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On irreducible conformally recurrent Riemannian manifolds

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Recently, A. Gębarowski [1] has determined decomposable conformally recurrent Riemannian manifolds. Then, it is natural to ask how irreducible conformally recurrent Riemannian manifolds are determined. In fact, taking account of the de Rham decomposion of Riemannian manifolds, a complete simply connected Riemannian manifold is either irreducible or decomposable. It is natural to assume that the Riemannian manifold has a complete metric and is simply connected by considering its universal covering space, if necessary.

The purpose of the present paper is to show

THEOREM. *An analytic irreducible conformally recurrent Riemannian manifold of dimension greater than 4 is conformally flat.*

§ 1. Preliminaries and notation.

Let M be an $n(\geqq 4)$-dimensional Riemannian manifold with a positive definite Riemannian metric g_{ji}. We denote by ∇_j, $K_{kji}{}^h$, K_{ji} and K the operator of covariant differentiation, the Riemannian curvature tensor, the Ricci tensor and the scalar curvature, respectively. Then the conformal curvature tensor C_{kjih} is given by

$$(1)\qquad C_{kjih}=K_{kjih}-\frac{1}{n-2}(K_{kh}g_{ji}-K_{jh}g_{ki}+g_{kh}K_{ji}-g_{jh}K_{ki}) +\frac{K}{(n-1)(n-2)}(g_{kh}g_{ji}-g_{jh}g_{ki}).$$

For the conformal curvature tensor C_{kjih} we know the following identities:

$$(2)\qquad C_{kjih}=-C_{jkih}=-C_{kjhi}=C_{ihkj}, \quad C_{kjih}+C_{jikh}+C_{ikjh}=0,\ g^{kh}C_{kjih}=0,$$

$$(3)\qquad \nabla_h C_{kji}{}^h=\frac{n-3}{n-2}C_{kji},\ \text{where } C_{kji}=\nabla_k K_{ji}-\nabla_j K_{ki} -\frac{1}{2(n-1)}(\nabla_k K g_{ji}-\nabla_j K g_{ki}),$$

$$(4)\qquad \nabla_l C_{kjih}+\nabla_k C_{jlih}+\nabla_j C_{lkih} = \frac{1}{n-2}(C_{kji}g_{lh}-C_{kjh}g_{li}+C_{jli}g_{kh}-C_{jlh}g_{ki}+C_{lki}g_{jh}-C_{lkh}g_{ji}).$$

M is called conformally recurrent if we have

$$(5)\qquad \nabla_l C_{kjih}=p_l C_{kjih},$$

for a non-zero 1-form p_l.

If M is of class C^ω and g_{ji} is so, we call M analytic. In this case, K_{kjih}, K_{ji}, K and hence C_{kjih} are also of class C^ω.

§ 2. Proof of Theorem.

LEMMA 1. *We have*

$$(6)\qquad p_l C_{kjih}+p_k C_{jlih}+p_j C_{lkih} = \frac{1}{n-3}p^t(C_{kjit}g_{lh}-C_{kjht}g_{li}+C_{jlit}g_{kh}-C_{jlht}g_{ki} +C_{lkit}g_{jh}-C_{lkht}g_{ji}).$$

PROOF. Substitute (3) and (5) into (4).

LEMMA 2. *There exists the equation,*

$$(7)\qquad p_l C_{kjih}C^{kjih}-\frac{2(n-2)}{n-3}C_{kjil}C^{kji}{}_h p^h=0.$$

PROOF. Teansvect (6) with C^{kjih} and take account of (2).

LEMMA 3 ([3]). *If there exists on an analytic Riemannian manifold a function f of class C^ω satisfying*

$$\nabla_i f=p_i f,$$

for a 1-form p_i, then either p_i is closed or f vanishes identically.

If an analytic Riemannian manifold M is conformally recurrent, then, by Lemma 3, either the recurrence vector p_i is closed or M is conformally flat. where as the function f of class C^ω we have adopted $\frac{1}{2}C_{kjih}C^{kjih}$. Hence, in the following, we assume that p_i is closed.

LEMMA 4. *If there exists a point where $C_{kjih}=0$, then C_{kjih} vanishes identically.*

The proof is similar to that of Lemma 7 in [3].

Hence, if M is not conformally flat, then C_{kjih} does not have zero points. In the following, we assume that M is not conformally flat.

LEMMA 5. *If we put*

$$p=\frac{1}{2}\log(C_{kjih}C^{kjih}),$$

then we obtain

$$(8)\qquad C_{kjih}C^{kji}{}_{l}=\frac{1}{n}e^{2p}g_{hl}\,.$$

PROOF. First of all we remark that

$$\nabla_m p=p_m\,.$$

Since we have

$$\nabla_m(e^{-2p}C_{kjih}C^{kji}{}_{l})=0\,,$$

there exists a constant c such that

$$(9)\qquad e^{-2p}C_{kjih}C^{kji}{}_{l}=cg_{hl}\,,$$

because of the facts that $e^{-2p}C_{kjih}C^{kji}{}_{l}$ is symmetric in h and l, and that M is irreducible. Transvecting (9) with g^{hl} we get

$$c=\frac{1}{n}e^{-2p}C_{kjih}C^{kjih}=\frac{1}{n}\,,$$

from which (8) follows.

Substituting (8) into (7) we have

$$e^{2p}p_l-\frac{2(n-2)}{(n-3)n}e^{2p}p_l=0$$

that is,

$$\frac{(n-1)(n-4)}{n(n-3)}e^{2p}p_l=0\,,\ \text{a contradiction for } n>4.$$

Therefore Theorem is proved.

§ 3. A note.

Let M be a Kaehlerian manifold with a complex structure $F_i{}^h$ and the Kaehlerian metric g_{ji}. Then M is analytic. The Bochner curvature tensor B_{kjih} is given by

$$\begin{aligned}B_{kjih}=K_{kjih}&-\frac{1}{n+4}(g_{kh}K_{ji}-g_{jh}K_{ki}+K_{kh}g_{ji}-K_{jh}g_{ki}+F_{kh}S_{ji}\\&-F_{jh}S_{ki}+S_{kh}F_{ji}-S_{jh}F_{ki}-2F_{kj}S_{ih}-2S_{kj}F_{ih})\\&+\frac{K}{(n+2)(n+4)}(g_{kh}g_{ji}-g_{jh}g_{ki}+F_{kh}F_{ji}-F_{jh}F_{ki}-2F_{kj}F_{ih})\,,\end{aligned}$$

where $S_{ji}=K_{kjih}F^{kh}=-\frac{1}{2}K_{khji}F^{kh}=-K_{jt}F_i{}^t$ is a closed 2-form.

B_{kjih} satisfies

$$B_{kjih}=-B_{jkih}=-B_{kjhi}=B_{ihkj}=B_{kjts}F_i^t F_h^s\,,$$

$$B_{kjih}+B_{jikh}+B_{ikjh}=0\,,\quad g^{kh}B_{kjih}=F^{kh}B_{kjih}=F^{kh}B_{khji}=0\,,$$

$$\nabla_h B_{kji}{}^h=\frac{n}{n+4}B_{kji}\,,$$

$$\begin{aligned}&\nabla_l B_{kjih}+\nabla_k B_{jlih}+\nabla_j B_{lkih}\\&=\frac{1}{n+4}(B_{kji}g_{lh}-B_{kjh}g_{li}+B_{jli}g_{kh}-B_{jlh}g_{ki}+B_{lki}g_{jh}-B_{lkh}g_{ji}\\&\quad+B_{kjt}F_h^t F_{li}-B_{kjt}F_i^t R_{lh}+B_{jlt}F_h^t F_{ki}-B_{jlt}F_i^t F_{kh}+B_{lkt}F_h^t F_{ji}\\&\quad-B_{lkt}F_i^t F_{jh}+2B_{iht}f_l^t F_{kj}+2B_{iht}F_k^t F_{jl}+2B_{iht}F_j^t F_{lk})\,,\end{aligned}$$

where

$$\begin{aligned}B_{kji}=\nabla_k K_{ji}-\nabla_j K_{ki}-\frac{1}{2(n-2)}(&\nabla_k K g_{ji}-\nabla_j K g_{ki}+F_k^t\nabla_t K F_{ji}\\&-F_j^t\nabla_t K F_{ki}-2F_{kj}F_j^t\nabla_t K)\,.\end{aligned}$$

M is said to have recurrent Bochner curvature tensor if B_{kjih} satisfies

$$\nabla_l B_{kjih}=p_l B_{kjih}\,,$$

for a non-zero 1-form p_l.

Then, as a Kaehlerian analogy of Theorem, we have

PROPOSITION. *An irreducible Kaehlerian manifold with recurrent Bochner curvature tensor of real dimension greater than 4 has vanishing Bochner curvature tensor.*

References

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