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Finitely generated projective modules over hereditary noetherian prime rings II

Dedicated to Professor Kentaro MURATA on his 60th birthday

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The purpose of this paper is to generalize [5, Theorem (2. 6)] as the following.

THEOREM. *Let R be a hereditary noetherian prime ring and M, N finitely generated projective modules such that $N\subset M$ and* rank $M=$ rank N. *Let $N=N_{0}\subset N_{1}\subset\cdots\subset N_{n}=M$ be a composition series of M/N, $S_{i}=N_{i}/N_{i-1}$ $(i=1, \cdots, n)$, $\mathscr{S}=\{S_{i}; i=1, \cdots, n\}$, and $\mathscr{P}=\{P; P$ is an idempotent maximal ideal such that $S_{i}P=0$ for some $S_{i}\in \mathscr{S}\}$. Then $M\sim N$ iff the following hold;*

1) *for an idempotent maximal ideal $P\notin \mathscr{S}$ and a simple right R-module S with $SP=0$, $\mathrm{Ext}_{R}^{1}(S_{i}, S)=0$ for every faithful simple module $S_{i}\in \mathscr{S}$,*

2) *for an idempotent maximal ideal $P\in \mathscr{P}$ which belongs to a cycle $\{P_{1}, \cdots, P_{k}\}$, $\mathscr{S}$ includes each simple right R-module T_{j} with $T_{j}P_{j}=0$ $(j=1, \cdots, k)$ by the same number,*

3) *for an idempotent maximal ideal $P\in \mathscr{P}$ which belongs to a strictly open cycle $\{P_{1}, \cdots, P_{k}\}$, $\mathscr{S}$ includes each simple right R-module T_{j} with $T_{j}P_{j}=0$ $(j=1, \cdots, k)$ and a faithful simple right R-module T with $\mathrm{Ext}_{R}^{1}(T, T_{k})\neq 0$ by the same number.*

Throughout the paper, let R be a hereditary noetherian prime ring and M, N finitely generated projective right R-modules. M and N are said to be of the same *genus* [3], denoted by $M\sim N$, if rank $M=$ rank N and $M/MP\cong N/NP$ for all maximal ideals P of R. In the previous paper [5], we studied the condition for $M\sim N$ when R has enough invertible ideals. We shall extend a portion of [5] to the general case.

Let Q be the maximal quotient ring of R. For a fractional R-ideal I, we put $O_{r}(I)=\{x\in Q; Ix\subset I\}$ and $O_{l}(I)=\{x\in Q; xI\subset I\}$. A finite set of distinct idempotent maximal ideals $\{P_{1}, \cdots, P_{k}\}$ of R is called a *cycle* if $O_{r}(P_{i})$

$=O_l(P_{i+1})$ $(i=1, \cdots, k-1)$ and $O_r(P_k)=O_l(P_1)$ [1], while it is called a *strictly open cycle* if $O_r(P_i)=O_l(P_{i+1})$ $(i=1, \cdots, k-1)$, $O_r(P_k)\neq O_l(P)$ and $O_r(P)\neq O_l(P_1)$ for any idempotent maximal ideal P of R [6]. Let $J(A)$ denote the intersection of all maximal submodules of a module A.

LEMMA 1. *Let P be an idempotent maximal ideal, S a simple right R-module with $SP=0$, and $N\subset M$, $M/N=T$ simple. Then the following hold,*

1) *If $T\cong S$, i.e., $TP=0$, then $M/MP\cong N/NP\oplus S$.*

2) *If $T\not\cong S$, i.e., $TP\neq 0$, then;*

i) *in case of either* a) *$TP_1=0$ for an idempotent maximal ideal P_1 with $O_r(P)=O_l(P_1)$ or* b) *T is faithful with* $\mathrm{Ext}_R^1(T, S)\neq 0$, *we have $N/NP\cong M/MP\oplus S$.*

ii) *otherwise, $N/NP\cong M/MP$.*

PROOF. 1) Since $MP=NP$, the exact sequence $0\to N/NP\to M/MP\to T\to 0$ splits, and then $M/MP\cong N/NP\oplus S$. 2) Since $MP+N=M$, then exact sequence $0\to(MP\cap N)/NP\to N/NP\to M/MP\to 0$ splits, and then $N/NP\cong M/MP\oplus(MP\cap N)/NP$. We shall determine the length of the finitely generated semisimple module $(MP\cap N)/NP$. Consider the exact sequence

$$(*)\quad 0\longrightarrow(MP\cap N)/NP\longrightarrow MP/NP\longrightarrow MP/(MP\cap N)\longrightarrow 0\,.$$

When the sequence (*) splits, we have $MP/NP\cong(MP\cap N)/NP\oplus T$, $(MP/NP)P=MP/NP$, and so $(MP\cap N)/NP=0$. In this case $N/NP\cong M/MP$ holds. Otherwise, the sequence (*) doesn't split, in particular, $(MP\cap N)/NP\neq 0$. Put $A=MP/NP$ and $B=(MP\cap N)/NP$. We have $AP=A$ and $BP=0$, and then B is isomorphic to the finite direct sum of S. We can assume $B\subset A$. We shall show $B=J(A)$. Let C be a maximal submodule of A with $C\neq B$. Then there exists a simple direct summand S' of B such that $S'\cong S$ and $C\oplus S'=A$. Thus $A=AP=CP$, a contradiction. Therefore, $B=J(A)$. Hence $B=S$ by [4, Lemma 5.5] and we have $N/NP\cong M/MP\oplus S$.

Next we shall show that (*) doesn't split iff $\mathrm{Ext}_R^1(T, S)\neq 0$. Assume that $\mathrm{Ext}_R^1(T, S)\neq 0$ and (*) splits. Then we have that $[O_r(P)/R]_R$ is a direct sum of copies of T by [3, Theorem 7] and $MP/NP\cong T\otimes_R P\cong T$. Let $X=\{K;\ K$ is a maximal right ideal of R such that $R/K\not\cong T\}$. Then $O_r(P)=R_X$ by the proof of [2, Theorem 5], where R_X is a localization of R constructed in [2]. Thus we have $T\otimes_R R_X=0$. Since $PR_X=P\cdot O_r(P)=P$, it holds that $0=T\otimes_R R_X\cong T\otimes_R P\otimes_R R_X\cong T\otimes_R P\cong T$, a contradiction. Hence $\mathrm{Ext}_R^1(T, S)\neq 0$ implies that (*) doesn't split. The converse is clear, so that the assertion is proved. By [3, Corollary 9, Theorem 11] we have that $\mathrm{Ext}_R^1(T, S)\neq 0$ iff the condition a) or b) holds. This completes the proof.

LEMMA 2. i) *Let* P_1, P_2 *be idempotent maximal ideals and* S_i, T_i $(i=1, 2)$ *simple right, resp. left R-modules with* $S_iP_i=0$, *resp.* $P_iT_i=0$. *Then we have that* $O_r(P_1)=O_l(P_2)\Leftrightarrow\mathrm{Ext}_R^1(S_2, S_1)\neq 0\Leftrightarrow\mathrm{Ext}_R^1(T_1, T_2)\neq 0$. ii) *Let* P_1, $P_2, \cdots$ *be idempotent maximal ideals, and assume that* $O_r(P_{i+1})=O_l(P_i)$ *for all* i. *Then there exists a positive integer* n *such that* $P_i=P_j$ *whenever* $i\equiv j \pmod n$.

PROOF. i) follows from [3, Theorem 8] and its left-handed version. ii) follows from i) and the left-handed version of [3, Theorem 20].

COROLLARY. *An idempotent maximal ideal of R belongs to either a cycle or a strictly open cycle,*

Now, Theorem follows from Corollary, lemma 1, and [5, Lemma 1. 3].

REMARKS. 1. A hereditary noetherian prime ring which has arbitrary finitely many strictly open cycles is found in [6].

2. In [5, (2. 7) and (2. 8)] we studied the generalization of a theorem of Roiter, however, the author has recently found that R. B. Warfield, Jr. obtained this in a more general case [7, Corollary 7. 3].

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