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On some exact sequences concerning with H-separable extensions

Dedicated to Prof. Kentaro Murata on his 60th birthday

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Introduction

In his paper [5] K. Hirata showed an exact sequence concerning with an H-separable extension A of B as follows

$$1 \longrightarrow \mathrm{Inn}(A|B) \longrightarrow \mathrm{Aut}(A|B) \longrightarrow P(A)$$

where $P(A)$ is the group of isomorphsm classes of some type of $A-A$-modules. But if we follow the same method as Azumaya algebras we can obtain also the following exact sequence

$$1 \longrightarrow \mathrm{Inn}(A|B) \longrightarrow \mathrm{Aut}(A|B) \longrightarrow \mathrm{Pic}(C)$$

where $\mathrm{Pic}(C)$ is the Picard group of the center C of A. Being stimulated by these facts the author tried to obtain some additional sequences. In this paper we will show that if A is an H-separable extension of B (*i.e.*, ${}_A A\otimes_B A_A \langle \oplus_A (A\oplus A\oplus\cdots\oplus A)_A$) such that $V_A(B)\subset B$, there exists an exact sequence of group homomorphisms

$$1 \longrightarrow \mathrm{Inn}(A|B) \longrightarrow \mathrm{Aut}(A|B) \longrightarrow \mathrm{Pic}(C) \longrightarrow \mathrm{Pic}(B')$$

where $B'=V_A(V_A(B))$. From this we can induce an exact sequence

$$1 \longrightarrow \mathrm{Inn}(A|S) \longrightarrow \mathrm{Aut}(A|S) \longrightarrow \mathrm{Pic}(C) \longrightarrow \mathrm{Pic}(S)$$

in the case where A is an Azumaya C-algebra and S is a maximal commutative subring of A such that A is left S-projective, that is, A is an S/C-Azumaya algebra.

Sequence of group homomorphisms

Throughout this paper A is a ring with the identity 1 and B is a subring of A such that $1\in B$. $\mathrm{Aut}(A|B)$ denotes the group of all automorphisms of A which fix all elements of B and $\mathrm{Inn}(A|B)$ denotes the subgroup of $\mathrm{Aut}(A|B)$ consisting of all inner automorphisms. For an $A-A$-module M and an automorphism σ of A we denote by ${}_{\sigma}M$ a new $A-A$-module such that ${}_{\sigma}M=M$ as right module and $am=\sigma(a)m$ for $m\in{}_{\sigma}M$ and $a\in A$ as left A-module. Similarly we can define M_{σ}. For each $A-A$-module M set $M^{A}=\{m\in M|ma=am$ for all $a\in A\}$. Then we have $({}_{\sigma}M)^{A}=(M_{\sigma^{-1}})^{A}$. Especially we will set ${}_{\sigma}J=({}_{\sigma}A)^{A}$ and $(A_{\sigma})^{A}=J_{\sigma}$.

The most part of the next lemma have already been known and appeared in the proofs of Prop. 3 [11] Prop. 5 [12] and in [5], though they are not stated as lemmas. But here we will state them definitely

LEMMA 1. *Let A be an H-separable extension of B, C the center of A and $D=V_{A}(B)$, the centralizer of B in A. Then we have*

(1) *For each $\sigma\in\mathrm{Aut}(A|B)$ the map g_{σ} of $D\otimes_{C}J_{\sigma}$ to D such that $g_{\sigma}(d\otimes d_{\sigma})=dd_{\sigma}$ for $d\in D$ and $d_{\sigma}\in J_{\sigma}$ is an isomorphism*

(2) *J_{σ} is a C-finitely generated projective module of rank* 1, *and ${}_{C}J_{\sigma}<\oplus_{C}D$, (a C-direct summand of D).*

(3) *$J_{\sigma}J_{\tau}=J_{\tau\sigma}\cong J_{\sigma}\otimes_{C}J_{\tau}$, and $J_{\sigma}J_{\sigma^{-1}}=C$ for any σ, $\tau\in\mathrm{Aut}(A|B)$*

(4) *σ is inner if and only if $J_{\sigma}=Cv$ for some $v\in J_{\sigma}$.*

(5) $\mathrm{Aut}(A|B)=\mathrm{Aut}(A|B')$ *and* $\mathrm{Inn}(A|B)=\mathrm{Inn}(A|B')$, *where* $B'=V_{A}(V_{A}(B))$.

PROOF. Since A is H-separable over B, D is C-finitely generated projective, and consequently ${}_{C}C<\oplus_{C}D$ (See Th. 2.1 [4] or Th. 1.2 [8]). (1). We can apply Th. 1.2 (c) [8] to an $A-A$-module A_{σ}, and we have $D\otimes_{C}J_{\sigma}=D\otimes_{C}(A_{\sigma})^{A}\cong(A_{\sigma})^{B}=D$. (2). Since ${}_{C}C<\oplus_{C}D$, $C\otimes_{C}J_{\sigma}<\oplus D\otimes_{C}J_{\sigma}$ as C-module. But $g_{\sigma}(C\otimes_{C}J_{\sigma})=J_{\sigma}$. Hence ${}_{C}J_{\sigma}<\oplus_{C}D$. Then J_{σ} is C-finitely generated projective. $D\otimes_{C}J_{\sigma}\cong D$ shows that J_{σ} is of rank 1. (3). Let σ, $\tau\in\mathrm{Aut}(A|B)$. Clearly $J_{\sigma}J_{\tau}\subset J_{\tau\sigma}$, and $J_{\sigma}J_{\sigma^{-1}}\subset C$. But by (1) we have $DJ_{\sigma}=D$, $DJ_{\sigma}J_{\sigma^{-1}}=DJ_{\sigma^{-1}}=D$. Then since ${}_{C}C<\oplus_{C}D$, $J_{\sigma}J_{\sigma^{-1}}=DJ_{\sigma}J_{\sigma^{-1}}\cap C=D\cap C=C$ $(=J_{\sigma^{-1}}J_{\sigma})$. Now there exist x_{i} in J_{σ} any y_{i} in $J_{\sigma^{-1}}$ such that $\sum x_{i}y_{i}=1$. Then for any d in $J_{\tau\sigma}$, $y_{i}d\in J_{\sigma^{-1}}J_{\tau\sigma}\subset J_{\tau}$ and $d=\sum x_{i}y_{i}d\in J_{\sigma}J_{\tau}$. Thus we have $J_{\tau\sigma}\subset J_{\sigma}J_{\tau}$, and $J_{\sigma}J_{\tau}=J_{\tau\sigma}$. Thus the map μ of $J_{\sigma}\otimes_{C}J_{\tau}$ to $J_{\tau\sigma}$ such that $\mu(d_{\sigma}\otimes d_{\tau})=d_{\sigma}d_{\tau}$ for $d_{\sigma}\in J_{\sigma}$ and $d_{\tau}\in J_{\tau}$ is a C-epimorphism. Then μ splits, since $J_{\tau\sigma}$ is C-projective. But both $J_{\sigma}\otimes_{C}J_{\tau}$ and $J_{\tau\sigma}$ are of rank 1. Hence μ is an isomorphism. (4). If $J_{\sigma}=Cv(v\in J_{\sigma})$, $D=DJ_{\sigma}=J_{\sigma}D=Dv=vD$. Hence v is a unit, and $\sigma(x)=v^{-1}xv$ for all $x\in A$. The converse is also clear. (5) is due to Th. 1 [11].

Let $P(A)$ be the group of isomorphism classes of $A-A$-modules M such that M is a left A-progenerator and $A\cong\mathrm{Hom}({}_{A}M,{}_{A}M)$, and denote by $|M|$ the class to which M belongs.

THEOREM 1. *Let A be an H-separable extension of B, $C=V_A(A)$ and $B'=V_A(V_A(B))$. Then we have the following sequence of group maps*

$$1 \longrightarrow \mathrm{Inn}\,(A|B) \xrightarrow[i]{} \mathrm{Aut}\,(A|B) \xrightarrow[j]{} P(C) \xrightarrow[t]{} P(B')$$

such that $i(\sigma)=\sigma$, $j(\sigma)=|{}_\sigma J|=|J_{\sigma^{-1}}|$ and $t(|E|)=|B'\otimes_C E|$ for $\sigma\in\mathrm{Aut}\,(A|B)$ and $|E|\in P(C)$. Furthermore we have

(1) $\mathrm{Ker}\, j=\mathrm{Im}\, i$ *and* $\mathrm{Ker}\, t\subset\mathrm{Im}\, j$

(2) *If furthermore $B\supset V_A(B)$ (i.e., $V_A(B)=$ the center of B), then the above sequence is exact.*

PROOF. By Lemma 1 j is a group homomorphism. It is also clear that t is a group homomorphism. (1). That $\mathrm{Im}\, i=\mathrm{Ker}\, j$ is also obvious by Lemma 1 (4). As for the rest we can assume that $B=B'$, since $\mathrm{Aut}\,(A|B)=\mathrm{Aut}\,(A|B')$ and $\mathrm{Inn}\,(A|B)=\mathrm{Inn}\,(A|B')$ by Lemma 1 and A is also H-separable over B' by Th. 1. 3′ [8]. Now let E be any rank 1 C-projective module such that $B\otimes_C E\cong B$ as $B-B$-module. Denote this isomorphism by φ. φ induces $B^B\otimes_C E=(B\otimes_C E)^B\simeq B^B$, since E is C-projective (See Lemma 2. 1 [10]). Set $C'=B^B$, the center of B. Then we have $A\otimes_C E=A\otimes_{C'} C'\otimes_C E=A\otimes_{C'} C'\cong A$ as $A-V_A(C')$-module. Denote this isomorphism also by φ. Thus $\varphi(x\otimes m)=x\varphi(1\otimes m)$ for $x\in A$ and $m\in E$. Let $E=\sum Cm_i$ (finite). Then there exist $c_i\in C'$ such that $\varphi(\sum c_i\otimes m_i)=1$. Hence for each x in A there exists a unique element, say $\sigma(x)$, in A such that $\sum c_i x\otimes m_i=\sum \sigma(x)c_i\otimes m_i$. But $\sum c_i x\otimes m_i=x^{(r)}(\sum c_i\otimes m_i)$, where $x^{(r)}$ is the left A-endomorphism of the right multiplication of an $A-A$-module $A\otimes_C E$ by x. Hence we have $\sigma(xy)=\sigma(x)\,\sigma(y)$ for x, $y\in A$. Thus σ is a ring-endomorphism of A which fixes all elements of B. Then σ is an automorphism by Th. 1 [11] (or Th. 2 [13]). Now set $K=[{}_\sigma(A\otimes_C E)]^A=\{\sum x_j\otimes n_j\in A\otimes_C E\,|\,\sum\sigma(x)\,x_j\otimes n_j=\sum x_j x\otimes n_j$ for all $x\in A\}$. Then by Theorem 1. 2 (c) [8], $D\cong D\otimes_C E=[{}_\sigma(A\otimes_C E)]^B=D\otimes_C[{}_\sigma(A\otimes_C E)]^A=D\otimes_C K$, where $D=V_A(B)$. Hence K is rank 1 C-projective. Next since $1\in\varphi(K)$ and ${}_C C<\oplus_C D$, we have ${}_C C<\oplus_C\varphi(K)$. Hence $\varphi(K)=C$. On the other hand since ${}_{C\sigma}J<\oplus_C D$, ${}_\sigma J\otimes_C E<\oplus D\otimes_C E$ as C-module. Hence ${}_C\varphi({}_\sigma J\otimes_C E)<\oplus_C\varphi(D\otimes_C E)=D$. But we have ${}_\sigma J\otimes_C E\subset K$. Hence $\varphi({}_\sigma J\otimes_C E)=\varphi(K)=C$. Thus we have $|E|=|{}_\sigma J|^{-1}=|J_\sigma|$. Therefore $\mathrm{Ker}\, t\subset\mathrm{Im}\, j$. (2). Suppose $D=C'$. Then for each $\sigma\in\mathrm{Aut}\,(A|B)$, the isomorphism g_σ of $D\otimes_C J_\sigma$ to D induces a $B-B$-isomorphism of $B\otimes_C J_\sigma$ to B, since $J_\sigma\subset D\subset B$. Hence $\mathrm{Im}\, j\subset\mathrm{Ker}\, t$.

COROLLARY 1. *Let A be an Azumaya C-algebra, and S a maximal commutative subalgebra of A such that A is a left S-projective. Then we* *have the following exact sequence defined by the same way as Theorem* 1.

$$1 \longrightarrow \mathrm{Inn}\,(A|S) \xrightarrow[i]{} \mathrm{Aut}\,(A|S) \xrightarrow[j]{} P(C) \xrightarrow[t]{} P(S)$$

PROOF. Set $B=S$. Since $V_A(S)=S$, $D=B=B'=S$. By Prop. 2.4 [2], $D\otimes_C A^0 \cong \mathrm{Hom}\,({}_SA, {}_SA)$. Since A is left S-finitely generated projective and S-faithful, A is a left S-generator and ${}_SS<\oplus_S A$. Hence D is C-finitely generated projective. Then by Cor. 3 [9], A is an H-separable extension of S. Now we can apply Theorem 1.

We can replace $P(B')$ by $P(C')$ in Theorem 1. Because $C'\otimes_C E\cong C'$ induces $A\otimes_C E\cong A$ as $A-V_A(C')$-module, and we can follow the same lines as Theorem 1 for the rest. Note that $\mathrm{Ker}\, f\subset j(\mathrm{Aut}\,(A|V_A(C')))$, since $\sum c_i x\otimes m_i=\sum xc_i\otimes m_i$ for all x in $V_A(C')$. Conversely let $\tau\in \mathrm{Aut}\,(A|V_A(C'))$. Then since $\tau|D=$identity, J_τ and $J_{\tau^{-1}}$ are contained in C' by Prop. 5 [12]. Hence $C'J_\tau\supset J_{\tau^{-1}}J_\tau\ni 1$, and we have $C'J_\tau=C'$. Then $C'\otimes_C J_\tau\cong C'$, since both are rank 1 C'-projective. Therefore we have

PROPOSITION 1. *Let A be an H-separable extension of B, and C' the center of $V_A(B)$. Then we have the following exact sequence*

$$1\longrightarrow \mathrm{Inn}\,(A|V_A(C'))\longrightarrow \mathrm{Aut}\,(A|V_A(C'))\longrightarrow P(C)\longrightarrow P(C')$$

Remark. By Prop. 2.13 [2] and Cor. 1 we have the following exact sequence in the case where A is an S/R-Azumaya algebra

$$1\longrightarrow \mathrm{Inn}\,(A|S)\longrightarrow \mathrm{Aut}\,(A|S)\longrightarrow P(R)\longrightarrow P(S)\longrightarrow A(S,R)\longrightarrow B(S/R)\longrightarrow 1$$

See for detail [2].

REMARK. In [3], S. Elliger proved the exactness of i and j under a different conditions. He added the condition that ${}_CC<\oplus_C A$. But we did not need this condition in this paper. He also defined Azumaya extension. But from his definition we can easily induce that A is an Azumaya extension of B if and only if $A=B\otimes_C D$ with an Azumaya C-algebra D and $C=V_B(B)=V_A(A)$ and ${}_CC<\oplus_C B$. Hence this is a special case of H-separable extensions.

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