of $m_k\times m_k$ symmetric matrices and $z\in H_1+\cdots+H_r$, $w\in H_{r+1}+\cdots+H_t$. The result on the linear independence of the $U(z^k, w^l)$ will follow if we can show that if $(z_1, \cdots, z_u)$ is a base for $H_1+\cdots+H_r$ and $(w_1, \cdots, w_v)$ is a base for $H_{r+1}+\cdots+H_t$ then the set of operators $\{U(z_p, w_q)\}$ are linearly independent. By using Proposition 1.7 we may assume $t=2$. Now it is known that the special universal envelope of the algebra $HM_n(F)$ of $n\times n$ symmetric matrices is $M_n(F)$ (Jacobson [2], p. 134). This applies to H_i and implies that $S(H_1)\otimes S(H_2)$ is simple. Since $U(e,f)\,H\neq 0$ (Lemma 2.1), this is a faithful module for $S(H_1)\otimes S(H_2)$ via (1.6). Then the operators $U(z_p, w_q)$ are linearly independent. □

2.11. REMARK. The proof of Theorem 2.3 is considerably more complicated than the proof of the corresponding result in the associative case. The latter follows from the fact that if A is central simple associative then $A^e = A \otimes A^{op}$ is simple and hence A is a faithful A^e-module. On the other hand, if H is central simple Jordan then H is generally not faithful as $S''(H)$-module. For example, let $H = H(A, j)$ where A is central simple of degree ≥ 3 and j is an involution in A. Assume char $F \neq 2$. Then $S''(H) \cong M_{n(n-1)/2}(F) \oplus M_{n(n+1)/2}(F)$ (Jacobson [2], p. 273). Since $H(A, j)$ is simple it is irreducible as $S''(H)$-module. Since $S''(H)$ is not simple, H is not faithful as $S''(H)$-module.

3. We prove next a result due to Azumaya (1956. unpublished).

3.1. THEOREM. *Let B be the subalgebra of $F[a] \otimes F[a]$ of fixed elements under the automorphism ε such that $b \otimes c \mapsto c \otimes b$. Then B is a Frobenius algebra if and only if either a is separable or char $F \neq 2$.*

PROOF. We recall first the well known result that an algebra A over F is Frobenius if and only if $A_{\bar{F}}$ is Frobenius for $\bar{F}$ the algebraic closure of F. Also it is clear that the exchange automorphism $\bar{\varepsilon}$ in $\bar{F}[a] \otimes \bar{F}[a]$ is the extension of ε in $F[a] \otimes F[a]$ and that the subalgebra of $\bar{\varepsilon}$-fixed elements is $B_{\bar{F}}$. Hence it suffice to prove the theorem for an algebraically closed base field. In this case we have $F[a] = F[a_1] \oplus \cdots \oplus F[a_r]$ where $a_i = \alpha_i e_i + z_i$, $\alpha_i \in F$, e_i the unit of $F[a_i]$ and z_i has minimum polynomial λ^{m_i}. Then $F[a] \otimes F[a] \cong \bigoplus_{i,j} (F[a_i] \otimes F[a_j])$. The automorphism ε interchanges $F[a_i] \otimes F[a_j]$ and $F[a_j] \otimes F[a_i]$. It follows that $B \cong \bigoplus_{i<j} (F[a_i] \otimes F[a_j]) \oplus (\bigoplus B_i)$ where B_i is the subalgebra of ε-fixed points of $F[a_i] \times F[a_i]$. Since the direct sum of algebras is Frobenius if and only if every component is Frobenius and the tensor product of Frobenius algebras is Frobenius, it suffices to prove the theorem for the case $r = 1$.

We now suppose $a = \alpha 1 + z$ where the minimum polynomial of z is λ^m. By replacing a by z we may assume a is nilpotent with minimum polynomial λ^m. If $m = 1$, $a = 0$ and $F[a] = F$. Then $B = F1$ is Frobenius. This and the preceding reductions show that B is Frobenius if a is separable.

Now assume a is nilpotent with minimum polynomial λ^m, $m > 1$. Then B has the base

$$(1\,;\ a^i \otimes a^i,\ 1 \leq i \leq m-1\,;\ a^i \otimes a^j + a^j \otimes a^i,\ 0 \leq i < j \leq m-1)\,. \tag{3.2}$$

These elements $\neq 1$ are nilpotent and hence span a nilpotent ideal N. We have $B/N \cong F$. Hence B is a local ring with N as its unique maximal ideal. It follows that B is Frobenius if and only if it has a unique minimal ideal (see e.g. Jacobson [1], p. 219). Since $B/N \cong F$ a B-module is completely reducible if and only if it is annihilated by N and irreducible B-modules are one dimensional. Hence the socle of B (= sum of the minimal ideals of B) is the annihilator M of N in B and B is Frobenius if and only if M is one dimensional.

Now N is generated as an ideal by the elements $a\otimes a$ and $1\otimes a^i+a^i\otimes 1$, $1\leq i\leq m-1$, since

$$(a\otimes a)^i = a^i\otimes a^i$$
$$a^i\otimes a^j+a^j\otimes a^i = (a\otimes a)^i\,(1\otimes a^{j-i}+a^{j-i}\otimes 1)$$

if $1\leq i<j\leq m-1$. We determine first the annihilator M_1 of $a\otimes a$. For this purpose we multiply the base (3.2) by $a\otimes a$. This multiplication gives 0 for the base elements $a^{m-1}\otimes a^{m-1}$ and $a^i\otimes a^{m-1}+a^{m-1}\otimes a^i$. Otherwise, we obtain distinct elements of the base (3.2). It follows that M_1 is the subspace spanned by $a^{m-1}\otimes a^{m-1}$ and $a^i\otimes a^{m-1}+a^{m-1}\otimes a^i$, $0\leq i<m-1$. We now multiply these elements by $1\otimes a+a\otimes 1$ to obtain successively 0 and $a^{i+1}\otimes a^{m-1}+a^{m-1}\otimes a^{i+1}$. It follows that the annihilator of $a\otimes a$ and $a\otimes 1+1\otimes a$ is spanned by $a^{m-1}\otimes a^{m-1}$ if char $F\neq 2$ and by $a^{m-1}\otimes a^{m-1}$ and $a^{m-2}\otimes a^{m-1}+a^{m-1}\otimes a^{m-2}$ if char $F=2$. Now

$$1\otimes a^i+a^i\otimes 1 \equiv (1\otimes a+a\otimes 1)^i \bmod \left((a\otimes a)\,B\right).$$

Hence $a^{m-1}\otimes a^{m-1}\in M$ for any F and $a^{m-2}\otimes a^{m-1}+a^{m-1}\otimes a^{m-2}\in M$ if char $F=2$. Thus

$$M=F(a^{m-1}\otimes a^{m-1}) \qquad \text{if char } F\neq 2$$
$$M=F(a^{m-1}\otimes a^{m-1})+F(a^{m-2}\otimes a^{m-1}+a^{m-1}\otimes a^{m-2}) \qquad \text{if char } F=2$$

and B is Frobenius if and only if char $F\neq 2$. □

4. We can now derive our main results on the central simple Jordan algebras listed at the beginning of 2.

4.1. THEOREM. *Let H be a Jordan algebra in the following list: 1. A^+, A central simple associative, 2. $H(A,j)$ where A is central simple and j is an involution, 3. $H(A,j)$ where A is simple with center a separable quadratic extension of the base field and j is an involution of second kind. Let $a\in H$ be of degree m and suppose a is separable if char $F=2$. Then there exists an element b in H such that $F[a]''b$ has the base a^iba^i, $a^iba^j+a^jba^i$ where $0\leq i<j\leq n-1$ and $H=F[a]''b\oplus M$, M an $F[a]''$-submodule of H.*

PROOF. The results of the last two sections show that $F[a]''$ is a Frobenius algebra and H is a faithful module for $F[a]''$. Hence $H=P\oplus M$ where P and M are $F[a]''$-submodules and $p\cong F[a]''$. Let b denote the image of 1 in a $F[a]''$ isomorphism of $F[a]''$ onto P. Since the elements $a_i\otimes a^i$, $a^i\otimes a^j+a^j\otimes a^i$ form a base for $F[a]''$ it follows that the elements $a^i ba^i$, $a^i ba^j+a^j ba^i$, $0\le i<j\le m-1$, form a base for P.

The dimensionality of $F[a]''b$ is $m(m+1)/2$. This coincides with $[H:F]$ if and only if $H=H(A,j)$ where A is central simple and j is an involution of orthogonal type, and a has maximal degree. The positive part of this result can be stated as

4.2. COROLLARY. *Let $H=H(A,j)$ where A is central simple and j is an involution of orthogonal type in A. Let a be an element of H of degree n where $[A:F]=n^2$ and assume a is separable if char $F=2$. Then there exists a $b\in H$ such that*

$$(a^i ba^i,\ a^i ba^j+a^j ba^i,\ 0\le i<j\le n-1) \tag{4.3}$$

is a base for H over F.

4.4. REMARKS. The foregoing theorem and corollary appear to constitute an adequate Jordan analogue for Corollary 0.2. One is tempted to define a Frobenius Jordan algebra as a Jordan algebra that possesses a non-degenerate symmetric associative bilinear form. If char $F\neq 2$ the analogue of a commutative associative algebra is a Jordan algebra that is associative in the multiplication $a.b=\frac{1}{2}U(a,b)1$. This leads to the question of the valdity of an analogue of Theorem 0.1 in which commutative Frobenius subalgebras are replaced by associative Frobenius subalgebras of H (for char $F\neq 2$).

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