Hokkaido Mathematical Journal Vol. 11 (1982) p. 295-300

An application of Evens' norm mapping

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(Received February 1, 1982; Revised February 12, 1982)

1. Introduction

Let B_0 be the principal block of kG, where k is the prime field of characteristic $p>0$ and G is a finite group such that $G_p\neq 1$. G_p means a Sylow p-subgroup of G. All modules are finite dimensional vector spaces over k.

If a simple kG-module M does not belong to B_0, then $\bigoplus_{i=1}^{\infty}H^i(G, M)=0$. Therefore, if $\bigoplus_{i=1}^{\infty}H^i(G, M)\neq 0$ is proved for any simple kG-module M lying in B_0, then B_0 is written as $\{M|M$ represents an isomorphic class of simple kG-modules such that $\bigoplus_{i=1}^{\infty}H^i(G, M)\neq 0.\}$ (cf. Barnes, Schmid and Stammbach [1, § 3, Remark]). This characterization of B_0 is known, only when G is a p-nilpotent group (classical), a p-solvable group with an abelian Sylow p-subgroup [3, Theorem 2] or a metabelian group [3, Theorem 3].

The aim of this note is to prove the following Theorem 1 which generalizes [3, Theorem 2], by using Evens' norm mapping [2]. Specifically we show that B_0 is written as above, when G is a Frobenius group whose Frobenius kernel has the order divisible by p.

THEOREM 1. *Let G be a finite group with a normal p-subgroup D. Suppose M be a projective $k[G/D]$-module. We regard M as a kG-module. If $M^*=\mathrm{Hom}_k(M, k)$ is isomorphic to a kG-submodule of $S^i(\Omega_1(A)^*)$ for some normal abelian subgroup A of G such that $A\leq D$, then $H^{2qi}(G, M)\neq 0$. Here $\Omega_1(A)=\langle x\in A|x^p=1\rangle$, $q=|D:A|$ and $S=\bigoplus_{i=0}^{\infty}S^i$ is the symmetric algebra functor over k.*

[3, Theorem 2] is deduced from the case of $D=A$. Next we specialize to a Frobenius group and have the following.

THEOREM 2. *Let G be a Frobenius group with the Frobenius kernel N such that $N_p\neq 1$. Then $\bigoplus_{i=1}^{|H|}H^{2qi}(G, M)\neq 0$ for every simple kG-module M lying in B_0, where $q=|N_p:Z(N_p)|$, $Z(N_p)$ is the center of N_p and H is a Frobenius complement of G. Namely B_0 is described as the set $\{M|M$ represents an isomorphic class of simple kG-modules such that $\bigoplus_{i=1}^{\infty}H^i(G, M)\neq 0\}$.*

All proofs are given in § 2. In § 3 we actually compute the cohomology of G with coefficients in the simple kG-modules lying in B_0, when $p=2$ and $G=S_4$ or A_4 (the symmetric group or the alternating group of degree four).

In the first draft of this note, Theorem 1 treated only a finite group of p-length 1. The author expresses his hearty thanks to Dr. Sasaki who recommended him the improvement of Theorem 1 standing on Sasaki [4, § 2, Step 2].

2. Proofs

Before proving Theorem 1, we recall Evens' norm mapping in the cohomology rings of finite groups [2]. The Norm is a multiplicative analogue of the usual transfer.

For a finite group G, put $a(G)=\bigoplus_{i=0}^{\infty} H^{2i}(G, k)$, where k is viewed as a trivial G-module. Then a is a contravariant functor

Finite groups$\rightarrow$Commutative graded k-algebras.

Let $G \geqq L \geqq H$, $G \geqq K$, $g \in G$, α, $\beta \in a(H)$, $\gamma \in a(G)$ and $n=|G:H|$. The inclusion map $H \rightarrow G$ defines $\mathrm{Res}_{G\rightarrow H}$: $a(G) \rightarrow a(H): \gamma \mapsto \gamma_H$, and a conjugation map $gHg^{-1} \rightarrow H$: $h \mapsto g^{-1}hg$ induces $a(H) \rightarrow a(gHg^{-1}): \alpha \mapsto \alpha^g$. If $g \in H$, then $\alpha^g=\alpha$. Evens constructed $\mathrm{Norm}_{H\rightarrow G}$: $a(H) \rightarrow a(G): \alpha \mapsto {}^G\alpha$, which satisfies the following properties:

$$\begin{cases} \text{If } \alpha \in H^0(H, k)=k, \text{ then } {}^G\alpha=\alpha^n \in H^0(G, k)=k\,. \\ \text{If } \alpha \in H^{2i}(H, k), \text{ then } {}^G\alpha \in H^{2ni}(G, k)\,. \\ {}^G(\alpha \cdot \beta)={}^G\alpha \cdot {}^G\beta\,. \\ {}^H\alpha=\alpha, \quad {}^G({}^L\alpha)={}^G\alpha\,. \\ {}^{L^g}(\alpha^g)=({}^L\alpha)^g\,. \\ ({}^G\alpha)_K=\prod_{g \in K\backslash G/H} {}^K(\alpha^g{}_{H^g \cap K}) \text{ (the double coset formula),} \\ \text{where } H^g=gHg^{-1}\,. \end{cases}$$

Here $K\backslash G/H$ is a transversal of the (K, H) double coset in G.

Proof of Theorem 1. First we prove two Claims, and then combine them.

Claim 1 (Sasaki [4, § 2, Step 2]).

$$H^i(G, M) \cong H^i(D, M)^{G/D} \qquad (i \geqq 0)\,.$$

Proof. Consider the spectral sequence

$$H^s\big(G/D, H^t(D, M)\big) \underset{s}{\Rightarrow} H^{s+t}(G, M)$$

associated with the group extension

$$1 \longrightarrow D \longrightarrow G \longrightarrow G/D \longrightarrow 1 \text{ (exact)}.$$

Since M is a projective $k[G/D]$-module, $H^t(D, M)$ is also a projective $k[G/D]$-module ($t \geqq 0$). Thus $H^s(G/D, H^t(D, M)) = 0$ for each $s > 0$, and we have $H^i(G, M) \cong H^i(D, M)^{G/D}$ for all $i \geqq 0$.

Claim 2. $H^{2qi}(D, k)$ has a $k[G/D]$-submodule M_1 isomorphic to M^*.

Proof. A is an abelian p-group, and so we will regard a symmetric algebra $\bigoplus_{j=0}^{\infty} S^j(\Omega_1(A)^*)$ as a graded k-subalgebra of $\bigoplus_{j=0}^{\infty} H^{2j}(A, k)$ (see for example [3, Proposition 1]). Observe that this inclusion is compatible with G-action. By the assumption there is a submodule of $S^i(\Omega_1(A)^*)$ isomorphic to M^*. We denote this by M_2, which is also a submodule of $H^{2i}(A, k)$.

Let $M_3 = \langle \mathrm{Norm}_{A \to D}(M_2) \rangle$ be the kG-submodule of $H^{2qi}(D, k)$ generated by $\mathrm{Norm}_{A \to D}(M_2)$. The double coset formula shows that

$$\mathrm{Res}_{D \to A}(M_3) = \left\langle \mathrm{Res}_{D \to A}\left(\mathrm{Norm}_{A \to D}(M_2)\right)\right\rangle = \langle \textstyle\prod_{d \in D/A} \alpha^d \mid \alpha \in M_2 \rangle .$$

Since D acts on M_2 trivially, $\mathrm{Res}_{D \to A}(M_3) = \langle \alpha^q \mid \in M_2 \rangle$. In addition the inclusion $M_2 \leqq \bigoplus_{i=0}^{\infty} S^i(\Omega_1(A)^*)$ implies that $\langle \alpha^q \mid \alpha \in M_2 \rangle \cong M_2$ as kG-modules. By $M_2 \cong M^*$, M^* is a homomorphic image of a $k[G/D]$-module M_3.

Now that M^* is a projective $k[G/D]$-module, M^* is isomorphic to a submodule of M_3. This is the required $k[G/D]$-submodule M_1 of $H^{2qi}(D, k)$.

Finally we combine Claims 1 and 2. There is an isomorphism of $k[G/D]$-modules

$$H^j(D, k) \otimes_k M \cong H^j(D, M)$$

for all $j \geqq 0$. Thus we have

$$\begin{aligned} H^{2qi}(G, M) &\cong H^{2qi}(D, M)^{G/D} \\ &\cong \left\{ H^{2qi}(D, k) \otimes_k M \right\}^{G/D} \\ &\geqq (M_1 \otimes_k M)^{G/D} \\ &\cong \mathrm{Hom}_{k[G/D]}(M, M) \neq 0 . \end{aligned}$$

Hence the proof of Theorem 1 is complete.

PROOF of Theorem 2. We apply Theorem 1 to the case of the Frobenius group, putting $D = N_p$ and $A = Z(N_p)$. Since N is the greatest normal p-nilpotent subgroup of G, N acts trivially on every simple kG-module M lying in B_0. Thus M can be regarded as a projective kH-module, for $G/N \cong H$ and H is a p'-group. Next $\Omega_1(A)^*$ is an H-faithful kH-module, and so the regular kH-module kH is isomorphic to a direct summand of $\bigoplus_{i=1}^{|H|} S^i(\Omega_1(A)^*)$ by [3, Theorem 1]. Specially M^* is isomorphic to a submodule of $\bigoplus_{i=1}^{|H|} S^i(\Omega_1(A)^*)$. Now the application of Theorem 1 completes the proof of Theorem 2.

REMARK. If $p=2$, the Norm can be defined by putting $a(G)=\bigoplus_{i=0}^{\infty} H^i(G, k)$. Thus $H^{2qi}(G, M)$ can be replaced by $H^{qi}(G, M)$ in Theorems 1 and 2, if A or N_p is an elementary abelian 2-group (see [3, § 3, Remark]).

3. Examples.

Setting $p=2$, we compute the cohomology of $G=A_4$ or S_4 with coefficients in the simple kG-modules. We will identify modules with their isomorphism class, and we write $M\cong M'$, $M\oplus M'$, $M\oplus\cdots\oplus M$ (i times) and $M\otimes_k M'$ as $M=M'$, $M+M'$, iM and $M\cdot M'$ respectively. Our results are stated in Table 1 below. We denote the first line in Table 1 by row 1, and so forth. Then row 1 shows that j runs through $3i$, $3i+1$ and $3i+2$ for each $i\geqq 0$.

Table 1.

j	$3i$	$3i+1$	$3i+2$
$S^j(U)$	$(i+1)k+iU$	$ik+(i+1)U$	$(i+1)k+(i+1)U$
$\dim_k H^j(A_4, k)$	$i+1$	i	$i+1$
$\dim_k H^j(A_4, U)$	$2i$	$2(i+1)$	$2(i+1)$
$[S^j(W)]$	$(i+1)[k]+i[W]$	$i[k]+(i+1)[W]$	$(i+1)[k]+(i+1)[W]$
$\dim_k H^j(S_4, W)$	i	$i+1$	$i+1$

(1) $G=A_4$.

G is the semi-direct product of $D=\langle(13)(24), (14)(23)\rangle$ by $C=\langle(123)\rangle$. kG has only one block and there are two simple kG-modules k and U. U is isomorphic to D with G-action given by inner automorphisms and $U^*=U$. k and U are also simple kC-modules. Then $kC=k+U$. Multiplying this equation by U, we have

$$\begin{aligned}(kC)\cdot U &= U+(U\cdot U)\\ &=2(kC)\\ &=2k+2U\,.\end{aligned}$$

Thus $U\cdot U=2k+U$.

Row 2 describes the decomposition of $S^j(U)$ to a direct sum of simple kC-modules. The proof is given by considering $S^j(U)\otimes_k F$, where F is the field of four elements.

Rows 3 and 4 give $\dim_k H^j(G, k)$ and $\dim_k H^j(G, U)$, which follow from row 2. For $\dim_k H^j(G, k)$ is the multiplicity of k in $S^j(U)$, and $\dim_k H^j(G, U)$ is twice the multiplicity of U in $S^i(U)$. Observe that $D^* = U$, $H^j(D, k) \cong S^j(U)$ and $U \cdot U = 2k + U$ and then use Claim 1.

(2) $G = S_4$.

G is the semi-direct product of $D = \langle (13)(24), (14)(23) \rangle$ by $H = \langle (123), (12) \rangle$. kG has a unique block and the simple kG-modules are given by letting D act on the simple kH-modules k and W. W is a projective kH-module defined by the reduction of an integral representation of H. We denote those kG-modules by k and W again. W is isomorphic to a kG-module D. Immediately $H^1(G, k) = \mathrm{Hom}(G, k) = \mathrm{Hom}(G/A_4, k) = k$. We shall compute $\dim_k H^j(G, W)$.

Table 2 provides the multiplicity of indecomposable kH-modules in their tensor product representations. Let V be a permutation representation $k[H/K]$, where $K = \langle (123) \rangle$. Then V is the projective cover of k and indecomposable kH-modules are given by k, V and W.

Table 2.

	k	V	W
k	k	V	W
V		$2V$	$2W$
W			$V+W$

Table 2 is proved as follows. Since W does not belong to $B_0(kH)$, we have $V^* = V$ and $W^* = W$. The trivial K-action on V implies that the K-action on $V \cdot V$ is also trivial. Thus $V \cdot V$ is a projective kH-module lying in $B_0(kH)$, and so $V \cdot V = 2V$. $V \cdot W = \mathrm{Hom}_k(V^*, W) = \mathrm{Hom}_k(V, W)$ deduces that $(V \cdot W)^H = \mathrm{Hom}_{kH}(V, W) = 0$. Hence $V \cdot W = 2W$. Finally $(W \cdot W)^H = \mathrm{Hom}_{kH}(W, W) = k$, as W is an absolutely simple kH-module. Therefore $W \cdot W = V + W$, as desired.

Row 5 refers the expression of $S^j(W)$ in the Grothendieck group of kH-modules. In other words row 5 shows how many times k and W appear in $S^j(W)$ as composition factors. To establish row 5, it suffices to use the Brauer character.

Row 6 follows from the fact that $\dim_k H^j(G, W)$ is equal to the multiplicity of W in $S^j(W)$, which is proved by Claim 1 and Table 2. Note that $H^j(D, k) = S^j(D^*) = S^j(W)$.

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