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Some multipliers on the space consisting of measures of analytic type

By Hiroshi YAMAGUCHI

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§ 1 Introduction

Let G be a LCA group with the dual group $\hat{G}$. m_G denotes the Haar measure of G. Let $M(G)$ and $M_s(G)$ denote the space of all bounded regular (complex-valued) measures on G and the subspace of $M(G)$ consisting of all singular measures respectively. $L^1(G)$ denotes the usual group algebra, and $Trig(G)$ denotes the space of all trigonometric polynomials on G. $M_c(G)$ and $M_d(G)$ denote the subspaces of $M(G)$ consisting of continuous measures and discrete measures respectively. For a subset E of $\hat{G}$, $M_E(G)$ denotes the space consisting of measures in $M(G)$ whose Fourier-Stieltjes transforms vanish off E. For a subset E of $\hat{G}$, E^0 and E^- denote its interior and closure. '$\wedge$' and '$\vee$' denote the Fourier-Stieltjes transform and the inverse Fourier transform respectively. When there exists a nontivial continuous homomorphism from $\hat{G}$ into R (the reals), we shall say that a measure $\mu\in M(G)$ is of analytic type if $\hat{\mu}(\gamma)=0$ for $\gamma\in\hat{G}$ with $\psi(\gamma)<0$. We denote by $M^a(G)$ the set of measures in $M(G)$ which are of analytic type. For a subset B of $M(G)$, $B^\wedge$ means the set $\{\hat{\mu}\,;\,\mu\in B\}$. For $\mu\in M(G)$, we signify $||\hat{\mu}||$ by $||\hat{\mu}||=||\mu||$.

For a discrete measure $\nu\in M_d(G)$, $\mu*\nu$ belongs to $M_s(G)$ for every $\mu\in M_s(G)$. For a compact abelian group G, Doss proved that a multiplier on $M_s(G)$ is given by convolution with a discrete measure ([4]). In [7], Graham and MacLean obtained an analogous result for a LCA group. In section 2 of this paper, we prove the following:

THEOREM 2.3. *Suppose an ordering of $\hat{G}$ is given by nontrivial continuous homomorphism ψ from $\hat{G}$ into R. Let δ be a positive real number and Φ a multiplier on $L^1_{-\delta}(R)$ (the definition of $L^1_{-\delta}(R)$ will be stated in Definition 2.1). Then $\Phi\circ\psi$ is also a multiplier on $M^a(G)$ with the following properties:*

$$\text{(I)}\qquad S\big(M^a(G)\cap L^1(G)\big)\subset M^a(G)\cap L^1(G)\,,$$

$$\text{(II)}\qquad S\big(M^a(G)\cap M_s(G)\big)\subset M^a(G)\cap M_s(G)\,,$$

(III) $\qquad ||\Phi\circ\phi|| \leq ||\Phi||$,

where S is the operator on $M^a(G)$ corresponding to $\Phi\circ\phi$.

By using above theorem, we construct a multiplier S on $M^a(G)$ such that it maps $M^a(G)\cap M_s(G)$ into itself and it is not given by convolution with a measure in $M(G)$ (Theorem 2. 4). In section 3, we prove that every multiplier on the space of analytic singular measures does not vanish at infinity (Theorem 3. 1). In section 4, we obtain an analogous result of Doss ([3]) by the method used in section 2 (Theorem 4. 2). In section 5, we shall show that the result obtained in section 3 is satisfied for a LCA group with the algebraically ordered dual (Theorem 5. 1). We use the ideas of Glicksberg ([6]) and the thory of disintegration.

DEFINITION 1. 1. *Let G be a LCA group and E a subset of $\hat{G}$. A function Φ on $\hat{G}$ which is continuous on E^0 is called a multiplier on $M_E(G)$ if $\Phi\hat{\mu}$ belongs to $M_E(G)^\wedge$ for each $\mu\in M_E(G)$. Let S be a bounded linear operator on $M_E(G)$ such that $S(\mu)^\wedge=\Phi\hat{\mu}$. S is also called a multiplier on $M_E(G)$. We denote a norm $||\Phi||$ by $||\Phi||=||S||$.*

We need the following lemma later on.

LEMMA (A) (R. Doss, Theorem 1 in [2]).
Let G be a LCA group. A continuous function ϕ on $\hat{G}$ is the Fourier-Stieltjes transform of a singular measure on G if and only if there exists a positive constant A such that

(i) *for every trigonometric polynomial $p(x)=\sum c_i(-x_i, \gamma_i)$, $\gamma_i\in\hat{G}$, the relation $||p||_\infty\leq 1$ implies $|\sum c_i\phi(\gamma_i)|\leq A$;*

(ii) *whatever be $\varepsilon>0$ and the compact set K in $\hat{G}$, there is a polynomial $p(x)=\sum c_i(-x, \gamma_i)$, $\gamma_i\in\hat{G}$, $\gamma_i\notin K$ such that $||p||_\infty\leq 1$ and $|\sum c_i\phi(\gamma_i)|>A-\varepsilon$.*

The following lemmas are well known.

LEMMA (B). *Let G be a LCA group and Λ an open subgroup of $\hat{G}$. Let H be the annihilator of Λ. For each $\mu\in M_s(G/H)$, there exists a measure $\nu\in M_s(G)$ such that $||\nu||=||\mu||$, $\hat{\nu}(\gamma)=\hat{\mu}(\gamma)$ on Λ and $\hat{\nu}(\gamma)=0$ on $\hat{G}\backslash\Lambda$.*

LEMMA (C). *Let G, H and Λ be as in Lemma (B). Let π be the natural homorphism from G onto G/H. Then, for each $\mu\in M_s(G)$ with supp$(\hat{\mu})\subset\Lambda$, $\pi(\mu)$ belongs to $M_s(G/H)$ and $||\mu||=||\pi(\mu)||$, where $\pi(\mu)$ denotes the continuous image of μ under π.*

The following lemma is easily obtained from ([11]; 1. 9. 1 Theorem, p. 32).

LEMMA (D). *Let G be a LCA group and Φ a continuous function on*

$\hat{G}$. Let A be a positive real number. If $\Phi|_{\Gamma}\in M(G/\Gamma^{\perp})^{\wedge}$ and $\|\Phi|_{\Gamma}\|\leq A$ for each σ-compact open subgroup Γ of $\hat{G}$, Φ belongs to $M(G)^{\wedge}$ and $\|\Phi\|\leq A$.

The following lemma is a useful one.

LEMMA (E). *Let G be a LCA group and E a subset of $\hat{G}$. Let Φ be a multiplier on $M_E(G)$ and S the operator on $M_E(G)$ corresponding to Φ. Then we have*

$$S\big(M_E(G)\cap L^1(G)\big)\subset M_E(G)\cap L^1(G)\,.$$

PROOF. For each $x\in G$ and $\mu\in M_E(G)$, we have

$$S(\mu)*\delta_x=S(\mu*\delta_x)\,,$$

where δ_x is the Dirac measure at x. Hence, if $\mu\in M_E(G)\cap L^1(G)$, we have

$$\begin{aligned}\lim_{x\to 0}\big\|S(\mu)*\delta_x-S(\mu)\big\| &= \lim_{x\to 0}\big\|S(\mu*\delta_x)-S(\mu)\big\|\\ &\leq \lim_{x\to 0}\|S\|\,\|\mu*\delta_x-\mu\|\\ &=0\,.\end{aligned}$$

Thus $S(\mu)$ belongs to $M_E(G)\cap L^1(G)$ and the proof is complete.

§ 2 Some multipliers on the space of analytic measures.

2. 1. The special case.

R and T denote the reals and the cricle group respectively. In this section we consider the case that G is the group of type $R\oplus K$ (or $T\oplus K$) and ψ is the projection from $\hat{G}$ onto R (or Z). Let $H^1(R)$ and $H^1(T)$ be the Hardy spaces. Then, by the F. and M. Riesz theorem, $H^1(R)=\{\mu\in M(R)\,;\ \hat{\mu}(x)=0 \text{ for } x<0\}$ and $H^1(T)=\{\mu\in M(T)\,;\ \hat{\mu}(n)=0 \text{ for } n<0\}$. Let K be a LCA group. We define $M^a(R\oplus K)$ and $M^a(T\oplus K)$ as follows:

$$M^a(R\oplus K)=\big\{\mu\in M(R\oplus K)\,;\ \hat{\mu}(x,\sigma)=0 \text{ for } x<0\big\}\,,$$

$$M^a(T\oplus K)=\big\{\mu\in M(T\oplus K)\,;\ \hat{\mu}(n,\sigma)=0 \text{ for } n<0\big\}\,.$$

PROPOSITION 2. 1. *Let K be a metrizable LCA group and ψ the projection from $R\oplus\hat{K}$ (or $Z\oplus\hat{K}$) onto R (or Z). Then the following are satisfied:*

(I. 1) *Let Φ be a multiplier on $H^1(R)$. Then $\Phi\circ\psi$ is also a multiplier on $M^a(R\oplus K)$ with the following properties:*

(I. 1. a) $$S\big(M^a(R\oplus K)\cap L^1(R\oplus K)\big)\subset M^a(R\oplus K)\cap L^1(R\oplus K)\,,$$

(I. 1. b) $\quad S\big(M^a(R\oplus K)\cap M_s(R\oplus K)\big)\subset M^a(R\oplus K)\cap M_s(R\oplus K)$,

(I. 1. c) $\quad \|S\|\leq\|\Phi\|$,

where S is the bounded linear operator on $M^a(R\oplus K)$ induced by $\Phi\circ\psi$.

(I. 2) *Let Φ be a multiplier on $H^1(T)$. Then $\Phi\circ\psi$ is also a multiplier on $M^a(T\oplus K)$ which satisfies the same properties in* (I. 1).

(II) *Let G be $R\oplus K$ or $T\oplus K$. Then there exists a multiplier S on $M^a(G)$ with the following properties:*

(II. a) $\quad S\big(M^a(G)\cap L^1(G)\big)\subset M^a(G)\cap L^1(G)$,

(II. b) $\quad S\big(M^a(G)\cap M_s(G)\big)\subset M^a(G)\cap M_s(G)$,

(II. c) *S is not given by convolution with a bounded regular measure on G.*

PROOF. (I. 1): First we consider the case that K is a metrizable σ-compact LCA group. Let π be the projection from $R\oplus K$ onto K. Let μ be a measure in $M^a(R\oplus K)$ and put $\eta=\pi(|\mu|)$. Then, by the theory of disintegration, there exists a family $\{\lambda_h\}_{h\in K}$ consisting of measures in $M(R\oplus K)$ with the following properties (see [6], p. 419~420):

(1) $h\mapsto\lambda_h(f)$ is a Borel measurable function of h for each bounded Borel function f on $R\oplus K$,

(2) $\operatorname{supp}(\lambda_h)\subset R\times\{h\}$,

(3) $\|\lambda_h\|\leq 1$,

(4) $\mu(g)=\int_K\lambda_h(g)\,d\eta(h)$
for each bounded Borel measurable function on $R\oplus K$.

From (2), we have $d\lambda_h(x,y)=d\nu_h(x)\times d\delta_h(y)$, where $\nu_h\in M(R)$ and δ_h is the Dirac measure at h. Then we have

(5) $\nu_h\in H^1(R)$ a. a. $h(\eta)$. (See the proof of Lemma 3 in [13].).

That is, there exists a Borel measurable set B in K satisfying the following:

(5)′ $\eta(\mathrm{B})=\|\eta\|$ and $\nu_h\in H^1(R)$ for $h\in B$.

Let S_Φ be a multiplier on $H^1(R)$ defined by $S_\Phi(f)^\wedge=\Phi\hat{f}$. We define a family $\{\xi_h\}_{h\in K}$ consisting of measures in $M(R\oplus K)$ as follows:

$$d\xi_h(x,y)=\begin{cases} dS_\Phi(\nu_h)(x)\times d\delta_h(y) & \text{for } h\in B\\ 0 & \text{for } h\notin B.\end{cases}$$

Then the following is satisfied.

(6) $h \mapsto \xi_h(f)$
is a Borel measurable function of h for each $f \in C_0(R \oplus K)$.

Indeed, for each $f \in C_c(\widehat{R \oplus K})$, we have

$$\begin{aligned}
\xi_h(\hat{f}) &= \int_{R \oplus K} \hat{f}(x, y)\, d\xi_h(x, y) \\
&= \int_{R \oplus K} \int_{\widehat{R \oplus K}} e^{-ixs}(-y, \sigma) f(s, \sigma)\, dm_{\widehat{R \oplus K}}(s, \sigma)\, d\xi_h(x, y) \\
&= \int_{\widehat{R \oplus K}} \int_{R \oplus K} e^{-ixs}(-y, \sigma) f(s, \sigma)\, d\xi_h(x, y)\, dm_{\widehat{R \oplus K}}(s, \sigma) \\
&= \begin{cases} \displaystyle\int_{\widehat{R \oplus K}} \Phi(s)\, \hat{\nu}_h(s)\,(-h, \sigma) f(s, \sigma)\, dm_{\widehat{R \oplus K}}(s, \sigma) & \text{for } h \in B \\ 0 & \text{for } h \notin B \end{cases} \\
&= \begin{cases} \displaystyle\int_{\widehat{R \oplus K}} \Phi(s)\, \lambda_h\big(e^{-is\cdot}(-\,\bullet, \sigma)\big) f(s, \sigma)\, dm_{\widehat{R \oplus K}}(s, \sigma) & \text{for } h \in B \\ 0 & \text{for } h \notin B\,. \end{cases}
\end{aligned}$$

Since $\Phi(s)\, \lambda_h(e^{-is\cdot}(-\,\bullet, \sigma)) f(s, \sigma)$ is a continuous function of (s, σ) for each $h \in B$ and a measurable function of h for each $(s, \sigma) \in \widehat{R \oplus K}$, $\xi_h(\hat{f})$ is a Borel measurable function of h. Since $C_c(\widehat{R \oplus K})^\wedge$ is dense in $C_0(R \oplus K)$, (6) is proved.

Now we define a measure $S(\mu)$ in $M(R \oplus K)$ as follows:

(7) $$S(\mu)(f) = \int_K \xi_h(f)\, d\eta(h) \qquad \text{for } f \in C_0(R \oplus K)\,.$$

Then we have

(8) $$\begin{aligned}
S(\mu)^\wedge(z, \sigma) &= \int_K \xi_h\big(e^{-iz\cdot}(-\,\bullet, \sigma)\big)\, \mathrm{d}\eta(h) \\
&= \int_K S_\Phi(\nu_h)(z)^\wedge(-h, \sigma)\, d\eta(h) \\
&= \int_K \Phi(z)\, \hat{\nu}_h(z)\,(-h, \sigma)\, d\eta(h) \\
&= \Phi(z)\, \hat{\mu}(z, \sigma) \\
&= \Phi \circ \psi(z, \sigma)\, \hat{\mu}(z, \sigma)\,.
\end{aligned}$$

Hence $\Phi \circ \psi$ is a multiplier on $M_a(R \oplus K)$. For $f \in C_0(R \oplus K)$, by (3), we have

$$\begin{aligned}\left|S(\mu)(f)\right| &= \left|\int_K \xi_h(f)\, d\eta(h)\right| \\ &\leq \int_K \left|\left\{S_\phi(\nu_h)\times\delta_h\right\}(f)\right| d\eta(h) \\ &\leq \int_K \|\Phi\|\,\|f\|_\infty d\eta(h) \\ &= \|\Phi\|\,\|f\|_\infty \|\mu\|\,.\end{aligned}$$

Thus we have $\|S\|\leq\|\Phi\|$. Hence (I. 1. c) is proved. (I. 1. a) is obtained from Lemma (E). Next we prove (I. 1. b).

Let μ be a measure in $M^a(R\oplus K)\cap M_s(R\oplus K)$. First we prove that $\eta=\pi(|\mu|)$ belongs to $M_s(K)$. Put $\eta=\eta_a+\eta_s$, where $\eta_a\in L^1(K)$ and $\eta_s\in M_s(K)$. For each Borel measurable subset F of $R\oplus K$ with $m_{R\oplus K}(F)=0$, there exists a Borel set F_2 in K with $m_K(F_2)=0$ such that $m_R(F_y)=0$ for $y\notin F_2$, where $F_y=\{x\in R\,;\,(x,y)\in F\}$. Put $F_o=F_2\subset(K\backslash B)$, where B is the subset of K appeared in (5′). Then $\eta_a(F_o)=0$. Since $\nu_h\in H^1(R)$ if $h\in B$, we have

$$\begin{aligned}\int_K \lambda_h(\chi_F)\, d\eta_a(h) &= \int_{F_0}(\nu_h\times\delta_h)(\chi_F)\, d\eta_a(h)+\int_{K\backslash F_0}(\nu_h\times\delta_h)(\chi_F)\, d\eta_a(h) \\ &=0\,.\end{aligned}$$

Hence the measure $\int_K\lambda_h d\eta_a(h)$ belongs to $L^1(R\oplus K)$. Evidently, the measure $\int_K\lambda_h d\eta_s(h)$ belongs to $M_s(R\oplus K)$. On the other hand, since $\mu\in M_s(R\oplus K)$, we have $\mu=\int_K\lambda_h d\eta_s(h)$. By (3), we have $\|\eta_a\|+\|\eta_s\|=\|\mu\|\leq\|\eta_s\|$. Hence we have $\eta=\eta_s\in M_s(K)$. Therefore, by the construction of $S(\mu)$, we can verify that $S(\mu)$ belongs to $M_a(R\oplus K)\cap M_s(R\oplus K)$ if $\mu\in M^a(R\oplus K)\cap M_s(R\oplus K)$. Thus (I. 1) is proved when K is a metrizable σ-compact LCA group. Next we consider the case that K is a metrizable LCA group. However, in this case, for $\mu\in M^a(R\oplus K)$, there exists a metrizable σ-compact open subgroup K_1 of K such that $\mathrm{supp}\,(\mu)$ is included in $R\oplus K_1$. Hence we can prove (I. 1) as in the same way as in the case that K is a metrizable σ-compact LCA group. This proves (I. 1).

(I. 2): We can prove (I. 2) by the same method used in (I. 1).

(II): Since we can prove (II) in the case that $G=T\oplus K$ as same as in $G=R\oplus K$, we prove only the case that $G=R\oplus K$. Let $\{a_n\}$ be a sequence of positive integers with $a_{m+1}/a_m>3$ $(m=1, 2, 3, \cdots)$. Put $F=\{a_m\,;\,m=1, 2, 3, \cdots\}$. Let $\varDelta(x)$ be a function in $L^1(R)^\wedge$ such that $\varDelta(x)=\max(1-3|x|, 0)$ and $\phi_o(n)=\chi_F(n)$, where χ_F is a characteristic function of F. Now we define a function Φ_o on R as follows:

$$\Phi_o(x)=\sum_{n\in Z}\phi_o(n)\,\varDelta(x-n)\,.$$

Then, as well known, Φ_0 is a multiplier on $H^1(R)$ which is not given by convolution with a bounded regular measure on R (see [12]), i.e., $\Phi_0|_{R^+} \notin M(R)^\wedge|_{R^+}$, where R^+ is the semigroup of positive real numbers. Hence we can verify that $\Phi_0 \circ \psi|_{P^0}$ does not belong to $M(R \oplus K)^\wedge|_{P^0}$, where $P = \{(x, \sigma) \in R \oplus k;\ x \geq 0\}$. Thus, by (I. 1), $\Phi_0 \circ \psi$ is such a multiplier. This completes the proof.

DEFINITION 2.1. *Let K be a LCA group. For a real number ε, we define $M^a_\varepsilon(R \oplus K)$ and $L^1_\varepsilon(R)$ as follows:*

$$M^a_\varepsilon(R \oplus K) = \left\{\mu \in M(R \oplus K);\ \hat{\mu}(s, \sigma) = 0 \text{ for } s < \varepsilon\right\},$$

$$L^1_\varepsilon(R) = \left\{f \in L^1(R);\ \hat{f}(x) = 0 \text{ for } x < \varepsilon\right\}.$$

REMARK 2.1. Let K be a metrizable LCA group. Then, by the same method used in Proposition 2.1, we can prove the following:

(I) Let Φ be a multiplier on $L^1_\varepsilon(R)$. Then $\Phi \circ \psi$ is also a multiplier on $M^a_\varepsilon(R \oplus K)$ which satisfies the following:

(I. a) $\quad S\left(M^a_\varepsilon(R \oplus K) \cap L^1(R \oplus K)\right) \subset M^a_\varepsilon(R \oplus K) \cap L^1(R \oplus K)$,

(I. b) $\quad S\left(M^a_\varepsilon(R \oplus K) \cap M_s(R \oplus K)\right) \subset M^a_\varepsilon(R \oplus K) \cap M_s(R \oplus K)$,

(I. c) $\quad \|S\| \leq \|\Phi\|$,

where S is a bounded linear operator on $M^a_\varepsilon(R \oplus K)$ induced by $\Phi \circ \psi$.

(II) There exists a multiplier S on $M^a_\varepsilon(R \oplus K)$ with the following properties:

(II. a) $\quad S\left(M^a_\varepsilon(R \oplus K) \cap L^1(R \oplus K)\right) \subset M^a_\varepsilon(R \oplus K) \cap L^1(R \oplus K)$,

(II. b) $\quad S\left(M^a_\varepsilon(R \oplus K) \cap M_s(R \oplus K)\right) \subset M^a_\varepsilon(R \oplus K) \cap M_s(R \oplus K)$,

(II. c) $\quad S$ is not given by convolution with a bounded regular measure on $R \oplus K$.

REMARK 2.2. Let K be a metrizable LCA group, and let G be $R \oplus K$ or $T \oplus K$. In Proposition 2.1, by the construction of S, we note that there exists a multiplier S' on $M^a(G)$ with the following properties:

(i) $\quad S'\left(M^a(G) \cap L^1(G)\right) \subset M^a(G) \cap L^1(G)$,

(ii) $\quad S'\left(M^a(G) \cap M_s(G)\right) \subset M^a(G) \cap M_s(G)$,

(iii) $\quad S'$ is given by convolution with some measure $\mu \in M(G)$ such that $\hat{\mu}|_{P^0} \notin M_d(G)^\wedge|_{P^0}$, where $P = \{(x, \sigma) \in \hat{G};\ x \geq 0\}$.

Indeed, let μ' be a nonzero function in $L^1(R)$ $(L^1(T))$. We define a measure $\mu\in M(G)$ by $\mu=\mu'\times\delta_0$. Then, by (I. 1) and (I. 2) in Proposition 2.1 we can verify that μ is such a measure.

2.2 The case that there exists a continuous homomorphism from $\hat{G}$ into R.

Let G be a LCA group and ψ a nontrivial continuous homomorphism from $\hat{G}$ into R. We may assume that there exists an element $\chi_0\in\hat{G}$ such that $\psi(\chi_0)=1$ without loss of generality*). Put $\Lambda=\{n\chi_0\,;\ n\in Z\}$. Let ϕ be the dual homomorphism of ψ, i. e., $(\phi(t),\gamma)=e^{i\psi(\gamma)t}$ for $t\in R$ and $\gamma\in\hat{G}$. Then ϕ is a continuous homomorphism from R into G.

Lemma 2.1. *Λ is a discrete subgroup of $\hat{G}$.*

Proof. Since ψ is continuous, it is sufficient to prove the closedness of Λ. Let γ be an element in $\hat{G}$. Suppose $n_\alpha\chi_0\mapsto\gamma$ $(n_\alpha\in Z)$. Then $n_\alpha=\psi(n_\alpha\chi_0)\to\psi(\gamma)$. Since $n_\alpha\in Z$, there exists an integer n_{α_0} such that $n_\alpha\chi_0=n_{\alpha_0}\chi_0$ for $\alpha\geq\alpha_0$. Hence we have $\gamma=n_{\alpha_0}\chi_0\in\Lambda$. This completes the proof.

Let $K=\Lambda^\perp$ (annihilator of Λ). We define a continuous homomorphism α from $R\oplus K$ into G as follows:

$$\alpha(t,u)=\phi(t)+u \qquad \text{for} \quad (t,u)\in R\oplus K\,. \tag{2.1}$$

Then α is an onto map (see Lemma 6.1 in [1]). Let $D=\ker(\alpha)$. Then we have

$$D=\Big\{\big(2\pi n,\ -\phi(2\pi n)\big)\in R\oplus K\,;\ n\in Z\Big\}. \quad \text{(cf. [1], p. 192).} \tag{2.2}$$

Lemma 2.2. $D^\perp=\{(\psi(\gamma),\gamma|_K)\,;\ \gamma\in\hat{G}\}$.

Proof. Let γ be an element in $\hat{G}$. We first prove that $(\psi(\gamma),\gamma|_K)\in D^\perp$. For each $(2\pi n,\ -\phi(2\pi n))\in D$, we have

$$\begin{aligned}\Big(\big(\psi(\gamma),\gamma|_K\big),\big(2\pi n,\ -\phi(2\pi n)\big)\Big)&=e^{i\psi(\gamma)2\pi n}\big(\gamma|_K,\ -\phi(2\pi n)\big)\\&=e^{i\psi(\gamma)2\pi n}\big(\gamma,\ -\phi(2\pi n)\big)\\&=e^{i\psi(\gamma)2\pi n}e^{-i\psi(\gamma)2\pi n}\\&=1\,.\end{aligned}$$

Conversely, let (t,σ) be an element in $D^\perp$. Then, for each $n\in Z$, we have

$$\begin{aligned}1&=\Big((t,\sigma),\big(2\pi n,\ -\phi(2\pi n)\big)\Big)\\&=e^{i2\pi nt}\big(\sigma,\ -\phi(2\pi n)\big)\,.\end{aligned}$$

*) The reason will be stated in Remark 2.5.

Let σ_* be an element in $\hat{G}$ such that $\sigma_*|_K=\sigma$. Then we have

$$\begin{aligned}1 &= e^{i2\pi nt}\Big(\sigma_*,\ -\phi(2\pi n)\\ &= e^{i2\pi nt}\, e^{-i\psi(\sigma_*)2\pi n}\\ &= e^{i(t-\psi(\sigma_*))2\pi n}\,.\end{aligned}$$

Since n is any integer, we have

$$t=\psi(\sigma_*)+m\,, \qquad \text{where } m \text{ is some integer.}$$

Put $\gamma=\mathrm{m}\chi_0+\sigma_*$. Then we have

$$\psi(\gamma)=t \quad \text{and} \quad \gamma|_K=\sigma_*|_K=\sigma\,.$$

Hence we have $(\psi(\gamma), \gamma|_K)=(t, \sigma)$. This completes the proof.

LEMMA 2. 3. *The following are satisfied.*
(I) $a((-\pi, \pi]\times K)=G$,
(II) *α is a homomorphism in the interior of $(-\pi, \pi]\times K$,*
(III) *α is an onto, open continuous homomorphism.*

PROOF. (I) and (II) can be proved by the same method used in ([1]; Lemma 6. 1). (III) is easily obtained from (I) and (II). This completes the proof.

By Lemma 2. 3 and ([8]; (5. 27) Theorem, p. 41), we have

(2. 3) $\qquad R\oplus K/D\cong G\,,$ and so $D^\perp\cong\hat{G}\,.$

DEFINITION 2. 2. *We define $M^a(G)$ as follows:*

$$M^a(G)=\Big\{\mu\in M(G)\,;\ \hat{\mu}(\gamma)=0 \text{ for } \gamma\in\hat{G} \text{ with } \psi(\gamma)<0\Big\}\,.$$

PROPOSITION 2. 2. *The following are satisfied.*

(i) $\qquad \alpha\Big(M^a(R\oplus K)\Big)\subset M^a(G)\,,$

(ii) $\qquad \alpha\Big(L^1(R\oplus K)\Big)\subset L^1(G)\,,$

(iii) $\qquad \alpha\Big(M_s(R\oplus K)\Big)\subset M_s(G)\,.$

PROOF. (i): Let μ be a measure in $M^a(R\oplus K)$. Then

$$\begin{aligned}\alpha(\mu)^\wedge(\gamma) &= \int_{R\oplus K}\Big(-\gamma, \alpha(t, u)\Big)\,d\mu(t, u)\\ &= \hat{\mu}\Big(\psi(\gamma), \gamma|_K\Big)\,.\end{aligned}$$

Hence (i) is proved.

(ii) and (iii) are obtained from Lemma 2. 3. This completes the proof.

Next we define a continuous homomorphism α_1 from $R\oplus\hat{G}$ into $R\oplus\hat{K}$ by $\alpha_1(t, \gamma)=(t+\psi(\gamma), \gamma|_K)$.

LEMMA 2. 4. *The following are satisfied.*

(i) $\quad ker(\alpha_1)=\left\{(n, -n\chi_0)\in R\oplus\hat{G}\,;\ n\in Z\right\}$,

(ii) $\quad \alpha_1\left(\left[-\frac{1}{2}, \frac{1}{2}\right)\times\hat{G}\right)=R\oplus\hat{K}$.

PROOF. (i):

$$\begin{aligned}\ker(\alpha_1)&=\left\{(t, \gamma)\in R\oplus\hat{G}\,;\ \left(t+\psi(\gamma), \gamma|_K\right)=0\right\}\\ &=\left\{(t, \gamma)\in R\oplus\hat{G}\,;\ \gamma\in\{n\chi_0\,;\ n\in Z\},\ t=-\psi(\gamma)\right\}\\ &=\left\{(n, -n\chi_0)\in R\oplus\hat{G}\,;\ n\in Z\right\}.\end{aligned}$$

(ii): Let (t, σ) be an element in $R\oplus\hat{K}$. Then there exists an element $\gamma\in\hat{G}$ such that $\gamma|_K=\sigma$. Let n be an integer such that $n-\frac{1}{2}\leq-\psi(\gamma)+t<n+\frac{1}{2}$. Put $t_1=-\psi(\gamma)+t-n$. Then we have $t_1\in\left[-\frac{1}{2}, \frac{1}{2}\right)$ and $\alpha_1(t_1, \gamma+n\chi_0)=(t, \sigma)$. This completes the proof.

Moreover the following lemma is satisfied.

LEMMA 2. 5. (a) $\alpha_1\left(\left(-\frac{1}{2}, \frac{1}{2}\right)\times\hat{G}\right)$ *is an open subset of* $R\oplus\hat{K}$,

(b) α_1 *is a homeomorphism on* $\left(-\frac{1}{2}, \frac{1}{2}\right)\times\hat{G}$,

(c) α_1 *is an open continuous homomorphism.*

PROOF. (a): By Lemma 2. 4, α_1 maps $\left(-\frac{1}{2}, \frac{1}{2}\right]\times G$ one to one, onto $R\oplus\hat{K}$. Hence, by Lemma 2. 2, we have

$$\begin{aligned}\alpha_1\left(\left(-\frac{1}{2}, \frac{1}{2}\right)\times\hat{G}\right)&=R\oplus\hat{K}\backslash\alpha_1\left(\left\{-\frac{1}{2}\right\}\times\hat{G}\right)\\ &=R\oplus\hat{K}\backslash\left\{\left(-\frac{1}{2}+\psi(\gamma), \gamma|_K\right);\ \gamma\in\hat{G}\right\}\\ &=R\oplus\hat{K}\backslash\left\{\left(-\frac{1}{2}, 0\right)+D^{\perp}\right\}.\end{aligned}$$

Since $\left(-\frac{1}{2}, 0\right)+D^{\perp}$ is a closed subset of $R\oplus\hat{K}$, (a) is proved.

(b): Suppose $\alpha_1(t_\alpha, \gamma_\alpha)$ converge to $\alpha_1(t_o, \gamma_0)$, where

$$(t_\alpha, \gamma_\alpha),\ (t_o, \gamma_0) \in \left(-\frac{1}{2}, \frac{1}{2}\right) \times \hat{G}\,.$$

Let $\{(t_{\alpha'}, \gamma_{\alpha'})\}$ be a subnet of $\{(t_\alpha, \gamma_\alpha)\}$. It is sufficient to prove that (t_o, γ_0) is an accumulation point of $\{(t_{\alpha'}, \gamma_{\alpha'})\}$. Since $|t_\alpha| \leq \frac{1}{2}$, there exists a subnet $\{t_\beta\}$ of $\{t_{\alpha'}\}$ and a real number t_1 with $|t_1| \leq \frac{1}{2}$ such that $t_\beta \xrightarrow{\beta} t_1$. Since $(t_\beta + \psi(\gamma_\beta), \gamma_\beta|_K) = \alpha_1(t_\beta, \gamma_\beta) \xrightarrow{\beta} \alpha_1(t_o, \gamma_0) = (t_o + \psi(\gamma_0), \gamma_0|_K)$, we have $(\psi(\gamma_\beta), \gamma_\beta|_K) \xrightarrow{\beta} (t_o - t_1 + \psi(\gamma_0), \gamma_0|_K)$. Since $(\psi(\gamma_\beta), \gamma_\beta|_K) \in D^\perp$, $(t_o - t_1 + \psi(\gamma_0), \gamma_0|_K) \in D^\perp$. Hence, by Lemma 2. 2, there exists an element $\gamma_1 \in \hat{G}$ such that $(t_0 - t_1 + \psi(\gamma_0), \gamma_0|_K) = (\psi(\gamma_1), \gamma_1|_K)$. Since $K^\perp = \Lambda$, we have $\psi(\gamma_0) - \psi(\gamma_1) = \psi(\gamma_0 - \gamma_1) \in Z$. On the other hand, since $|t_o - t_1| < 1$, we have $t_o = t_1$, and so $\gamma_0 = \gamma_1$. That is, (t_β, γ_β) converges to (t_o, γ_0). This proves (b).

(c): (c) is easily obtained from (a) and (b).
This completes the proof.

Definition 2. 3. *For* $0 < \varepsilon < \frac{1}{6}$, *we define a function* $\Delta(x, \sigma)$ *on* $R \oplus \hat{K}$ *by* $\Delta(x, \sigma) = \max\left(1 - \frac{1}{\varepsilon}|x|, 0\right)$ *for* $\sigma = 0$ *and* $\Delta(x, \sigma) = 0$ *for* $\sigma \neq 0$.

Lemma 2. 6. *Let G be a compact abelian group and ψ a nontrivial homomorphism from $\hat{G}$ into R. Let ε be a positive real number such that* $0 < \varepsilon < \frac{1}{6}$. *For $\mu \in M(G)$, by regarding μ as a measure in $M(R \oplus K/D)$ (cf. (2. 3)), we define a function $\Phi_\mu(t, \sigma)$ on $R \oplus \hat{K}$ as follows:*

$$(2.4) \qquad \Phi_\mu(t, \sigma) = \sum_{\gamma \in \hat{G}} \hat{\mu}(\gamma)\, \Delta^2\Big((t, \sigma) - \big(\psi(\gamma), \gamma|_K\big)\Big)\,.$$

Then the following are satisfied.

(I) $\quad \Phi_\mu \in M(R \oplus K)^\wedge \quad$ *for* $\mu \in M(G)$ *and* $\|\Phi_\mu\| = \|\mu\|$,

(II) $\quad \Phi_\mu \in L^1(R \oplus K)^\wedge \quad$ *if* $\mu \in L^1(G)$,

(III) $\quad \Phi_\mu \in M_s(R \oplus K)^\wedge \quad$ *if* $\mu \in M_s(G)$.

Proof. Let $V_\varepsilon = \{(s, 0) \in R \oplus \hat{K}\,;\ |s| \leq \varepsilon\}$. Then V_ε is a compact neigborhood of 0 in $R \oplus K$ such that $V_\varepsilon \cap D^\perp = \{0\}$. Moreover Δ is a positive definite function such that $\|\check{\Delta}\|_1 = 1$ and $\operatorname{supp}(\Delta) \subset V_\varepsilon$. Hence, by ([5]; Theorem 1), for each $\mu \in M(G)$, we have

(1) $\Phi_\mu \in M(R \oplus K)^\wedge$ and $\|\Phi_\mu\| \leq \|\mu\|$.

Claim 1: $\|\Phi_\mu\| = \|\mu\|$.

By (1), it is sufficient to prove that $\|\Phi_\mu\| \geq \|\mu\|$. Let $\{p_n\}$ be a sequence in $\mathrm{Trig}(G)$ with $\|p_n\|_\infty \leq 1$ such that $\|\mu\| = \lim_{n\to\infty} |\int_G p_n(y)\, d\mu(y)|$. We define $\tilde{p}_n \in \mathrm{Trig}(R \oplus K)$ by $\tilde{p}_n(x, u) = p_n(\alpha(x, u))$. Since $\Phi_\mu(\psi(\gamma), \gamma|_K) = \hat{\mu}(\gamma)$, we have

$$\int_{R\oplus K} \tilde{p}_n(x, u)\, d\check{\Phi}_\mu(x, u) = \int_G p_n(y)\, d\mu(y)\,.$$

Hence we have

$$\begin{aligned} \|\mu\| &= \lim_{n\to\infty} \left| \int_G p_n(y)\, d\mu(y) \right| \\ &= \lim_{n\to\infty} \left| \int_{R\oplus K} \tilde{p}_n(x, u)\, d\check{\Phi}_\mu(x, u) \right| \\ &\leq \overline{\lim_{n\to\infty}}\, \|\tilde{p}_n\|_\infty \|\Phi_\mu\| \\ &\leq \|\Phi_\mu\|\,. \end{aligned}$$

Thus Claim 1 is proved and (I) is proved.

Claim 2: $\check{\Phi}_\mu$ belongs to $M_s(R \oplus K)$ if $\mu \in M_s(G)$.

Put $A = \|\mu\|$. Then, by Claim 1, we have $\|\check{\Phi}_\mu\| = A$. Let $\varepsilon' > 0$ and K' a compact subset of $R \oplus \hat{K}$. Since $D^\perp \subset K'$ is compact in $D^\perp$ and $\mu \in M_s(G)$, by Lemma (A), there exists $p(y) = \sum c_i(-y, \gamma_i) \in \mathrm{Trig}(G)$ with $(\psi(\gamma_i), \gamma_i|_K) \in D^\perp \backslash (D^\perp \cap K')$ such that

(2) $\|p\|_\infty \leq 1$ and $\left|\sum c_i \hat{\mu}(\gamma_i)\right| > A - \varepsilon'$.

Let $\tilde{p}$ be a trigonometric polinomial on $R \oplus K$ such that $\tilde{p}(t, u) = \sum c_i e^{-i\psi(\gamma_i)t}(-u, \gamma_i|_K)$. Then $\|\tilde{p}\|_\infty \leq 1$. Since $\Phi_\mu(\psi(\gamma), \gamma|_K) = \hat{\mu}(\gamma)$, we have

$$\begin{aligned} \left|\sum c_i \Phi_\mu\big(\psi(\gamma_i), \gamma_i|_K\big)\right| &= \left|\sum c_i \hat{\mu}(\gamma_i)\right| \\ &> A - \varepsilon'\,. \end{aligned}$$

Hence, by Lemma (A), we have $\check{\Phi}_\mu \in M_s(R \oplus K)$. Thus Claim 2 is proved.

Claim 3: $\check{\Phi}_\mu$ belongs to $L^1(R \oplus K)$ if $\mu \in L^1(G)$.

Let μ be a measure in $L^1(G)$. Then there exists a sequence $\{\mu_n\}$ in $L^1(G)$ such that $\hat{\mu}_n$ has a compact support and $\lim_{n\to\infty} \|\mu - \mu_n\| = 0$. Then, since $\Phi_{\mu_n}(s, \sigma) = \sum_{\gamma\in\hat{G}} \hat{\mu}_n(\gamma)\, \Delta^2((s, \sigma) - (\psi(\gamma), \gamma|_K))$ has a compact support, $\check{\Phi}_{\mu_n}$ belongs to $L^1(R \oplus K)$. Hence, by Claim 1, we have $\check{\Phi}_\mu \in L^1(R \oplus K)$. This completes the proof.

THEOREM 2.1. *Let G be a LCA group and ψ a nontrivial continuous* *homomorphism from* $\hat{G}$ *into* R. *We may assume that there exists* $\chi_0 \in \hat{G}$ *such that* $\psi(\chi_0)=1$*). *Let* ε *be a positive real number such that* $0<\varepsilon<\frac{1}{6}$. *Let* K *and* D *be as after Lemma* 2.1. *For* $\mu \in M(G)$, *by regarding* μ *as a measure in* $M(R\oplus K/D)$, *we define a function* $\Phi_\mu(t, \sigma)$ *on* $R\oplus\hat{K}$ *as follows*:

$$(2.5) \qquad \Phi_\mu(t, \sigma)=\sum_{\gamma\in\hat{G}} \hat{\mu}(\gamma)\, \Delta^2\Big((t, \sigma)-\big(\psi(\gamma), \gamma|_K\big)\Big).$$

Then the following are satisfied:

(I) $\quad \Phi_\mu \in M(R\oplus K)^\wedge \quad$ *for* $\mu\in M(G)$ *and* $\|\Phi_\mu\|=\|\mu\|$,

(II) $\quad \Phi_\mu \in L^1(R\oplus K)^\wedge \quad$ *if* $\mu \in L^1(G)$,

(III) $\quad \Phi_\mu \in M_s(R\oplus K)^\wedge \quad$ *if* $\mu \in M_s(G)$.

PROOF. We may consider only the case that G is noncompact. Let $\bar{G}$ be the Bohr compactification of G and $\bar{K}$ the closure of K in $\bar{G}$. Then $\bar{K}$ is the annihilator of Λ in $\bar{G}$, where Λ is the discrete subgroup of $\hat{G}$ generated by χ_0. Let ψ_* be the homomorphism from $\hat{G}_d$ into R such that $\psi_*(\gamma)=\psi(\gamma)$ and ϕ_* the dual homomorphism of ψ_*, where $\hat{G}_d$ is the group $\hat{G}$ with the discrete topology. Let μ be a measure in $M(G)$. We regard μ as a measure in $M(\bar{G})$. We define a function ${}_*\Phi_\mu(t, \sigma)$ on $R\oplus\hat{\bar{K}}$ by

$$(1) \qquad {}_*\Phi_\mu(t, \sigma)=\sum_{\delta\in\hat{G}_d} \hat{\mu}(\gamma)\, \Delta^2\Big((t, \sigma)-\big(\psi_*(\gamma), \gamma|_{\bar{K}}\big)\Big).$$

Since $\hat{\bar{K}}=\hat{K}_d$, we have $\Phi_\mu(t, \sigma)={}_*\Phi_\mu(t, \sigma)$ for $(t, \sigma)\in R\oplus\hat{K}$. Hence, by Lemma 2.6, we have

$$(2) \qquad \Phi_\mu \in M(R\oplus\bar{K})^\wedge \quad \text{and} \quad \|\Phi_\mu\|=\|\mu\|$$
(regarding μ as a measure in $M(\bar{G})$).

Claim: Φ_μ is a continuous function on $R\oplus\hat{K}$.
Put $I=\{(t, \gamma)\in R\oplus\hat{G};\ |t|\leq 2\varepsilon\}$ and $\mathring{I}=\{(t, \gamma)\in R\oplus\hat{G};\ |t|\leq 2\varepsilon\}$. Let α_1 be a continuous homomorphism from $R\oplus\hat{G}$ into $R\oplus\hat{K}$ such that $\alpha_1(t, \gamma)=(t+\psi(\gamma), \gamma|_K)$. Then, by Lemma 2.5, $\alpha_1(\mathring{I})$ is an open subset of $R\oplus\hat{K}$. We define a function Φ'_μ on $R\oplus\hat{G}$ as follows:

$$(3) \qquad \Phi'_\mu(t, \gamma)=\Delta_\varepsilon^2(t)\, \hat{\mu}(\gamma), \qquad \text{where } \Delta_\varepsilon(t)=\max\left(1-\frac{1}{\varepsilon}|t|, 0\right).$$

Step 1. For $(t, \gamma)\in\left[-\frac{1}{2}, \frac{1}{2}\right)\times\hat{G}$, we have $\Phi'_\mu(t, \gamma)=\Phi_\mu(\alpha_1(t, \gamma))$. Indeed,

$$\Phi_\mu\big(\alpha_1(t, \gamma)\big)=\Phi_\mu\big(t+\psi(\gamma), \gamma|_K\big)$$

*) If there is not $\chi_0\in\hat{G}$ such that $\psi(\chi_0)=1$, we define Λ, K and D as in Remark 2.5.

$$
\begin{aligned}
&= \sum_{\lambda\in\hat{G}} \hat{\mu}(\lambda)\, \Delta^2\Big(\big(\psi(\gamma)+t, \gamma|_K\big)-\big(\psi(\lambda), \lambda|_K\big)\Big) \\
&= \sum_{n\in Z} \hat{\mu}(\gamma+n\chi_0)\, \Delta^2(t-n, 0) \\
&= \hat{\mu}(\gamma)\, \Delta^2(t, 0) \qquad\qquad (n-t\in(-\varepsilon, \varepsilon)\Leftrightarrow n=0) \\
&= \hat{\mu}(\gamma)\, \Delta^2_\varepsilon(t)\,.
\end{aligned}
$$

Step 2. Φ_μ vanishes on $\alpha_1\Big(\Big[-\frac{1}{2}, -\frac{3}{2}\varepsilon\Big]\times\hat{G}\Big)\cup\alpha_1\Big(\Big[\frac{3}{2}\varepsilon, \frac{1}{2}\Big)\times\hat{G}\Big)$. This is obtained from Step 1.

By Step 2, in order to prove that Φ_μ is continuous, it is sufficient to prove that Φ_μ is continuous on $\alpha_1(\mathring{I})$. Suppose $\alpha_1(t_\alpha, \gamma_\alpha)\rightarrow\alpha_1(t_0, \gamma_0)$, where $(t_\alpha, \gamma_\alpha)$, $(t_0, \gamma_0)\in\mathring{I}$. Then, by Lemma 2.5, we have $(t_\alpha, \gamma_\alpha)\rightarrow(t_0, \gamma_0)$. Hence we have

$$
\begin{aligned}
\lim_\alpha \Phi_\mu\big(\alpha_1(t_\alpha, \gamma_\alpha)\big) &= \lim_\alpha \Phi'_\mu(t_\alpha, \gamma_\alpha) \qquad\qquad \text{(by Step 1)} \\
&= \Phi'_\mu(t_0, \gamma_0) \\
&= \Phi_\mu\big(\alpha_1(t_0, \gamma_0)\big)\,.
\end{aligned}
$$

Thus Claim is proved. Therefore, since $\Phi_\mu\in M(R\oplus\bar{K})^\wedge$ and Φ_μ is continuous on $R\oplus\hat{K}$, Φ_μ belongs to $M(R\oplus K)^\wedge$. Thus (I) is proved. (II) and (III) are obtained by the same method used in Lemma 2. 6. This completes the proof.

REMARK 2. 3. *Let G be a LCA group and ψ a nontrivial continuous homomorphism from $\hat{G}$ into R. Let K be as before. Then the following are equivalent.*

(I) $\quad M^a(G)\cap M_s(G)\neq\{0\}$;

(II) $\quad$ *K is not discrete.*

PROOF. (I)$\Rightarrow$(II): Suppose K is discrete. Let μ be a nonzero measure in $M^a(G)\cap M_s(G)$. Then, by Theorem 2. 1, Φ_μ belongs to $M(R\oplus K)^\wedge$. Moreover, by the construction of Φ_μ, we have

(1) $\quad \Phi_\mu(t, \sigma)=0 \quad$ for $\quad t<-\varepsilon$.

Since K is discrete, there exist $\{\nu_n\}\subset M(R)$ and $\{x_n\}\subset K$ such that

(2) $\quad \check{\Phi}_\mu=\sum_{n=1}^\infty \nu_n\times\delta_{x_n}$ and $\|\check{\Phi}_\mu\|=\sum_{n=1}^\infty \|\nu_n\|$.

By (1), we have

(3) $\quad \hat{\nu}_n(t)=0 \quad$ for $\; t<-\varepsilon \quad (n=1, 2, 3, \cdots)$.

Hence, by the F. and M. Riesz theorem, $\check{\Phi}_\mu$ belongs to $L^1(R\oplus K)$. Thus we have $\mu=\alpha(\check{\Phi}_\mu)\in L^1(G)$ by Proposition 2. 2. This contradicts the choice of μ. Hence '(I)$\Rightarrow$(III)' is proved.

(II)$\Rightarrow$(I): Suppose K is not discrete. Let f be a function in $H^1(R)$ such that $\hat{f}(\psi(\gamma_0))\neq 0$ for some $\gamma_0\notin\hat{G}$ with $\psi(\gamma_0)>0$. Then $f\times\delta_0$ is a measure in $M^a(R\oplus K)\cap M_s(R\oplus K)$. Hence, by Proposition 2. 2, $\phi(f)=\alpha(f\times\delta_0)$ belongs to $M^a(G)\cap M_s(G)$. Since $\hat{f}(\psi(\gamma_0))\neq 0$, $\phi(f)$ is a nonzero measure in $M^a(G)\cap M_s(G)$. This completes the proof.

THEOREM 2. 2. *Let G be a metrizable LCA group and ψ a nontrivial continuous homomorphism from $\hat{G}$ into R. Suppose $M^a(G)\cap M_s(G)\neq\{0\}$. Let δ be a positive real number and Φ a multiplier on $L^1_{-\delta}(R)$. Then $\Phi\circ\psi$ is also a multiplier on $M^a(G)$ with the following properties*:

$$\text{(i)}\qquad S\big(M^a(G)\cap L^1(G)\big)\subset M^a(G)\cap L^1(G)\,,$$

$$\text{(ii)}\qquad S\big(M^a(G)\cap M_s(G)\big)\subset M^a(G)\cap M^s(G)\,,$$

$$\text{(iii)}\qquad \|S\|\leq\|\Phi\|\,,$$

where S is a bounded linear operator on $M^a(G)$ induced by $\Phi\circ\psi$.

PROOF. We choose ε so that $0<\varepsilon<\min\left(\delta,\dfrac{1}{6}\right)$. Then, by the construction of Φ_μ, the following is satisfied:

$$\text{(1)}\qquad \check{\Phi}_\mu\in M^a_{-\varepsilon}(R\oplus K)\qquad\text{for}\quad \mu\in M^a(G)\,.$$

We define a bounded linear operator S_1 from $M^a(G)$ into $M^a_{-\varepsilon}(R\oplus K)$ by $S_1(\mu)=\check{\Phi}_\mu$. Let ψ_0 be the projection from $R\oplus\hat{K}$ onto R. Then, by Remark 2. 1, $\Phi\circ\psi_0$ is a multiplier on $M^a_{-\varepsilon}(R\oplus K)$. Let S_2 be the bounded linear operator on $M^a_{-\varepsilon}(R\oplus K)$ induced by $\Phi\circ\psi_0$. Moreover we define an operator S on $M^a(G)$ as follows:

$$\text{(2)}\qquad S=\alpha\circ S_2\circ S_1\,.$$

Then, by Remark 2. 1, Proposition 2. 2 and Theorem 2. 1, we can verify that (i)$\sim$(iii) are satisfied. Finally we prove that S is a multiplier on $M^a(G)$ corresponding to $\Phi\circ\psi$. For $\mu\in M^a(G)$ and $\gamma\in\hat{G}$, we have

$$\begin{aligned} S(\mu)^\wedge(\gamma) &= S_2\circ S_1(\mu)^\wedge\big(\psi(\gamma),\gamma|_K\big) \\ &= \Phi\circ\psi_0\big(\psi(\gamma),\gamma|_K\big)\,S_1(\mu)^\wedge\big(\psi(\gamma),\gamma|_K\big) \\ &= \Phi\big(\psi(\gamma)\big)\,\Phi_\mu\big(\psi(\gamma),\gamma|_K\big) \\ &= \Phi\big(\psi(\gamma)\big)\,\hat{\mu}(\gamma)\,. \end{aligned}$$

Hence S is a multiplier on $M^a(G)$ corresponding to $\Phi\circ\psi$. This completes the proof.

THEOREM 2.3. *Let G be a LCA group and ψ a nontrivial continuous homomorphism from $\hat{G}$ into R. Suppose $M^a(G)\cap M_s(G)\neq\{0\}$. Let δ be a positive real number and Φ a multiplier on $L^1_{-\delta}(R)$. Then $\Phi\circ\psi$ is also a multiplier on $M^a(G)$ which satisfies* (i)$\sim$(iii) *in Theorem* 2.2.

PROOF. Put $\Phi_*(\gamma)=\Phi(\psi(\gamma))$. First we prove that Φ_* is a multiplier on $M^a(G)$. Let μ be a measure in $M^a(G)$. Let Γ be a σ-compact open subgroup of $\hat{G}$. Then, by Theorem 2.2, we have $\Phi_*|_\Gamma\cdot\hat{\mu}|_\Gamma\in M^a(G/\Gamma^\perp)^\wedge$ and $\|\Phi_*|_\Gamma\cdot\hat{\mu}|_\Gamma\|\leq\|\Phi\|\,\|\hat{\mu}|_\Gamma\|\leq\|\Phi\|\,\|\mu\|$. Hence, by Lemma (D), we have $\Phi_*\hat{\mu}\in M^a(G)^\wedge$ and $\|\Phi_*\hat{\mu}\|\leq\|\Phi\|\,\|\mu\|$. This shows that Φ_* is a multiplier on $M^a(G)$ and $\|S\|\leq\|\Phi\|$, where S is a bounded linear operator on $M^a(G)$ corresponding to Φ_*. By Lemma (E), S maps $M^a(G)\cap L^1(G)$ into itself.

Finally we prove that S maps $M^a(G)\cap M_s(G)$ into itself. Let ν be a measure in $M^a(G)\cap M_s(G)$. Suppose $S(\nu)\notin M_s(G)$. Put $S(\nu)=\zeta+f$, where $\zeta\in M_s(G)$ and $f\in L^1(G)$. Then $f\neq 0$. Since $\hat{f}\in C_0(\hat{G})$, there exists a σ-compact open subgroup F of $\hat{G}$ such that $\hat{f}(\gamma)=0$ for $\gamma\notin F$. Since ν, $\zeta\in M_s(G)$, there exist σ-compact subsets E_ν and E_ζ of G such that $|\nu|(E_\nu^c)=0$, $|\zeta|(E_\zeta^c)=0$ and $m_G(E_\nu\cup E_\zeta)=0$. Hence, by Lemma 4 in [10], there exists a σ-compact open subgroup Γ of $\hat{G}$ such that (a) $\Gamma\supset F$ and (b) $m_G(E_\nu\cup E_\zeta+\Gamma^\perp)=0$. Let π_1 be the natural homomorphism from G onto $G/\Gamma^\perp$. Then, by (b), we have $\pi_1(\nu)$, $\pi_1(\zeta)\in M_s(G/\Gamma^\perp)$. Since $G/\Gamma^\perp$ is a metrizable LCA group, by Theorem 2.2, $\Phi_*|_\Gamma$ is a multiplier on $M^a(G/\Gamma^\perp)$ which satisfies the following:

$$\text{(c)}\qquad S_1\Big(M^a(G/\Gamma^\perp)\cap M_s\big((G/\Gamma^\perp)\big)\Big)\subset M^a(G/\Gamma^\perp)\cap M_s(G/\Gamma^\perp)\,,$$

where S_1 is a bounded linear operator on $M^a(G/\Gamma^\perp)$ corresponding to $\Phi_*|_\Gamma$. Then, since $\pi_1(S(\nu))=S_1(\pi_1(\nu))$, we have $S_1(\pi_1(\nu))=\pi_1(f)+\pi_1(\zeta)$. By (c), $S_1(\pi_1(\nu))$ belongs to $M_s(G/\Gamma^\perp)$. Since $\pi_1(\zeta)\in M_s(G/\Gamma^\perp)$ and $\pi_1(f)$ is a non-zero function in $L^1(G/\Gamma^\perp)$, we have a contradiction. Thus S maps $M^a(G)\cap M_s(G)$ into itself. This completes the proof.

By Theorem 2.3, we obtain the following theorem.

THEOREM 2.4. *Let G be a LCA group and ψ a nontrivial continuous homomorphism from $\hat{G}$ into R. Suppose $M^a(G)\cap M_s(G)\neq\{0\}$. Then there exists a multiplier S on $M^a(G)$ which satisfies the following*:

$$\text{(i)}\qquad S\big(M^a(G)\cap L^1(G)\big)\subset M^a(G)\cap L^1(G)\,,$$

$$\text{(ii)}\qquad S\big(M^a(G)\cap M_s(G)\big)\subset M^a(G)\cap M_s(G)\,,$$

(iii) *S is not given by convolution with a bounded regular measure on G.*

PROOF. Let $\{a_n\}$ be a sequence consisting of positive integers such that $a_{m+1}/a_m>3$ $(m=1, 2, 3, \cdots)$. Put $F=\{a_m\,;\ m=1, 2, 3, \cdots\}$. Let $\Delta(x)$ be a function in $L^1(R)^\wedge$ such that $\Delta(x)=\max(1-3\,|x|, 0)$ and $\phi_0(n)=\chi_F(n)$. Now we define a function Φ on R as follows:

$$\Phi(x)=\sum_{n\in Z}\phi_0(n)\,\Delta(x-n)\,.$$

Let δ be a positive real number. Then Φ is a multiplier on $L^1_{-\delta}(R)$ (see [12]). Hence, by Theorem 2. 3, $\Phi\circ\psi$ is a multiplier on $M^a(G)$ which satisfies (i) and (ii). Let S be a bounded linear operator on $M^a(G)$ corresponding to $\Phi\circ\psi$. We note that $\phi_0|_{Z^+}\notin M(T)^\wedge|_{Z^+}$, where Z^+ is the semigroup consisting of nonnegative integers. Hence we can verify that S is not given by convolution with a bounded regular measure on G. This completes the proof.

REMARK 2. 4. Under the assumption of Theorem 2. 4, we note that there exists a multiplier S' on $M^a(G)$ which satisfies the following:

(i) $S'\big(M^a(G)\cap L^1(G)\big)\subset M^a(G)\cap L^1(G)$,

(ii) $S'\big(M^a(G)\cap M_s(G)\big)\subset M^a(G)\cap M_s(G)$,

(iii) S' is given by convolution with a measure $\xi\in M(G)$ such that $\hat{\xi}|_{P^0}\notin M_d(G)^\wedge|_{P^0}$, where $P=\{\gamma\in\hat{G}\,;\ \psi(\gamma)\geq 0\}$.

Indeed, let f be a nonzero function in $L^1(R)$ such that $\hat{f}(\psi(\gamma_0))\neq 0$ for some $\gamma_0\in\hat{G}$ with $\psi(\gamma_0)>0$. Put $\xi=\phi(f)$, where ϕ is the dual homomorphism of ψ. Then ξ satisfies the above conditions.

REMARK 2. 5. Let ψ be a nontrivial continuous homomorphism from $\hat{G}$ into R. Suppose there is not $\gamma\in\hat{G}$ such that $\psi(\gamma)=1$. Since ψ is nontrivial, there exists $\chi_0\in\hat{G}$ such that $\psi(\chi_0)\geq 1$. We fix it and put $d_o=\psi(\chi_0)$. Let Λ be the discrete subgroup of $\hat{G}$ generated by χ_0 and we put $K=\Lambda^\perp$. We define α: $R\oplus K\mapsto G$ and α_1: $R\oplus\hat{G}\mapsto R\oplus\hat{K}$ as before. Then

$$(2.\ 2)'\qquad (D=)\ \ker(\alpha)=\Big\{\Big(2\pi n/d_o,\ -\phi(2\pi n/d_o)\Big)\,;\ n\in Z\Big\}$$

and Lemma 2. 2 is satisfied. Lemma 2. 3 is held if we replace $(-\pi, \pi]\times K$ by $(-\pi/d_o, \pi/d_o]\times K$. Hence (2. 3) and Proposition 2. 2 are also satisfied. Moreover Lemmas 2. 4 and 2. 5 are held if we exchange $\left[-\frac{1}{2}, \frac{1}{2}\right)\times\hat{G}$ by $\left[-\frac{1}{2}d_o, \frac{1}{2}d_o\right)\times\hat{G}$. Therefore Lemma 2. 6 can be proved as same as before, hence Theorem 2.1 is satisfied. Other theorems in this section are also satisfied.

§ 3 Some property of multipliers on the space of analytic singular measures.

In [4], for a compact abelian group G, Doss proved that a multiplier on $M_s(G)$ is given by convolution with a discrete measure on G. Graham and MacLean obtained an analogous result for a LCA group in [7]. Therefore, in particular, a multiplier on $M_s(G)$ does not belong to $C_0(\hat{G})$. In this section, we shall prove that every multiplier on the space of analytic singular measures does not vanish at infinity. $M_0(G)$ denotes the Banach algebra of all bounded regular measures on G whose Fourier-Stieltjes transforms vanish at infinity.

DEFINITION 3.1. *Let G be a LCA group. Suppose there exists a nontrivial continuous homomorphism ψ from $\hat{G}$ into R. We define $M^a(G)_s$ by $M^a(G)\cap M_s(G)$. A function Φ on $\hat{G}$ which is continuous on $\{\gamma\in\hat{G};\ \psi(\gamma)\geq 0\}^0$ is called a multiplier on $M^a(G)_s$ if $\Phi\hat{\mu}\in M^a(G)_s\hat{}$ for each $\mu\in M^a(G)_s$. Let S be the bounded linear operator on $M^a(G)_s$ such that $S(\mu)\hat{}=\Phi\hat{\mu}$. S is also called a multiplier on $M^a(G)_s$.*

The following lemma is well known.

LEMMA 3.1. *Let G be a LCA group and μ a measure in $M_0(G)$. Let ν be a measure in $M(G)$ such that it is absolutely continuous with respect to $|\mu|$. Then ν belongs to $M_0(G)$.*

Let G be a LCA group and ψ a nontrivial continuous homomorphism from $\hat{G}$ into R. We may assume that there exists an element $\chi_0\in\hat{G}$ such that $\psi(\chi_0)=1$. Let ϕ be the dual homomorphism of ψ. Let Λ be a discrete subgroup of $\hat{G}$ generated by χ_0 and K the annihilator of Λ. Let α and D be as in 2.2.

For $\nu\in M(G)$, Φ_ν is a function on $R\oplus\hat{K}$ such that

$$\Phi_\nu(t,\sigma)=\sum_{\gamma\in\hat{G}}\hat{\nu}(\gamma)\,\Delta^2\Big((t,\sigma)-\big(\psi(\gamma),\gamma|_K\big)\Big)\,.$$

Then, as seen in 2.2, Φ_ν belongs to $M(R\oplus K)\hat{}$. Moreover we remember that Φ_ν belongs to $L^1(R\oplus K)\hat{}$ if $\nu\in L^1(G)$ and Φ_ν belongs to $M_s(R\oplus K)\hat{}$ if $\nu\in M_s(G)$ respectively. The following lemma is obtained from the fact that $\alpha(\check{\Phi}_\nu)=\nu$.

LEMMA 3.2. (cf. Proposition 2.2).

(a) *α maps $L^1(R\oplus K)$ onto $L^1(G)$;*

(b) *α maps $M_s(R\oplus K)$ onto $M_s(G)$.*

The following lemmas are easily obtained.

LEMMA 3.3. *For μ, $\nu\in M(R\oplus K)$, we have $\alpha(\mu*\nu)=\alpha(\mu)*\alpha(\nu)$. In particular, for $\mu_1\in M(R)$ and $\mu_2\in M(K)$, we have*

$$\alpha(\mu_1\times\mu_2)=\alpha(\mu_1\times\delta_0)*\alpha(\delta_0\times\mu_2)=\alpha(\mu_1)*\mu_2\,.$$

LEMMA 3.4. *Let G be a LCA group and ψ a nontrivial continuous homomorphism from $\hat{G}$ into R. Then the following are satisfied.*

(1) $\ker(\psi)$ *is open if and only if*

$$\{\gamma\in\hat{G}\,;\,\psi(\gamma)>0\}^-=\{\gamma\in\hat{G}\,;\,\psi(\gamma)>0\}\,;$$

(2) $\ker(\psi)$ *is not open if and only if*

$$\{\gamma\in\hat{G}\,;\,\psi(\gamma)>0\}^-=\{\gamma\in\hat{G}\,;\,\psi(\gamma)\geq 0\}\,.$$

THEOREM 3.1. *Let G be a LCA group and ψ a nontrivial continuous homomorphism from $\hat{G}$ into R. Suppose $M^a(G)_s\neq\{0\}$. Then, for each nonzero multiplier Φ on $M^a(G)_s$, we have $\Phi|_{P^0}\notin C_0(P^0)$, where $P=\{\gamma\in\hat{G}\,;\,\psi(\gamma)\geq 0\}$.*

PROOF. Case 1. We first consider the case that there exists an element $\gamma_0\in\hat{G}$ with $\psi(\gamma_0)>0$ such that $\Phi(\gamma_0)\neq 0$. Suppose $\Phi|_{P^0}\in C_0(P^0)$. Then there exists a σ-compact open subgroup Γ of $\hat{G}$ such that $\gamma_0\in\Gamma$ and $\Phi(\gamma)=0$ for $\gamma\in P^0\cap(\hat{G}\backslash\Gamma)$. Let $\psi|_\Gamma$ be the restriction of ψ to Γ and put $M^a(G/\Gamma^\perp)_s=\{\mu\in M(G/\Gamma^\perp)_s\,;\,\hat{\mu}(\gamma)=0$ for each $\gamma\in\Gamma$ with $\psi|_\Gamma(\gamma)<0\}$. Then, by Lemmas (B) and (C), we have $M^a(G/\Gamma^\perp)_s\neq\{0\}$ and $\Phi|_\Gamma$ is a nonzero multiplier on $M^a(G/\Gamma^\perp)_s$. Moreover $\Phi|_\Gamma(\gamma_0)\neq 0$ $(\psi|_\Gamma(\gamma_0)>0)$. Since Γ is σ-compact, $G/\Gamma^\perp$ is metrizable. Thus we may assume that G is metrizable without loss of generality. Let f_o be a function in $H^1(R)$ such that $\hat{f}_o(\psi(\gamma_0))=1$. Then, since $\alpha(f_o\times\delta_0)=\phi(f_o)$, $\phi(f_o)$ belong to $M^a(G)_s$. Hence there exists a measure $\nu\in M^a(G)_s$ such that $\hat{\nu}(\gamma)=\Phi(\gamma)\hat{f}_o(\psi(\gamma))$. Since $\hat{\nu}\in C_o(\hat{G})$, we have $\Phi_\nu\in C_o(\widehat{R\oplus K})$. Let π_K be the projection from $R\oplus K$ onto K.

Claim 1. $\pi_K(|\check{\Phi}_\nu|)\in M_c(K)$.

For each $(\theta,\omega)\in\widehat{R\oplus K}$, we have $\pi_K((\theta,\omega)\,\check{\Phi}_\nu)\hat{\ }(\gamma_2)=\Phi_\nu(-\theta,\gamma_2-\omega)$. Hence we have $\pi_K((\theta,\omega)\,\check{\Phi}_\nu)\hat{\ }\in C_o(\hat{K})$, and so $\pi_K((\theta,\omega)\,\check{\Phi}_\nu)\in M_c(K)$. Since there exists a sequence $\{p_n\}$ in $\mathrm{Trig}\,(R\oplus K)$ such that $\lim\|p_n\check{\Phi}_\nu-|\check{\Phi}_\nu|\,\|=0$, we have $\pi_K(|\check{\Phi}_\nu|)\in M_c(K)$. Thus Claim 1 is proved.

Put $\eta=\pi_K(|\check{\Phi}_\nu|)$. Then, by ([7]; Theorems 1 and 2), there exists a measure $\xi\in M_s(K)\cap M^+(K)$ such that $\eta*\xi\neq 0$ and $\eta*\xi\in L^1(K)$. Let σ_0 be an element in $\hat{K}$ such that $(\sigma_0\xi)\hat{\ }(\gamma_0|_K)\neq 0$. We define a measure $\mu\in M(R\oplus K)$ as follows:

$$d\mu(x, y) = f_0(x)\, dx \times d(\sigma_0 \xi)(y) .$$

Then, since $\mu \in M_s(R \oplus K)$, we have $\alpha(\mu) \in M^a(G)_s$. Since G is metrizable, K is so. Hence, by the theory of disintegration, there exists a family $\{\lambda_h\}_{h \in K}$ consisting of measures in $M(R \oplus K)$ satisfying the following:

(1) $h \mapsto \lambda_h(f)$ is a Borel measurable function of h for each bounded Borel measurable function f on $R \oplus K$,

(2) $\operatorname{supp}(\lambda_h) \subset R \times \{h\}$,

(3) $\|\lambda_h\| \leq 1$,

(4) $\check{\Phi}_\nu(g) = \int_K \lambda_h(g)\, d\eta(h)$

for each bounded Borel function g on $R \oplus K$.

From (2), we have $d\lambda_h(x, y) = d\nu_h(x) \times d\delta_h(y)$, where $\nu_h \in M(R)$ and δ_h is the Dirac measure at h. We note the following (cf. [13], Claims 2 and 3 in the proof of Theorem 1):

(5) $$\mu * \check{\Phi}_\nu = \int_K \int_K \left(\left(\sigma_0(h_1) f_o \right) * \nu_{h_2} \right) \times \delta_{(h_1 + h_2)}\, d(\xi \times \eta)(h_1, h_2) .$$

Since $(\sigma_0(h_1) f_0 * \nu_{h_2}) \in L^1(R)$ for each $(h_1, h_2) \in K \times K$ and $\xi * \eta \in L^1(K)$, we can verify that $\mu * \check{\Phi}_\nu \in L^1(R \oplus K)$. Hence, by Lemma 3. 2 and Lemma 3. 3, we have $\alpha(\mu) * \nu \in L^1(G)$. Since $\hat{\mu}(\psi(\gamma_0), \gamma_0|_K) \neq 0$, $\alpha(\mu) * \nu \neq 0$. On the other hand, by using Lemma 3. 3, we have

$$\begin{aligned}
\left(\alpha(\mu) * \nu\right)^\wedge(\gamma) &= \alpha(\mu)^\wedge(\gamma)\, \hat{\nu}(\gamma) \\
&= \alpha(\mu)^\wedge(\gamma)\, \phi(f_0)^\wedge(\gamma)\, \Phi(\gamma) \\
&= \alpha\left(\mu * (f_o \times \delta_0)\right)^\wedge(\gamma)\, \Phi(\gamma) \\
&= \alpha\left((f_o * f_o) \times (\sigma_0 \xi)\right)^\wedge(\gamma)\, \Phi(\gamma) .
\end{aligned}$$

Since $(f_o * f_o) \times (\sigma_0 \xi) \in M_s(R \oplus K)$, we have $\alpha((f_o * f_o) \times (\sigma_0 \xi)) \in M^a(G)_s$. This contradicts the hypothesis that Φ is a multiplier on $M^a(G)_s$. Hence, in this case, we have $\Phi|_{P^0} \notin C_o(P^0)$.

Case 2. Next we consider the case that $\Phi(\gamma) = 0$ for $\gamma \in \hat{G}$ with $\psi(\gamma) > 0$. In this case we prove the theorem by dividing three cases.

Case 2. 1. $\ker(\psi)$ is not open.
In this case, by Lemma 3. 4, Φ is a zero multiplier on $M^a(G)_s$. Hence this contradicts the hypothesis.

Case 2. 2. $\ker(\psi)$ is open and compact.
In this case, Φ is a zero multiplier. Hence we have a contradiction.

Case 2. 3. $\ker(\psi)$ is open and noncompact.
In this case, by Lemmas (B) and (C), $\Phi|_{\ker(\psi)}$ is a nonzero multiplier on $M_s(G/\ker(\psi)^{\perp})$. Hence we have $\Phi|_{P^0}\notin C_0(P^0)$. This completes the proof.

REMARK 3. 1. From the proof of Theorem 3. 1, the following is satisfied:

Let G and ψ be as in Theorem 3. 1. We put ${}_{+}M^a(G)_s=\{\mu\in M_s(G);\ \hat{\mu}(\gamma)=0$ for $\gamma\in\hat{G}$ with $\psi(\gamma)\leq 0\}$. Then, for each nonzero multiplier Φ on ${}_{+}M^a(G)_s$, we have $\Phi|_S\notin C_0(S)$, where $S=\{\gamma\in\hat{G};\ \psi(\gamma)>0\}$.

§ 4 Application.

The F. and M. Riesz theorem was generalized by Helson and Lowdenslarger as follows:

THEOREM A (cf. [11], 8. 2. 3 Theorem).
Let G be a compact abelian group such that $\hat{G}$ is ordered. Suppose $\mu\in M(G)$ is of analytic type (i. e. $\hat{\mu}(\gamma)=0$ for $\gamma<0$). Then the following are satisfied:

(1) μ_a *and* μ_s *are of analytic type*,

(2) $\hat{\mu}_s(0)=0$.

In [3], Doss extended Theorem A for a LCA group with the algebraically ordered dual. Moreover he obtained the following theorem:

THEOREM B ([3], Main Theorem).
Let G be a LCA group such that $\hat{G}$ is algebraically ordered. Let μ be a measure in $M(G)$. Put $\mu=\mu_a+\mu_s$, where $\mu_a\in L^1(G)$ and $\mu_s\in M_s(G)$. Suppose there exists a function f in $L^r(G)$ $(1\leq r\leq 2)$ such that $\hat{\mu}(\gamma)=\hat{f}(\gamma)$ a. e. for $\gamma<0$. Then $\hat{\mu}_s(\gamma)=0$ for $\gamma<0$ and $\hat{\mu}_s(0)=0$.

On the other hand, in [1], DeLeeuw and Glicksberg obtained an analogous result of Theorem A for a compact abelian group such that there exists a nontrivial homomorphism from $\hat{G}$ into R. In this section, we shall prove that an analogous result of Theorem B is satisfied for a LCA group G such that there exists a nontrivial continuous homomorphism from $\hat{G}$ into R. In this section, we use notations appeared in 2. 2.

PROPOSITION 4. 1. *Let G be a LCA group and ψ a nontrivial continuous homomorphism from $\hat{G}$ into R. Let ε be a positive real number such that $0<\varepsilon<\frac{1}{6}$. For $f\in L^r(G)$ $(1\leq r\leq 2)$, we define a function Φ_f on $R\oplus\hat{K}$ as follows*:

$$(*)\qquad \Phi_f(t,\sigma)=\sum_{\gamma\in\hat{G}}\hat{f}(\gamma)\,\Delta_\varepsilon^2\Big((t,\sigma)-\big(\psi(\gamma),\gamma|_K\big)\Big),$$

where K is the closed subgroup of G defined in 2. 2 *and Δ_ε is a function on $R\oplus\hat{K}$ such that $\Delta_\varepsilon(t,\sigma)=\max\left(1-\frac{1}{\varepsilon}|t|,0\right)$ for $\sigma=0$ and $\Delta_\varepsilon(t,\sigma)=0$ for $\sigma\neq 0$. Then we have $\Phi_f\in L^r(R\oplus K)^\wedge$.*

PROOF. Put $\mathscr{A}=L^1(G)\cap L^2(G)$. Then $\mathscr{A}$ is dense in $L^1(G)$, $L^r(G)$ and $L^2(G)$ respectively. We define an operator U from $\mathscr{A}$ into $L^1(R\oplus K)$ as follows:

(1) $\quad U(h)=\check{\Phi}_h$, where Φ_h is a function on $R\oplus\hat{K}$ appeared in 2. 2, i. e. $\Phi_h(t,\sigma)=\sum_{\gamma\in\hat{G}}\hat{h}(\gamma)\,\Delta_\varepsilon^2((t,\sigma)-(\psi(\gamma),\gamma|_K))$.

Then, as seen in 2. 2, we have $||U(h)||_1=||h||_1$. By Lemma 2. 4, Lemma 2. 5, ([8], (5. 27) Theorem) and the definition of Φ_h, we have

(2) $\quad U(h)\in L^2(R\oplus K)$, and

$$\Big\|U(h)\Big\|_2=\Big\|U(h)^\wedge\Big\|_2\leq\sqrt{\varepsilon}\,B||\hat{h}||_2=\sqrt{\varepsilon}\,B||h||_2,$$

where B is a positive constant independent of h. Hence, by the Riesz-Thorin theorem, there exists a positive number A_r such that

(3) $\quad \Big\|U(h)\Big\|_r\leq A_r||h||_r \quad$ for $\quad h\in\mathscr{A}$.

This completes the proof.

LEMMA 4. 1. *Let G be a metrizable LCA group and ψ a nontrivial continuous homomorphism from $\hat{G}$ into R. Let μ be a measure in $M^a(G)$. Then μ_a and μ_s belong to $M^a(G)$, where μ_a and μ_s are absolutely continuous part of μ and the singular part of μ respectively.*

PROOF. Let ϕ be the dual homomorphism of ψ. Let χ_0, Λ and K be as in 2. 2. Let γ_0 be an element in $\hat{G}$ such that $\psi(\gamma_0)<0$. We choose a positive real number ε so that $\varepsilon<\min\left(\frac{1}{6},\left|\psi(\gamma_0)\right|\right)$. We note that $\Phi_\mu(t,\sigma)=0$ for $t\leq-\varepsilon$. Since G is metrizable, K is so. Let π be the projection from $R\oplus K$ onto K. Put $\eta=\pi(|\check{\Phi}_\mu|)$. Then, by the theory of disintegration, there exists a family $\{\lambda_h\}_{h\in K}$ consisting of measures in $M(R\oplus K)$ with the following properties:

(1) $\quad h\mapsto\lambda_h(f)$ is a Borel measurable function for each bounded Borel measurable function f on $R\oplus K$,

(2) $\quad \operatorname{supp}(\lambda_h)\subset R\times\{h\}$,

(3) $\quad ||\lambda_h||\leq 1$,

$$(4) \qquad \check{\Phi}_\mu(g) = \int_K \lambda_h(g)\, d\eta(h)$$

for each bounded Borel function g on $R \oplus K$.

By (2), there exists a measure $\nu_h \in M(R)$ such that $d\lambda_h(x, y) = d\nu_h(x) \times d\delta_h(y)$, where δ_h is the Dirac measure at h. Since $\Phi_\mu(t, \sigma) = 0$ for $t \leq -\varepsilon$, we obtain the following (see the proof of Lemma 3 in [13]):

$$(5) \qquad \hat{\nu}_h(t) = 0 \quad \text{on} \quad (-\infty, -\varepsilon] \qquad \text{a. a. } h\ (\eta)\,.$$

Hence, by the F. and M. Riesz theorem, we have

$$(6) \qquad \nu_h \in L^1(R) \qquad \text{a. a. } h\ (\eta)\,.$$

Put $\eta = \eta_a + \eta_s$, where $\eta_a \in L^1(K)$ and $\eta_s \in M_s(K)$. We define measures $\xi_1, \xi_2 \in M(R \oplus K)$ as follows:

$$(7) \qquad \xi_1(g) = \int_K \lambda_h(g)\, d\eta_a(h)\,,$$

$$\xi_2(g) = \int_K \lambda_h(g)\, d\eta_s(h) \qquad \text{for} \quad g \in C_0(R \oplus K)\,.$$

Then $\check{\Phi}_\mu = \xi_1 + \xi_2$ and we can easily verify that $\xi_2 \in M_s(R \oplus K)$. Next we show that ξ_1 belongs to $L^1(R \oplus K)$. Let E be a Borel measurable set in $R \oplus K$ with $m_{R \oplus K}(E) = 0$. Then there exists a Borel measurable set E_2 in K with $m_k(E_2) = 0$ and $m_R(E_y) = 0$ for $y \notin E_2$, where $E_y = \{x \in R\,;\ (x, y) \in E\}$. By (6), there exists a Borel measurable set F in K such that $\eta(F^c) = 0$ and $\nu_h \in L^1(R)$ for $h \in F$. Hence we have

$$\begin{aligned} \xi_1(E) &= \int_K \lambda_h(\chi_E)\, d\eta_a(h) \\ &= \int_F \{\nu_h \times \delta_h\}\ (\chi_E)\, d\eta_a(h) \\ &= \int_{F \setminus E_2} \nu_h(E_h)\, d\eta_a(h) + \int_{F \cap E_2} \nu_h(E_h)\, d\eta_a(h) \\ &= 0\,, \end{aligned}$$

where χ_E is the characteristic function of E. Hence ξ_1 belongs to $L^1(R \oplus K)$. Evidently $\check{\Phi}_\mu = \check{\Phi}_{\mu_a} + \check{\Phi}_{\mu_s}$. By Theorem 2. 1, we have $\check{\Phi}_{\mu_a} \in L^1(R \oplus K)$ and $\check{\Phi}_{\mu_s} \in M_s(R \oplus K)$. Hence we have $\check{\Phi}_{\mu_a} = \xi_1$ and $\check{\Phi}_{\mu_s} = \xi_2$. By (5), we can verify that $\hat{\xi}_2(t, \sigma) = 0$ for $t \leq -\varepsilon$. Hence we have $\Phi_{\mu_s}(t, \sigma) = 0$ for $t \leq -\varepsilon$. On the other hand, since $\psi(\gamma_0) < -\varepsilon$, we have $\hat{\mu}_s(\gamma_0) = \Phi_{\mu_s}(\psi(\gamma_0), \gamma_0|_K) = 0$. Hence $\mu_s \in M^a(G)$, and so $\mu_a \in M^a(G)$. This completes the proof.

THEOREM 4. 1. *Let G be a LCA group and ψ a nontrivial continuous homomorphism from $\hat{G}$ into R. Let μ be a measure in $M^a(G)$. Then μ_a and μ_s belong to $M^a(G)$.*

PROOF. It is sufficient to prove that $\mu_s \in M^a(G)$. Suppose there exists an element $\gamma_0 \in \hat{G}$ with $\psi(\gamma_0)<0$ such that $\hat{\mu}_s(\gamma_0)\neq 0$. Since $\mu_s \in M_s(G)$, there exists a σ-compact subset S of G such that $|\mu_s|(S^c)=0$ and $m_G(S)=0$. Hence, by ([10], Lemma 4), there exists a σ-compact, noncompact open subgroup Γ of $\hat{G}$ which contains γ_0 such that (*) $m_G(S+\Gamma^{\perp})=0$. Let π be the natural homomorphism from G onto $G/\Gamma^{\perp}$. Put $G_1=G/\Gamma^{\perp}$. Then, by (*), we have $\pi(\mu_s)\in M_s(G_1)$. Evidently, $\pi(\mu_a)$ belongs to $L^1(G_1)$. We put $M^a(G_1)=\{\nu\in M(G_1)\,;\ \hat{\nu}(\gamma)=0$ for $\gamma\in\Gamma$ with $\psi|_\Gamma(\gamma)<0\}$. Then $\pi(\mu)$ belongs to $M^a(G_1)$. Since Γ is σ-compact, G_1 is metrizable. Hence, by Lemma 4.1, we have $\pi(\mu_s)\in M^a(G_1)$. That is, $\hat{\mu}_s(\gamma_0)=\pi(\mu_s)\hat{\ }(\gamma_0)=0$. This contradicts the choice of γ_0. Hence we have $\hat{\mu}_s(\gamma)=0$ for $\gamma\in\hat{G}$ with $\psi(\gamma)<0$. This completes the proof.

Next we prove that Theorem B is satisfied for a LCA group G such that there exists a nontrivial continuous homomorphism ψ from $\hat{G}$ into R. Let $M^a(R\oplus K)$ and $M^a(T\oplus K)$ be the spaces defined in 2. 1. Let ψ be the projection from $R\oplus\hat{K}$ onto R. Then, from Theorem 4. 1, we obtain the following lemma.

LEMMA 4. 2. *Let K be a LCA group. Let μ be a measure in $M^a(R\oplus K)$. Then μ_a and μ_s belong to $M^a(R\oplus K)$.*

The following lemma can be proved by the same method used in ([3], Lemma 2).

LEMMA 4. 3. *Let K be a LCA group and $P=\{(x,\theta)\in R\oplus\hat{K}\,;\ x\geq 0\}$. Let σ be a positive measure in $M(R\oplus K)$. Put $d\sigma(x)=ds(x)+w(x)\,dx$, where $s\in M_s(R\oplus K)$ and $w\in L^1(R\oplus K)$. Let M be a compact subset of $\widehat{R\oplus K}$ and $\Omega=\{p(x)\in \mathrm{Trig}\,(R\oplus K)\,;\ p(x)=\sum a_\gamma(x,\gamma), \gamma\notin P\cup M\}$. Let ϕ be the unique function in the closure of Ω in $L^2(d\sigma)$ such that*

$$\int_{R\oplus K}|1-\phi|^2 d\sigma=\inf_{p\in\Omega}\int_{R\oplus K}|1-p|^2 d\sigma\,.$$

Then we have

$$\int_{R\oplus K}|1-\phi|^2 d\sigma\leq\int_{R\oplus K} w(x)\,dx\,.$$

By Lemma 4. 3, we can prove the following lemma as in the same way as in ([3], Main Theorem).

LEMMA 4. 4. *Let K be a LCA group. Let $\mu\in M(R\oplus K)$ and $f\in L^r(R\oplus K)$ $(1\leq r\leq 2)$. If $\hat{\mu}(x,\sigma)=\hat{f}(x,\sigma)$ a. e. on $\{(x,\sigma)\in R\oplus\hat{K}\,;\ x<0\}$, we have $\mu_s\in M^a(R\oplus K)$, where μ_s is the singular part of μ.*

LEMMA 4.5. *Let K be a LCA group and $P=\{(n,\sigma)\in Z\oplus\hat{K}\,;\ n\geq 0\}$. Let $\mu\in M(T\oplus K)$ and $f\in L^r(T\oplus K)$ $(1\leq r\leq 2)$. If $\hat{\mu}(n,\sigma)=\hat{f}(n,\sigma)$ a.e. on P^c, we have $\mu_s\in M^a(T\oplus K)$.*

PROOF. Let $\Delta(t,\sigma)$ be a function on $R\oplus\hat{K}$ such that $\Delta(t,\sigma)=\max(1-3|t|,0)$ for $\sigma=0$ and $\Delta(t,\sigma)=0$ for $\sigma\neq 0$. For $\nu\in M(T\oplus K)$ and $g\in L^r(T\oplus K)$, we define functions $\Psi_\nu(t,\sigma)$ and $\Psi_g(t,\sigma)$ on $R\oplus\hat{K}$ as follows:

$$\Psi_\nu(t,\sigma)=\sum_{(n,\tau)\in Z\oplus\hat{K}}\hat{\nu}(n,\tau)\,\Delta^2\Big((t,\sigma)-(n,\tau)\Big),$$

$$\Psi_g(t,\sigma)=\sum_{(n,\tau)\in Z\oplus\hat{K}}\hat{g}(n,\tau)\,\Delta^2\Big((t,\sigma)-(n,\tau)\Big).$$

Then we can verify that $\Psi_\nu\in M(R\oplus K)^\wedge$ and $\Psi_g\in L^r(R\oplus K)^\wedge$. We define $\xi\in M(R\oplus K)$ and $h\in L^r(R\oplus K)$ as follows:

$$\hat{\xi}(t,\sigma)=\Psi_\mu\Big(t-\frac{1}{3},\sigma\Big),\qquad \hat{h}(t,\sigma)=\Psi_f\Big(t-\frac{1}{3},\sigma\Big).$$

Then, since $\hat{\mu}(n,\sigma)=\hat{f}(n,\sigma)$ a.e. on P^c, we have

$$\hat{\xi}(t,\sigma)=h(t,\sigma)\qquad \text{a.e. on } \Big\{(t,\sigma)\in R\oplus\hat{K}\,;\ t\leq 0\Big\}$$

Hence, by Lemma 4.4, we have $\hat{\xi}_s(t,\sigma)=0$ for $t\leq 0$, where ξ_s is the singular part of ξ. Hence we have $\hat{\mu}_s(n,\sigma)=0$ for $n\leq -1$, i.e., $\mu_s\in M^a(T\oplus K)$. This completes the proof.

THEOREM 4.2. *Let G be a LCA group and ψ a nontrivial continuous homomorphism from $\hat{G}$ into R. Put $P=\{\gamma\in\hat{G}\,;\ \psi(\gamma)\geq 0\}$. Let $\mu\in M(G)$ and $f\in L^r(G)$ $(1\leq r\leq 2)$.* If $\hat{\mu}(\gamma)=\hat{f}(\gamma)$ *a.e. on P^c, we have $\mu_s\in M^a(G)$, where μ_s is the singular part of μ.*

PROOF. Let Λ, K and α be as in 2.2. We first consider the case that $\psi(\hat{G})$ is not dense in R with respect to the usual topology. In this case, $\psi(\hat{G})$ is isomorphic to Z and $\hat{G}=Z\oplus\ker(\psi)$. Hence, by Lemma 4.5, we obtain $\mu_s\in M^a(G)$. Next we consider the case that $\psi(\hat{G})$ is dense in R with respect to the usual topology. Let γ_0 be an element in $\hat{G}$ such that $\psi(\gamma_0)<0$. It is sufficient to prove that $\hat{\mu}_s(\gamma_0)=0$. Since $\psi(\hat{G})$ is dense in R, there exist $\gamma_1,\gamma_2\in\hat{G}$ such that $\psi(\gamma_0)<\psi(\gamma_1)<\psi(\gamma_2)<0$. Put $\delta=|\psi(\gamma_2)|$. We define $\mu_1\in M(G)$ and $f_1\in L^r(G)$ by $\mu_1=(-\gamma_1)\mu$ and $f_1=(-\gamma_1)f$ respectively. Then we have $\hat{\mu}_1(\gamma)=\hat{f}_1(\gamma)$ a.e. on $\{\gamma\in\hat{G}\,;\ \psi(\gamma)<\delta\}$. We choose a positive number ε so that $\varepsilon<\min\Big(\frac{1}{6},\delta\Big)$. Let $\Delta_\varepsilon(t,\sigma)$ be the function on $\widehat{R\oplus K}$ defined in Proposition 4.1. We define functions $\Phi_{\mu_1}(t,\sigma)$ and $\Phi_{f_1}(t,\sigma)$ on $\widehat{R\oplus K}$ as follows:

$$\Phi_{\mu_1}(t, \sigma) = \sum_{\gamma\in\hat{G}} \hat{\mu}_1(\gamma)\, \Delta_\epsilon^2\Big((t, \sigma) - \big(\psi(\gamma), \gamma|_K\big)\Big),$$

$$\Phi_{f_1}(t, \sigma) = \sum_{\gamma\in\hat{G}} \hat{f}_1(\gamma)\, \Delta_\epsilon^2\Big((t, \sigma) - \big(\psi(\gamma), \gamma|_K\big)\Big).$$

Then, by Theorem 2.1 and Proposition 4.1, we have $\Phi_{\mu_1}\in M(R\oplus K)^\wedge$ and $\Phi_{f_1}\in L^r(R\oplus K)^\wedge$. Since $\hat{\mu}_1(\gamma)=\hat{f}_1(\gamma)$ a.e. on $\{\gamma\in\hat{G}\,;\ \psi(\gamma)<\delta\}$, we can verify that $\Phi_{\mu_1}(t, \sigma)=\Phi_{f_1}(t, \sigma)$ a.e. on $\{(t, \sigma)\in R\oplus K\,;\ t<0\}$. We note that $\check{\Phi}_{(\mu_1)_s}=(\check{\Phi}_{\mu_1})_s$, where $(\mu_1)_s$ and $(\check{\Phi}_{\mu_1})_s$ are singular parts of μ_1 and $\check{\Phi}_{\mu_1}$ respectively. Then, by Lemma 4.4, we have $\Phi_{(\mu_1)_s}(t, \sigma)=0$ for $t<0$. Hence $(\mu_1)_s^\wedge(\gamma)=\Phi_{(\mu_1)_s}(\psi(\gamma), \gamma|_K)=0$ if $\psi(\gamma)<0$. Thus we have $\hat{\mu}_s(\gamma_0)=\hat{\mu}_s(\gamma_0-\gamma_1+\gamma_1)=(\mu_1)_s^\wedge(\gamma_0-\gamma_1)=0$, because $\psi(\gamma_0-\gamma_1)<0$. This completes the proof.

§ 5 Appendix.

In this section we prove that Theorem 3.1 is satisfied for a LCA group with the algebraically ordered dual.

DEFINITION 5.1. *Let Γ be a LCA group. Γ is called an algebraically ordered group if and only if there exists a semigroup P in Γ with the (AO)-condition, namely* (i) $P\cup(-P)=\Gamma$ *and* (ii) $P\cap(-P)=\{0\}$. *We do not assume the closedness of P.*

DEFINITION 5.2. *Let G be a LCA group and E a subset of $\hat{G}$. We denote $M_E(G)_s$ by $M_E(G)\cap M_s(G)$. If P is a semigroup in $\hat{G}$ with the (AO)-condition, we denote especially $M_P^a(G)_s$ by $M_P(G)_s$. A function Φ on $\hat{G}$ which is continuous on E^o is called a multiplier on $M_E(G)_s$ if $\Phi\hat{\mu}\in M_E(G)_s^\wedge$ for each $\mu\in M_E(G)_s$. Let S be the bounded linear operator on $M_E(G)_s$ corresponding to Φ. S is also called a multiplier on $M_E(G)_s$.*

LEMMA 5.1. *Let G be a LCA group and E a subset of $\hat{G}$. Let Φ be a nonzero multiplier on $M_E(G)_s$. Suppose there exist a noncompact open subgroup F of $\hat{G}$ and $\gamma_0\in\hat{G}$ such that $\gamma_0+F\subset E^o$ and $\Phi|_{\gamma_0+F}\neq 0$. Then we have $\Phi|_{\gamma_0+F}\notin C_o(\gamma_0+F)$.*

PROOF. We may assume that $\gamma_0=0$ without loss of generality. By Lemmas (B) and (C), $\Phi|_F$ is a nonzero multiplier on $M_s(G/F^\perp)$. Hence, by ([7]; Theorems 1 and 2), we have $\Phi|_F\in M_d(G/F^\perp)^\wedge$. Therefore $\Phi|_F\notin C_o(F)$ and the proof is complete.

THEOREM 5.1. *Let G be a LCA group such that $\hat{G}$ is algebraically ordered. Let P be a semigroup in $\hat{G}$ with the (AO)-condition such that it is not dense in $\hat{G}$. Suppose $M_P^a(G)_s\neq\{0\}$. Then, for each nonzero multiplier Φ on $M_P^a(G)_s$, we have $\Phi|_{P^0}\notin C_o(P^0)$.*

PROOF. Put $\Lambda = P^- \cap (-P)^-$ and $A = \{\gamma \in P^- \backslash \Lambda ; \Phi(\gamma) \neq 0\}$. By Lemmas (B) and (C), we may assume that $\hat{G}$ coincides with the group generated by Λ and A. Let π be the natural homomorphism from $\hat{G}$ onto $\hat{G}/\Lambda$ and put $\tilde{P} = \pi(P^-)$. Then $\tilde{P}$ is a closed semigroup in $\hat{G}/\Lambda$ with the (AO)-condition (cf. [9], Lemma 3). Hence, by ([11], 8.1.5 Theorem), $\hat{G}/\Lambda \cong R \oplus D$ or $\hat{G}/\Lambda \cong D$, where D is a discrete ordered group. We prove the theorem by dividing several cases.

Case 1. $\hat{G}/\Lambda \cong D$ and $\pi(P^-)$ induces an archimedean order on D.
In this case, there exists an order preserving isomorphism ψ^* from $\hat{G}/\Lambda$ into R. We put $\psi = \psi^* \circ \pi$. Let μ be a measure in $M_P^a(G)_s$. Then $\hat{\mu}$ vanishes on Λ because $P^c \cap \Lambda$ is dense in Λ. Hence we have $M_P^a(G)_s = {}_+M^a(G)_s\, (= \{\mu \in M_s(G) ; \hat{\mu}(\gamma) = 0$ for $\gamma \in \hat{G}$ with $\psi(\gamma) \leq 0\})$. Hence, by Remark 3.1, we have $\Phi|_{P^0} \notin C_0(P^0)$.

Caes 2. $\hat{G}/\Lambda \cong D$ and $\pi(P^-)$ induces a nonarchimedean order on D.

Claim: There exist $\gamma_1 \in P^- \backslash \Lambda$ and $\gamma_0 \in A$ such that $n\pi(\gamma_1) <_{\tilde{P}} \pi(\gamma_0)$ for all $n \in N$.

Let $\pi(\gamma_2)$ be an element in $\pi(A)$ and $S = \{\gamma \in P^- \backslash \Lambda ; \pi(\gamma) <_{\tilde{P}} \pi(\gamma_2)\}$.

Case 2. (a). We suppose that there exists $\gamma' \in S$ such that $n\pi(\gamma') <_{\tilde{P}} \pi(\gamma_2)$ for all $n \in N$.
In this case, Claim is easily obtained.

Case 2. (b). For each $\gamma \in S$, suppose that there exists a positive integer n_γ such that $n_\gamma \pi(\gamma) >_{\tilde{P}} \pi(\gamma_2)$.
Let F_* be an open subgroup of $\hat{G}$ generated by S, Λ and γ_2. Then $\pi(F_* \cap P^-)$ induces an archimedean order on $\pi(F_*)$. Thus, by the hypothesis of Case 2, we have $F_* \subsetneqq \hat{G}$. Since $\hat{G}$ is generated by Λ and A, there exists $\gamma_* \in A \backslash F_*$. Then we have $n\pi(\gamma_2) <_{\tilde{P}} \pi(\gamma_*)$ for all $n \in N$. Thus, in this case, Claim is proved. Therefore, in each case, Claim is obtained.
Let $\tilde{F}$ be the subgroup of $\hat{G}/\Lambda$ generated by $\pi(\gamma_1)$ and put $F = \pi^{-1}(\tilde{F})$. Then, by Claim, F is a noncompact open subgroup of $\hat{G}$ such that $\gamma_0 + F \subset P^0$. Hence, by Lemma 5.1, we have $\Phi|_{P^0} \notin C_0(P^0)$.

Case 3. $\hat{G}/\Lambda \cong R$.
In this case, we have $\Phi|_{P^0} \notin C_0(P^0)$ by Theorem 3.1 and Remark 3.1.

Case 4. $\hat{G}/\Lambda \cong R \oplus D$ and D is nontrivial.
Let $F = \pi^{-1}(R)$. Then F is a noncompact open subgroup of $\hat{G}$.

Case 4.1. Suppose $\Phi(\gamma) = 0$ on $P \cap F^c$.
In this case, $\Phi|_F$ is a nonzero multiplier on $M^a(G/F^\perp)_s$ or ${}_+M^a(G/F^\perp)_s$. Hence, by Theprem 3.1 and Remark 3.1, we have $\Phi|_{P^0} \notin C_0(P^0)$.

Case 4.2. Suppose there ecists $\gamma_0 \in \hat{G}$ such that $\pi(\gamma_0) \notin R$, $\pi(\gamma_0) >_{\tilde{P}} 0$ and $\Phi(\gamma_0) \neq 0$.

In this case, γ_0+F is included in P^0. Hence, by Lemma 5.1, we have $\Phi|_{P^0}\notin C_0(P^0)$. This completes the proof.

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Department of Mathematics
Josai University