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A characterization of certain weak*-closed subalgebras of $L^{\infty}(G)$

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1. Introduction

Let G be a locally compact Hausdorff group, and let $L^{\infty}(G)$ be the usual Banach algebra. Let X be a non-zero weak*-closed linear subspace of $L^{\infty}(G)$ which is (i) left and right translation invariant, (ii) self-adjoint, and (iii) an algebra. Such subspaces X were characterized by Pathak and Shapiro [5] for LCA groups G, and by Crombez and Govaerts [1] for general locally compact Hausdorff groups G (not necessarily abelian) under the assumption that X contains the constant functions. In this paper we consider the property (ii)′ complemented, instead of (ii), and characterize weak*-closed linear subspaces of $L^{\infty}(G)$ with the properties (i), (ii)′, and (iii) for LCA groups G and compact Hausdorff groups G, not necessarily abelian. Pathak-Shapiro Theorem ([5]) and our result show that if G is a LCA group, and if X is a weak*-closed translation invariant subalgebra of $L^{\infty}(G)$, then X is complemented if and only if X is self-adjoint. Also, Crombez-Govaerts Theorem ([1]) and our result show that if G is a compact Hausdorff group, not necessarily abelian, and if X is a weak*-closed left and right translation invariant subalgebra of $L^{\infty}(G)$, then X is complemented if and only if X is self-adjoint. (See Remark 3 in section 3).

Let G be a locally compact Hausdorff group and fix left Haar measure dx on G. Let $L^{\infty}(G)$ be the class of all complex-valued essentially bounded Haar-measurable functions on G, and let $L^{1}(G)$ be the class of all complex-valued Haar-integrable functions on G. $L^{\infty}(G)$ is a commutative Banach algebra under pointwise multiplication of functions as the product. As is well-known, $L^{\infty}(G)$ is the Banach space dual of $L^{1}(G)$. For $s\in G$, left and right translation of a function f on G by s are denoted by $(L_{s}f)(x)=f(sx)$ and $(R_{s}f)(x)=f(xs)$ $(x\in G)$, respectively. A linear subspace X of $L^{\infty}(G)$ is said to be left [right, left and right] translation invariant if $L_{s}f\in X$ [$R_{s}f\in X$, $L_{s}f$ and $R_{s}f\in X$] for all $s\in G$ and $f\in X$. If G is abelian, left (and hence left and right) translation invariant subspaces of $L^{\infty}(G)$ are simply said to be translation invariant. A subset X of $L^{\infty}(G)$ is said to be self-adjoint if

$f\in X$ implies $\bar{f}\in X$, where $\bar{f}$ denotes the complex conjugate of f. A closed linear subspace X of $L^\infty(G)$ is said to be complemented if there exists a bounded projection P (*i.e.*, a bounded linear operator with $P^2=P$) of $L^\infty(G)$ onto X.

Given a closed normal subgroup H of G, we put $X_H=\{f\in L^\infty(G);\ L_sf=R_sf=f$ for all $s\in H\}$. We can easily see that every X_H is a weak*-closed linear subspace of $L^\infty(G)$ which is left and right translation invariant and an algebra containing the constant functions. Also, if G is a LCA group or a compact Hausdorff group, not necessarily abelian, then X_H is complemented. This is verified as follows; If G is a LCA group, then it follows immediately from Gilbert Theorem ([2]) that X_H is complemented. (See Remark 1 in section 2). If G is a compact Hausdorff group, not necessarily abelian, and if we define P: $L^\infty(G)\to L^\infty(G)$ by $(Pf)(x)=\int_H f(x\xi)d\xi$ $(f\in L^\infty(G))$, where $d\xi$ is the normalized Haar measure on H, then P is a bounded projection $L^\infty(G)$ onto X_H (See [4] (28. 54)). Hence X_H is complemented.

We prove the following converse Theorems.

THEOREM 1. *Let G be a LCA group, and let X be a non-zero weak*-closed linear subspace of $L^\infty(G)$ which is* (i) *translation invariant,* (ii)′ *complemented, and* (iii) *an algebra. Then there exists a unique closed subgroup H of G such that $X=X_H$.*

THEOREM 2. *Let G be a compact Hausdorff group, not necessarily abelian, and let X be a nonzero weak*-closed linear subspace of $L^\infty(G)$ which is* (i) *left and right translation invariant,* (ii)′ *complemented, and* (iii) *an algebra. Then there exists a unique closed normal subgroup H of G such that $X=X_H$.*

We will prove Theorem 1 and 2 in section 2 and 3, respectively.

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2. Proof of Theorem 1

Throughout this section, G will be a LCA group unless the contrary is explicitly specified. The group operation in G will be written additively. The dual group of G is denoted by $\hat{G}$. We need two Lemmas to obtain the proof of Theorem 1.

For an abelian group G, the coset-ring of G is the smallest ring of sets of G containing all the cosets of G. The coset-ring of G is denoted by $\mathscr{R}(G)$.

LEMMA 1. *Let Z be the additive group of the integers, S a subsemigroup of Z. Suppose that S differs from* $\bigcup_{j=1}^{k}(tZ+u_j)$ *in at most finitely many places, where $t, u_j \in Z$, $0<t$, $0 \leq u_j < t$ $(1 \leq j \leq k)$. Then S is a subgroup of Z.*

PROOF. Let $c=\min\{n \in S;\ n>0\}$ and $d=\max\{m \in S;\ m<0\}$. Indeed, there exist such elements by the form of S. Then $c+d=0$ because of $S+S \subset S$ and our choice of c and d. Hence $0 \in S$ and $-c=d \in S$, and so we have $cZ \subset S$. If x is a positive element in S, then there exists a unique $n \in Z$ such that $(n+1)(-c) \leq -x < n(-c)$. Thus $0<n(-c)+x \leq c$. Since $n(-c)+x \in S$, we have $n(-c)+x=c$ by our choice of c. Hence $x \in cZ$. If x is a negative element in S, by the same argument we have $x \in cZ$. Hence we conclude $S=cZ$. This completes the proof of Lemma 1.

LEMMA 2. *Let G be an abelian group, and let E be a non-empty subset of G in $\mathscr{R}(G)$. If E is a subsemigroup of G, then E is a subgroup of G.*

PROOF. It suffices to show that $x \in E$ implies $-x \in E$. Let H be the subgroup generated by x. Then $H \cap E$ is a subsemigroup of G. If the order of x is finite, then $H \cap E$ is a finite subsemigroup of G. Since a finite subsemigroup of every group is a subgroup, $H \cap E$ is a subgroup of G. Hence $-x \in E$. If the order of x is Infinite, then $H \cong Z$. Since $H \cap E \in \mathscr{R}(H)$ and $H \cong Z$, we may consider $H \cap E \in \mathscr{R}(Z)$. Since $H \cap E$ is infinite, it follows from Helson Theorem ([8]. p. 61) that $H \cap E$ must be the form described in Lemma 1. Hence $H \cap E$ is a subgroup, and so $-x \in E$. This completes the proof of Lemma 2.

Let X be a weak*-closed translation invariant subspace of $L^\infty(G)$. Then the spectrum of X, written $sp(X)$, is defined as the set of all elements of $\hat{G}$ which belong to X ([8]. 7. 8).

The following Theorem is due to J. E. Gilbert ([2]).

THEOREM 3 (J. E. Gilbert). *Let X be a weak*-closed translation invariant subspace of $L^\infty(G)$. Then X is complemented if and only if $sp(X) \in \mathscr{R}(\hat{G})$.*

REMARK 1. In section 1 we described that X_H is complemented. This fact follows immediately from Theorem 3 since $sp(X_H)=H^\perp \in \mathscr{R}(\hat{G})$. Here $H^\perp$ denotes the annihilator of H, *i. e.*, $H^\perp=\{\gamma \in \hat{G};\ (x,\gamma)=1 \text{ for all } x \in H\}$.

PROOF OF THEOREM 1. Let X be a non-zero weak*-closed linear subspace of $L^\infty(G)$ with the properties (i), (ii)', and (iii). By (i) and (iii), $sp(X)$

is non-empty and is a closed subsemigroup of $\hat{G}$. Since X has the property (ii)′, it follows from Theorem 3 that $sp(X)\in\mathscr{R}(\hat{G})$. Hence by Lemma 2, $sp(X)$ is a closed subgroup of $\hat{G}$. Putting $H=(sp(X))^\perp=\{x\in G\,;\ (x,\gamma)=1$ for all $(\gamma\in sp(X)\}$, we have $X=X_H$. Noting $sp(X_H)=H^\perp$ for every closed subgroup of G, we obtain the uniqueness of H with $X=X_H$. This completes the proof of Theorem 1.

We conclude this section with three examples which show that all the conditions in Theorem 1 are really necessary.

EXAMPLE 1. Let $G=T$ be the circle group. Then $\hat{G}=Z$ (the additive group of the integers). Let X be a non-zero weak*-closed translation invariant subspace of $L^\infty(T)$ such that $sp(X)$ belongs to $\mathscr{R}(Z)$ and is not subsemigroup of Z. Indeed, there exists such X. For example, let X be the weak*-colsed translation invariant subspace with $sp(X)=\{2n+1\,;\ n\in Z\}$. Then X satisfies (i), (ii)′ but not (iii).

EXAMPLE 2. Let $G=T$ and $X=H^\infty(T)=\{f\in L^\infty(T)\,;\ \hat{f}(n)=0$ for all negative integers $n\}$. Here $\hat{f}$ denotes the Fourier transform of f. Then X is a weak*-closed linear subspace of $L^\infty(T)$, and satisfies (i), (iii) but not (ii)′ ([2]).

EXAMPLE 3. Let $G=T$ and m the normalized Haar (Lebesgue) measure on T. Let $E\subset T$ be a Borel set such that $0<m(E)<1$. Put $X=\{f\in L^\infty(T)\,;$ $f(x)=0$ on $E^c\}$, where E^c denotes the complement of E relative to T. Then X is a non-zero weak*-closed linear subspace and satisfies (ii)′, (iii) but not (i).

3. Proof of Theorem 2

Throughout this section G will be a compact Hausdorff group, not necessarily abelian, with the normalized left Haar measure dx unless the contrary is explicitly specified. The identity element of G is denoted by e. Given a function f on G, we put $\tilde{f}(x)=f(x^{-1})$ $(x\in G)$. Let $C(G)$ be the Banach algebra of all complex-valued continuous functions on G, and $M(G)$ the Banach space of all bounded regular complex Borel measure on G with total variation norm. Then, as is well-known, $M(G)$ is the Banach space dual of $C(G)$. Self-adjoint subsets and complemented linear subspaces of $C(G)$ are defined in the same way, except that $L^\infty(G)$ in the definitions of those of $L^\infty(G)$ is replaced by $C(G)$.

For two functions f and g in $L^1(G)$, the convolution $f*g$ is defined by

$$f*g(x)=\int_G f(xy)\,g(y^{-1})\,dy=\int_G f(y)\,g(y^{-1}x)\,dy \qquad (x\in G)\,.$$

For $f\in L^1(G)$ and $\mu\in M(G)$, the convolution $\mu*f$ and $f*\mu$ are defined by

$$\mu*f(x)=\int_G f(y^{-1}x)\,d\mu(y)$$

and

$$f*\mu(x)=\int_G \Delta(y^{-1})f(xy^{-1})\,d\mu(y),$$

respectively. Here Δ is the modular function of G. Since every compact Hausdorff group is unimodular, *i.e.*, $\Delta(x)\equiv 1$ ($x\in G$) ([6]. p. 62), in the present case we have

$$f*\mu(x)=\int_G f(xy^{-1})\,d\mu(y).$$

To prove Theorem 2 we need some Lemmas. Lemma 6 and 7 leading to Theorem 2 are also of interest in their own right.

LEMMA 4. *Let X be a weak*-closed right translation invariant complemented subspace of $L^\infty(G)$. Then there exists a bounded projection T of $L^\infty(G)$ onto X such that $TR_s=R_sT$ for all $s\in G$.*

PROOF. We can prove this Lemma by using an argument similar to one of the proof of Theorem 1.1 in [7]. Let M denote the bounded linear functional on $L^\infty(G)$ defined by $M(f)=\int_G f(x)dx$. Thus M satisfies the following,

(a) $M(1)=1$,

(b) $M(R_sf)=M(f)$ for all $s\in G$ and $f\in L^\infty(G)$,

(c) $|M(f)|\leq\|f\|_\infty$ for all $f\in L^\infty(G)$.

Let (,) denote the usual pairing between $L^1(G)$ and $L^\infty(G)$. Thus if $f\in L^1(G)$ and $g\in L^\infty(G)$, then $(f,g)=\int_G f(x)\,g(x^{-1})\,dx$.

Since X is complemented, there exists a bounded projection P of $L^\infty(G)$ onto X. Now fix $g\in L^\infty(G)$. For each $f\in L^1(G)$, consider $(f, R_{x^{-1}}PR_xg)$ an element of $L^\infty(G)$. Then $f\rightarrow M((f, R_{x^{-1}}PR_xg))$ defines a bounded linear functional on $L^1(G)$. Let Tg be the unique element of $L^\infty(G)$ representing this functional. Then T is a bounded linear operator of $L^\infty(G)$ into $L^\infty(G)$ with the norm $\|T\|\leq\|P\|$. To see that T is a projection of $L^\infty(G)$ onto X, it suffices to show that $T(L^\infty(G))\subset X$ and that $g\in X$ implies $Tg=g$. Since X is weak*-closed, we have $X=\{g\in L^\infty(G);\ (f,g)=0$ for all $f\in X^\perp\}$, where $X^\perp=\{f\in L^1(G);\ (f,g)=0$ for all $g\in X\}$. Let $g\in L^\infty(G)$. Then $(f, R_{x^{-1}}PR_xg)=0$ for each $x\in G$ and $f\in X^\perp$ since $R_{x^{-1}}PR_xg\in X$. Thus $(f,$

$Tg)=M((f, R_{x^{-1}}PR_xg))=0$, and so $Tg\in X$. Hence $T(L^\infty(G))\subset X$. Next, let $g\in X$. Then $PR_xg=R_xg$ for each $x\in G$. By (a), we have $(f, Tg)=M((f, R_{x^{-1}}R_xg))=M((f,g))=(f,g)$ for all $f\in L^1(G)$. Hence $Tg=g$. Finally, to see $TR_s=R_sT$ for all $s\in G$, let $g\in L^\infty(G)$, $f\in L^1(G)$, and $s\in G$. Noting G is unimodular, we have $(f, R_sTg)=(L_sf, Tg)$. Since M satisfies (b),

$$\begin{aligned}(f, TR_sg) &= M\big((f, R_{x^{-1}}PR_xR_sg)\big)\\ &= M\big((f, R_sR_{(xs)^{-1}}PR_{xs}g)\big)\\ &= M\big((L_sf, R_{(xs)^{-1}}PR_{xs}g)\big)\\ &= (L_sf, Tg)=(f, R_sTg)\,.\end{aligned}$$

Hence we have $\mathrm{TR}_s=R_sT$ for all $s\in G$. This completes the proof of Lemma 4.

LEMMA 5. *Let X be a weak*-closed right translation invariant subspace of* $L^\infty(G)$. *Put* $X*L^1=\{g*f\,;\, f\in L^1(G),\, g\in X\}$. *Then* $X*L^1\subset X$.

PROOF. This Lemma can be proved by using the same argument as that in the proof of Lemma 2 in [1] if we note the following equation;

$$\int_G k(x)\,(g*f)\,(x)\,dx=\int_G f(x)\,(\tilde{g}*k)\,(x)\,dx$$

for every $f,\ k\in L^1(G)$ and $g\in L^\infty(G)$.

REMARK 2. Lemma 5 holds for every unimodular locally compact Hausdorff group G.

In view of Lemma 4 and 5, we can extend the result known for compact abelian groups to compact Hausdorff groups, not necessarily abelian.

LEMMA 6. *Let X be a weak*-closed right translation invariant complemented subspace of* $L^\infty(G)$. *Then there exists a weak*-closed left translation invariant subspace Y of* $L^\infty(G)$ *such that* $L^\infty(G)=X\oplus Y$.

PROOF. By Lemma 4, there exists a bounded projection P of $L^\infty(G)$ onto X such that $PR_s=R_sP$ for all $s\in G$. Then $f\in C(G)$ implies $Pf\in C(G)$. For if $f\in C(G)$, then

$$\|R_sPf-Pf\|_\infty=\|PR_sf-Pf\|_\infty\le\|P\|\,\|R_sf-f\|_\infty\to 0 \text{ as } s\to e$$

in G. So $f\to(Pf)(e)$ defines a bounded linear functional on $C(G)$. Consequently, there exists a $\mu\in M(G)$ such that $(Pf)(e)=\int_G f(y^{-1})\,d\mu(y)$ for every $f\in C(G)$. But for $x\in G$,

$$(Pf)(x)=(R_xPf)(e)=(PR_xf)(e)=\int_G f(y^{-1}x)\,d\mu(y)=\mu^*f(x).$$

Hence we conclude that $Pf=\mu^*f$ for each $f\in C(G)$.

Now we consider T: $L^1(G)\to L^1(G)$ defined by $Tf=f-f^*\mu$. Put $X^\perp=\{f\in L^1(G);\ (f,g)=0$ for all $g\in X\}$, where $(f,g)=\int_G f(x)g(x^{-1})\,dx$. Then we claim that T is a bounded projection of $L^1(G)$ onto $X^\perp$ and that $L_sT=TL_s$ for all $s\in G$. It is clear that $L_sT=TL_s$ for all $s\in G$. To see that $T(L^1(G))\subset X^\perp$, let $f\in L^1(G)$ and $g\in X$. Then $g^*f\in C(G)$. So

$$\begin{aligned}(f-f^*\mu, g)&=(g^*f)(e)-(\mu^*g^*f)(e)\\&=(g^*f)(e)-P(g^*f)(e).\end{aligned}$$

By Lemma 5, we have $X^*L^1\subset X$, and so $g^*f\in X$. Hence $P(g^*f)=g^*f$, and $(f-f^*\mu, g)=0$. We obtain $T(L^1(G))\subset X^\perp$. Next, to see that $f\in X^\perp$ implies $Tf=f$, *i.e.*, $f^*\mu=0$, let $f\in X^\perp$ and $g\in C(G)$. Since

$$(f^*\mu, g)=(\mu^*g^*f)(e)=(Pg^*f)(e)=(f, Pg)$$

and $f\in X^\perp$ and $Pg\in X$, we have $(f^*\mu, g)=0$. Since $C(G)$ is weak*dense in $L^\infty(G)$, $f\in X^\perp$ implies $f^*\mu=0$. Hence we conclude that T is a bounded projection of $L^1(G)$ onto $X^\perp$ such that $L_sT=TL_s$ for all $s\in G$.

Let T^* be the adjoint operator of T, *i.e.*, T^* is the bounded linear operator of $L^\infty(G)$ into $L^\infty(G)$ which satisfies $(Tf, g)=(f, T^*g)$ for all $f\in L^1(G)$ and $g\in L^\infty(G)$. Put $Y=\{g\in L^\infty(G);\ (I-T^*)g=0\}$. Here I denotes the identity operator on $L^\infty(G)$. Then Y is a weak*-closed left translation invariant subspace. Since $I-T^*$ is weak*-continuous, Y is weak*-closed. Let $g\in Y$, $f\in L^1(G)$, and $s\in G$. Then by a direct computation, we have $(f, L_sg-T^*L_sg)=(R_sf, g-T^*g)$, and so $(f, L_sg-T^*L_sg)=0$. Hence $L_sg\in Y$ for all $s\in G$ and $g\in Y$. By the definition of Y, it is clear that $L^\infty(G)=X\oplus Y$. This completes the proof of Lemma 6.

The following Lemma 7 is of interest from viewpoint of constructing a complemented subalgebra of $C(G)$ from a complemented one of $L^\infty(G)$.

LEMMA 7. *Let X be a weak*-closed left and right translation invariant complemented subalgebra of $L^\infty(G)$. Then $L^1{}^*X$ is a closed complemented subalgebra of $C(G)$.*

PROOF. If we note that G is a compact Hausdorff group, by Lemma 4 in [1], we have $L^1{}^*X=C(G)\cap X$. Hence $L^1{}^*X$ is a closed subalgebra of $C(G)$. By Lemma 6, there exists a weak*closed left translation invariant subspace Y such that $L^\infty(G)=X\oplus Y$. Since by Corollary 2 in [1] $L^1{}^*X\subset X$, $L^1{}^*Y\subset Y$, and $C(G)=L^1{}^*L^\infty$ ([4]. 32. 45 (b).), we have $C(G)=L^1{}^*L^\infty=$

$(L^1*X)\oplus(L^1*Y)$. Since L^1*X and L^1*Y are closed in $C(G)$, it follows that L^1*X is complemented in $C(G)$. This completes the proof of Lemma 7.

The following result is due to I. Glicksberg ([3]). It is used to prove Lemma 9 below.

THEOREM 8 (I. Glicksberg). *Let X be a closed left and right translation invariant subalgebra of $C(G)$. Then X is complemented in $C(G)$ if and only if X is self-adjoint.*

In view of Lemma 7, Theorem 8, and Lemma 4 in [1], we obtain the following result.

LEMMA 9. *Let X be a weak*-closed left and right translation invariant complemented subalgebra of $L^\infty(G)$. Then L^1*X is a closed self-adjoint subalgebra of $C(G)$.*

In the following Lemma 10 we prove Theorem 2 under the assumption that X contains the constant functions.

LEMMA 10. *Let X be a weak*-closed linear subspace of $L^\infty(G)$ which has the properties* (i), (ii)′ *and* (iii). *If X contains the constant functions, then there exists a unique closed normal subgroup H of G such that $X=X_H$.*

PROOF. Let X be a weak*-closed linear subspace of $L^\infty(G)$ which satisfies the assumption of Lemma. Then we first note that L^1*X is a closed self-adjoint subalgebra of $C(G)$, by Lemma 9. Once we obtain this, we can proceed in the same method as that of Crombez-Govaerts ([1]).

PROOF OF THEOREM 2. In view of Lemma 10 the proof of Theorem 2 will be completed if we show that X contains the constant function 1 under the assumption of Theorem 2. As we saw in the proof of Lemma 6, there exist a bounded projection P of $L^\infty(G)$ onto X and a $\mu\in M(G)$ such that $Pf=\mu*f$ for each $f\in C(G)$.

Case 1. $\mu(G)\neq 0$.

Since $1=\mu*\left(\frac{1}{\mu(G)}\right)=P\left(\frac{1}{\mu(G)}\right)\in X$, X contains the constant function 1.

Case 2. $\mu(G)=0$.

We show that Case 2 cannot occur. Using the notations in the proof of Lemma 6, we have $L^\infty(G)=X\oplus Y$, where $Y=\{g\in L^\infty(G)\,;\ (I-T^*)g=0\}$ and $Tf=f-f*\mu$ $(f\in L^1(G))$. Then Y contains the constant function 1. For if $f\in L^1(G)$,

$$\begin{aligned}\big(f,(I-T^*)\,1\big)&=(f,1)-(f,T^*1)=(f,1)-(Tf,1)\\&=(f,1)-(f,1)+(f*\mu,1)=\mu(G)\int_G f(x)\,dx=0.\end{aligned}$$

Let Y_1 be the set of all the constant functions in $L^\infty(G)$. Then it is easy to verify that Y_1 is a weak*-closed left and right translation invariant subalgebra of $L^\infty(G)$ and is complemented in Y. Thus it follows that $X\oplus Y_1$ is a weak*-closed left and right translation invariant complemented subalgebra of $L^\infty(G)$ containing the constant functions. Hence, by Lemma 10, we have $X\oplus Y_1=X_H$ for some closed normal subgroup H of G. Since X is non-zero, there exists $f\in X$ such that $f\neq 0$ in $L^\infty(G)$. Noting that X_H is self-adjoint and X is a two-sided ideal of X_H, we have $|f|^2=f\cdot\bar{f}\in X$. Then it follows from Corollary 2 in [1] that $1*|f|^2\in X$. Since $(1*|f|^2)(x)=\int_G |f(y^{-1})|^2dy\neq 0$, $1*|f|^2$ is a non-zero constant function in X. Hence we have $1\in X$. But this is impossible. Consequently Case 2 cannot occur.

REMARK 3. For compact Hausdorff groups, Crombez-Govaerts Theorem holds if we assume that X is non-zero instead of the assumption that X contains the constant functions. For since X is non-zero, there exists $f\in X$ such that $f\neq 0$ in $L^\infty(G)$. Then $|f|^2\in X$ because X is self-adjoint and an algebra. Hence $1*|f|^2\in X$ (Corollary 2 in [1]), and so X contains the constant functions. From this fact and our Theorem 2, it is easily verified that if G is a compact Hausdorff group and if X is a weak*-closed left and right translation invariant subalgebra of $L^\infty(G)$, then X is complemented if and only if X is self-adjoint.

References

[1] G. CROMBEZ and W. GOVAERTS: A characterization of certain weak*-closed subalgebras of $L^\infty(G)$. J. Math. Anal. Appl. 72. (1979), 430–434.

[2] J. E. GILBERT: On projections of $L^\infty(G)$ onto translation invariant subspaces. Proc. London Math. Soc. 19. (1969), 69–88.

[3] I. GLICKSBERG: Some uncomplemented function algebras. Trans. Amer. Math. Soc. 111. (1964), 121–137.

[4] E. HEWITT and K. ROSS: Abstract harmonic analysis Vol. II. Springer-Verlag. (1971).

[5] P. K. PATHAK and H. S. SHAPIRO: A characterization of certain weak*-closed subalgebras of L^∞. J. Math. Anal. Appl. 58. (1977), 174–177.

[6] H. REITER: Classical harmonic analysis and locally compact groups. Oxford University Press. (1968).

[7] H. P. ROSENTHAL: Projections onto translation-invariant subspaces of $L_p(G)$. Memoirs Amer. Math. Soc. No. 63. (1966).

[8] W. RUDIN: Fourier analysis on groups. Interscience. (1962).

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