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On the number of irreducible characters in a finite group

By Tomoyuki WADA

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1. Introduction

Let F be an algebraically closed field of characteristic p, and G be a finite group with a Sylow p-subgroup P. Let B be a block ideal of the group algebra FG which can be regarded as an indecomposable direct summand of FG as an $F(G\times G)$-module. We denote by $k(G)$ and $l(G)$ the number of irreducible ordinary and modular characters in G, respectively (also by $k(B)$ and $l(B)$ the number of those in the block associated with B).

In [15] the author introduced the invariant $n(B)$ that is the number of indecomposable direct summands of $B_{P\times P}$. In the present paper, we show that the inequality "$l(B)\leqq n(B)$" holds, and this inequality is closely related to the well-known result that $k(G)\leqq|G:H|k(H)$ for any subgroup H of G (see [5], [7], [14]). In section 2, we shall obtain a modular version of the above well-known result that $l(G)\leqq|G:H|l(H)$ for any subgroup H of G. When $H=P$, our result $l(B)\leqq n(B)$ provides a more explicit consequence that $l(G)\leqq|P\backslash G/P|$ (the number of (P, P)-double cosets in G) which is proved in section 3. Furthermore, in section 3, we will investigate the case that the above equality holds. In this case, for example, every projective indecomposable FG-module in B has dimension $|P|$, and every irreducible FG-module in B has dimension a power of p.

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2. Let M be a right FG-module, and H be a subgroup of G. We denote by $\mathrm{rad}_H(M)$ and $\mathrm{soc}_H(M)$ the radical and the socle of M as an FH-module. Let $r_H(M)$ and $s_H(M)$ denote the number of irreducible FH-constituents of $M/\mathrm{rad}_H(M)$ and $\mathrm{soc}_H(M)$, respectively.

LEMMA 1. *Let F be an algebraically closed field of arbitrary char-*

acteristic, and let $\{L_1, L_2, \cdots, L_{l(G)}\}$ *and* $\{M_1, M_2, \cdots, M_{l(H)}\}$ *be the sets of all non-isomorphic irreducible* FG *and* FH*-modules, respectively. Then the following hold;*

1) $r_G(V)=\sum_j \dim_F \mathrm{Hom}_{FG}(V, L_j)$ *and* $s_G(V)=\sum_j \dim_F \mathrm{Hom}_{FG}(L_j, V)$ *for any* FG*-module* V,

2) $\sum_i r_G(M_i^G)=\sum_j s_H(L_j)$ *and* $\sum_i s_G(M_i^G)=\sum_j r_H(L_j)$.

PROOF. 1) is clear, and 2) is easy observation from 1) and Frobenius reciprocity theorem: $\mathrm{Hom}_{FG}(M_i^G, L_j)\simeq \mathrm{Hom}_{FH}(M_i, L_{jH})$ and $\mathrm{Hom}_{FG}(L_j, M_i^G)\simeq \mathrm{Hom}_{FH}(L_{jH}, M_i)$.

LEMMA 2. *Under the same notation as above, it holds that* $r_G(M_i^G)\leqq |G:H|$ *and* $s_G(M_i^G)\leqq|G:H|$ *for all* i.

PROOF. Let $M_i^G/\mathrm{rad}(M_i^G)=\bigoplus_j a_{ij}L_j$, and $\mathrm{soc}(M_i^G)=\bigoplus_j b_{ij}L_j$. Then, from Frobenius reciprocity theorem, $a_{ij}\neq 0$ means that $M_i\leqq \mathrm{soc}_H(L_j)$, and also $b_{ij}\neq 0$ means that $M_i\leqq L_j/\mathrm{rad}_H(L_j)$. In particular, $a_{ij}\neq 0$ or $b_{ij}\neq 0$ implies that $\dim M_i\leqq \dim L_j$. Now, since

$$|G:H|\dim M_i=\dim M_i^G\geqq \dim\left(M_i^G/\mathrm{rad}(M_i^G)\right)=\sum_j a_{ij}\dim L_j,$$

and $$|G:H|\dim M_i=\dim M_i^G\geqq \dim\left(\mathrm{soc}_G(M_i^G)\right)=\sum_j b_{ij}\dim L_j,$$

we have that

$$|G:H|\geqq \sum_j a_{ij}\dim L_j/\dim M_i\geqq \sum_j a_{ij}=r_G(M_i^G),$$

and $$|G:H|\geqq \sum_j b_{ij}\dim L_j/\dim M_i\geqq \sum_j b_{ij}=s_G(M_i^G).$$

THEOREM 1. It holds that $l(G)\leqq|G:H|l(H)$ *for any subgroup* H *of* G. *Furthermore, suppose that equality holds, then* $H\lhd G$, G/H *is abelian* p'*-group and* $G=C_G(h)H$ *for any* p'*-element* h *of* H.

PROOF. First statement follows from Lemmas 1, 2, since

$$l(G)\leqq \sum_j r_H(L_j)=\sum_i s_G(M_i^G)\leqq |G:H|\,l(H), \quad \text{or}$$

$$l(G)\leqq \sum_j s_H(L_j)=\sum_i r_G(M_i^G)\leqq |G:H|\,l(H).$$

It is easy to find that equality holds if and only if M_i^G is completely reducible for all i, and M_i^G has exactly $t=|G:H|$ distinct irreducible constituents $L_{i1}, \cdots, L_{it}$, where $L_{ij|H}=M_i$. Let M_1 be the trivial FH-module, then each L_{1j} must be one dimensiona. Hence, $\bigcap_j \mathrm{Ker}(L_{1j})=H\geqq G'$ and we have $H\lhd G$, G/H is abelian. Since $O_p(G/H)$ is contained in the kernel of every irreducible $F(G/H)$-module, this forces that G/H is a p'-group. By Clifford's theorem, G acts trivially on each M_i. Then, G fixes each p-regular classes of H. This implies that $G=C_G(h)H$ for any p'-element h of H.

Note that Theorem 1 includes the well-known result $k(G)\leqq|G:H|k(H)$.

LEMMA 3. *Let $K\leqq H$ be subgroups of G, U and V be an FH and FK-module, respectively, and F be any field. Then*

1) $r_H(U)\leqq r_K(U)$, $s_H(U)\leqq s_K(U)$,

2) $r_K(V)\leqq r_H(V^H)$, $s_K(V)\leqq s_H(V^H)$.

PROOF. 1). Let $\bar{U}=U/\mathrm{rad}_H(U)\simeq X_1\oplus\cdots\oplus X_{r_H(U)}$, where X_i is an irreducible FH-module. Set J_K to be the inverse image of $\mathrm{rad}_K(\bar{U})$ by the natural homomorphism from U to $\bar{U}$. Since $\mathrm{rad}_K(\bar{U})\simeq\mathrm{rad}_K(X_1)\oplus\cdots\oplus\mathrm{rad}_K(X_{r_H(U)})$, U/J_K has at least $r_H(U)$ irreducible constituents. On the other hand, since U/J_K is completely reducible FK-module, $U/\mathrm{rad}_K(U)$ contains at least as many irreducible constituents as U/J_K does. Then, we have that $r_H(U)\leqq r_K(U)$. Second statement is clear from $\mathrm{soc}_K(\mathrm{soc}_H(U))\leqq\mathrm{soc}_K(U)$.

2). Let $\overline{V^H}=V^H/\mathrm{rad}_K(V)^H\simeq(V/\mathrm{rad}_K(V))^H$ and J_H be the inverse image of $\mathrm{rad}_H(\overline{V^H})$ by the natural homomorphism from V^H to $\overline{V^H}$. Then V^H/J_H has at least $r_K(V)$ irreducible constituents. On the other hand, since V^H/J_H is completely reducible, $V^H/\mathrm{rad}_H(V^H)$ contains at least as many irreducible constituents as V^H/J_H. This shows that $r_K(V)\leqq r_H(V^H)$. Second statement is clear from $\mathrm{soc}_H((\mathrm{soc}_K(V))^H)\leqq\mathrm{soc}_H(V^H)$.

It follows from Lemma 3 that the following holds, but it may be well-known, since it holds by another easy observation.

COROLLARY 1. *Let* $\mathrm{Irr}(G)$ *and* $\mathrm{IBr}(G)$ *be the set of all irreducible ordinary and Brauer characters of G. Then*

$$\sum_{\zeta\in\mathrm{Irr}(H)}\zeta(1)\leqq\sum_{\chi\in\mathrm{Irr}(G)}\chi(1)\leqq|G:H|\sum_{\zeta\in\mathrm{Irr}(H)}\zeta(1),\quad\textit{and}$$

$$\sum_{\psi\in\mathrm{IBr}(H)}\psi(1)\leqq\sum_{\phi\in\mathrm{IBr}(G)}\phi(1)\leqq|G:H|\sum_{\psi\in\mathrm{IBr}(H)}\psi(1).$$

PROOF. Let F be an algebraically closed field of any characteristic, then it suffices to show the second statement. From Lemma 3, it holds that

$$r_H(FH)\leqq r_G(FH^G)=r_G(FG)\leqq r_H(FG)=|G:H|r_H(FH).$$

And, $r_H(FH)$, $r_G(FG)$ coincides with the desired term in the second inequality.

3. Firstly, we will show the following theorem, which may be an unknown result in finite group theory.

THEOREM 2. *Let B be a block of G, then it holds that $l(B)\leqq n(B)$,* *in particular* $l(G) \leqq |P \backslash G/P|$.

We can take some way to prove this theorem, and at first we consider $r_{G\times G}(B)$ and $s_{G\times G}(B)$ as a block ideal B is an $F(G\times G)$-module. Next, in the proof of Theorem 3, we will show a more brief method which is owed to Dr. Okuyama.

LEMMA 4. *Let F be a field of characteristic p, and B be a block ideal of FG. Then* $n(B)=r_{P\times P}(B)=s_{P\times P}(B)$.

PROOF. Let $[PxP]$ denote the $F(P\times P)$-module whose basis consists of all elements of a (P, P)-double coset PxP of G. Then, every indecomposable direct summand of $B_{P\times P}$ is isomorphic to some $[PxP]$ (see [8], p. 105). Since $[PxP]$ is a transitive permutation module over $F(P\times P)$, we have that $s_{P\times P}([PxP])=1$ and hence $s_{P\times P}(B)=n(B)$. Furthermore, since $[PxP]$ is cyclic over $F(P\times P)$, it is a homomorphic image of $F(P\times P)$. As $F(P\times P)$ has the unique maximal submodule $\mathrm{rad}_{P\times P}(F(P\times P))$, a homomorphic image does so. Therefore, $r_{P\times P}([PxP])=1$, and hence $r_{P\times P}(B)=n(B)$.

PROOF OF THEOREM 2. Let F be an algebraically closed field of characteristic p, and $J(B)$ be the Jacobson radical of the ring B, then $J(B)=\mathrm{rad}_{1\times G}(B) \geqq \mathrm{rad}_{G\times G}(B)$. Therefore, $l(B)=r_{G\times G}(B/J(B)) \leqq r_{G\times G}(B)$. Then, Lemmas 3, 4 imply that $r_{G\times G}(B) \leqq r_{P\times P}(B)=n(B)$.

By another consideration of socle, it holds that $l(B)=s_{G\times G}(B)$. Because, let $l(J(FG))=I$ be the left annihilator of $J(FG)$ which is a two-sided ideal, and let e be a primitive idempotent of FG, then $Ie \simeq \widehat{eI}$ as a left FG-module (where $\widehat{eI}$ is the dual of eI), since FG is a symmetric algebra. Therefore, we have that $IeI \simeq \widehat{eI} \otimes_F eI$ as an $F(G\times G)$-module. Then B contains exactly $l(B)$ non-isomorphic irreducible $F(G\times G)$-submodules $Ie_1I, \cdots, Ie_{l(B)}I$. Thus, we establishes that $s_{G\times G}(B)=l(B)$. Hence, Lemmas 3, 4 imply that $l(B)=s_{G\times G}(B) \leqq s_{P\times P}(B)=n(B)$.

In the following, we shall investigate the structure of a block B when equality $l(B)=n(B)$ holds. For example, if $G=S_4$, $p=2$ and B is the principal 2-block, then $l(B)=n(B)=2$, and furthermore, $\phi_1(1)=1$, $\phi_2(1)=2$ for $\phi_i \in \mathrm{IBr}(B)$ and $\Phi_1(1)=\Phi_2(1)=8$, where Φ_i is the character afforded by the projective indecomposable FG-module corresponding to ϕ_i. Now, we have the following theorem.

THEOREM 3. *Let F_P be the trivial FP-module, and e be the block idempotent corresponding to B. Then, the following are equivalent;*

1) $l(B)=n(B)$,

2) *$F_P{}^G \cdot e$ is completely reducible and multiplicity-free,*

3) $\dim_F U=|P|$ *for all projective indecomposable FG-module U in B. Furthermore, if one of the above conditions holds, then*

4) $\dim_F L=$ *a power of p for all irreducible FG-module L in B.*

PROOF. Firstly, in order to prove our theorem, we will review that $l(B)\leqq n(B)$ by a different way from the proof of Theorem 2. Let us set $M=F_P{}^G$. Then, we have that

$$n(B)=\dim_F \mathrm{Hom}_{FG}(Me, Me).$$

For, it holds that $\mathrm{Hom}_{FG}(Me, Me)\simeq \mathrm{Hom}_{FG}(Me, M)\simeq \mathrm{Hom}_{FP}(Me_P, F_P)$, by Frobenius reciprocity theorem. Since $FGe\simeq \bigoplus_{i=1}^{n(B)}[Px_iP]$ for some Px_iP in G, we have from Mackey decomposition that

$$Me_P\simeq F_P\underset{FP}{\otimes}\Big(\bigoplus_i [Px_iP]\Big)\simeq \bigoplus_i (F_{P^{x_i}\cap P})^P.$$

Thus, our assertion holds, since $\dim_F \mathrm{Hom}_{FP}((F_{P^{x_i}\cap P})^P, F_P)=1$.

Let $L_1, \cdots, L_{l(B)}$ denote all non-isomorphic irreducible FG-modules in B, then $\mathrm{soc}_G(Me)$ and $Me/\mathrm{rad}_G(Me)$ contains every L_i, respectively. Therefore, $\dim_F\mathrm{Hom}_{FG}(Me/\mathrm{rad}(Me), \mathrm{soc}(Me))\geqq l(B)$, and hence we have the following composite homomorphism

$$Me\xrightarrow{nat.} Me/\mathrm{rad}(Me)\longrightarrow \mathrm{soc}(Me)\xrightarrow{inc.} Me,$$

where *nat.* is the natural epimorphism, and *inc.* is the inclusion map. Thus we have that $n(B)=\dim_F\mathrm{Hom}_{FG}(Me, Me)\geqq \dim_F\mathrm{Hom}_{FG}(Me/\mathrm{rad}(Me), \mathrm{soc}(Me))\geqq l(B)$.

1)$\Leftrightarrow$2). Above argument implies that $l(B)=n(B)$ if and only if $Me/\mathrm{rad}(Me)\simeq \mathrm{soc}(Me)\simeq L_1\oplus\cdots\oplus L_{l(B)}$ (multiplicity-free) and $Me=\mathrm{soc}(Me)$.

2)$\Rightarrow$3). Suppose that $Me\simeq L_1\oplus\cdots\oplus L_{l(B)}$. Then, Nakayama's relation (see p. 603 in [3]) implies that $U_P\simeq FP$ for all projective indecomposable FG-module U in B. Hence, 3) holds.

3)$\Rightarrow$2). Suppose that $\dim_F U=|P|$ for all U in B. Then, from Nakayama's relation, Me contains every L_i, as composition factor, exactly once. Therefore, it follows from Frobenius reciprocity theorem that Me must be completely reducible and isomorphic to $L_1\oplus\cdots\oplus L_{l(B)}$.

The last statement is proved as follows. Suppose that $Me\simeq L_1\oplus\cdots\oplus L_{l(B)}$, then Fronemius reciprocity theorem means that L_{iP} is an indecomposable FP-module. Since $L_i|F_P{}^G$, we have from Mackey decomposition that $L_{iP}|F_P{}^G{}_P\simeq \bigoplus_x (F_{P^x\cap P})^P$. Therefore, $L_{iP}\simeq(F_{P^x\cap P})^P$ for some x in G. Hence, it holds that $\dim_F L_i=|P:P^x\cap P|$. This completes the proof of Theorem 3.

REMARK 1. In the proof of Theorem 3, it is showed that if $l(B)=n(B)$, then $L_P \simeq (F_{P^x \cap P})^P$ for all irreducible FG-module L in B. This means that, in our situation $l(B)=n(B)$, every irreducible FG-module in B has a vertex $P \cap P^x$ for some x in G.

COROLLARY 2. *Let B be a block of G with defect group D such that $D \lhd P$ for some $P \in \mathrm{Syl}_p(G)$. Suppose that $l(B)=n(B)$, then the following hold.*

1) $Z(D) \leqq O_p(G \bmod \mathrm{Ker}\, B)$, *in particular, if D is abelian, then* $D\,\mathrm{Ker}\, B \lhd G$,

2) *there exists a p-solvable subgroup $N \lhd G$ such that $D \in \mathrm{Syl}_p(N)$, in particular, if $D=P$, then G is p-solvable.*

PROOF. Our assumption $l(B)=n(B)$ implies that for every irreducible FG-module L in B, $L_P \simeq F_Q{}^P$, where Q is a vertex of L in P. By Knörr we can choose a defect group D as $C_D(Q) \leqq Q \leqq D$, in particular $Z(D) \leqq Q$ (see [11]). In our situation, we may take $P \rhd D$. Hence $Z(D) \lhd P$, and this follows that $Z(D) \leqq \bigcap_{L \in B} \mathrm{Ker}\, L = O_P(G \bmod \mathrm{Ker}\, B)$ from Mackey decomposition. Thus 1) holds.

2). Let us set $H=O_p(G \bmod \mathrm{Ker}\, B)$ and $\bar{G}=G/H$. Then, every block $\bar{B}$ of $\bar{G}$ which is contained in B satisfies that $l(\bar{B})=n(\bar{B})$. For, let τ be the cannonical algebra homomorphism from FG onto $F\bar{G}$, and e be the block idempotent of B, then there exists an FG-homomorphism from $F_P{}^G \cdot e$ onto $F_{\bar{P}}{}^{\bar{G}} \tau(e)$ (i.e., $id \otimes \tau$). On the other hand, since $l(B)=n(B)$, we have that $F_P{}^G \cdot e$ is completely reducible and multiplicity-free. This means that $F_{\bar{P}}{}^{\bar{G}} \tau(e)$ is so as an $F\bar{G}$-module. Let $\bar{e}$ be the block idempotent of $\bar{B}$, then $F_{\bar{P}}{}^{\bar{G}} \cdot \bar{e}$ is also completely reducible and multiplicity-free, since it is a direct summand of $F_{\bar{P}}{}^{\bar{G}} \tau(e)$. Hence our assertion holds. Therefore, if we take $\bar{B}$ with defect group $\bar{D}$, then the same argument in 1) shows that $Z(\bar{D}) \leqq O_p(\bar{G} \bmod \mathrm{Ker}\, \bar{B})$. Repeating this argument, we have 2).

On the converse that 4)$\Leftarrow$1) in theorem 3, we have the following.

COROLLARY 3. *Let B be a block of G with abelian defect group D such that $D \lhd P$ for some $P \in \mathrm{Syl}_p(G)$. Then the following are equivalent.*

1) $\dim_F L = |P:D|$ *and* $\dim_F U = |P|$ *for all irreducible FG-module L and projective indecomposable FG-module U in B.*

2) $\dim_F L = |P:D|$ *for all L in B.*

3) $\dim_F U = |P|$ *for all U in B.*

PROOF. 1)$\Rightarrow$2) is clear. 2)$\Rightarrow$3). Our assumption implies that L_P is indecomposable and isomorphic tc $F_D{}^P$. Since $D \lhd P$, it follows from Mackey decomposition that $D\,\mathrm{Ker}\, B \lhd G$ (see Theorem (4 A) in [15]). We may assume that Ker $B=1$. Let $\bar{G}=G/D$, then it is easy to see that every L must be contained in a block of defect 0 of $\bar{G}$. Therefore, every projective cover $\bar{U}$ of L as an $F\bar{G}$-module has dimension $|P:D|$. Hence, every U in B has dimension $|D|\dim_F\bar{U}=|P|$. 3)⇨1). From Corollary 2, 1) we have that $D\,\mathrm{Ker}\,B\lhd G$. This implies that $n(B)=v(B)$ (see Theorem (3 A) in [15]). Then it follows from Theorem 3 that $l(B)=v(B)$, and this means that our assertion 1) holds (see Proposition (2 C) in [15]).

COROLLARY 4. *Let G be a p-solvable group. Then the statements* 1), 2), 3) *and* 4) *are equivalent.*

PROOF. 4)⇨3) immediately follows from Theorem (2 B) of Fong's [6].

REMARK 2. If $D\ntriangleleft P$, then there exists an example that Corollary 2 does not hold. Let $G=S_5$, $p=2$ and B be the block of defect 1, then $l(B)=n(B)$. but $Z(D)$ is not normal in G.

Further results on completely reducibility of $F_P{}^G\cdot e$ are investigated in [10], [12] and [12]. In [12], the group in which $F_P{}^G$ is completely reducible is called p-radical group.

EXAMPLE. 4)⇨1) in Theorem 3 need not hold in general. Let $G=SL(2, 2^n)$, $p=2$ and B be the principal block, then $\phi(1)$ is a power of 2 for all $\phi\in\mathrm{IBr}(B)$ (see p. 588 in [2]). However $l(B)=2^n-1<n(B)=2^{n+1}-3$ for $n\geqq 2$ (it is verified from Proposition (2 B) in [15] and character table of $SL(2, 2^n)$).

In p-solvable group G, it is interesting to determine the structure of G whose principal block B has the property that $l(B)=n(B)$. It is hoped to obtain something about p-length of G, but we have only the following.

THEOREM 4. *Let G be a p-solvable group, B_0 be the principal block of G. Let $1\leqq O_{p'}(G)\leqq O_{p'p}(G)\leqq\cdots\leqq G$ (*) be the lower p-series of G. Then the following are equivalent.*

1) *$\phi(1)$ is a power of p for all $\phi\in\mathrm{IBr}(B_0)$.*

2) *Let $\bar{G}=G/O_{p'}(G)$. Then, each p'-factor $\bar{H}/\bar{K}$ appeared in (*) is abelian, and for each p'-composition factor $\bar{L}/\bar{N}$ of $\bar{G}$ which is afforded by a refinement of (*), $\bar{L}$ acts trivially on $\mathrm{IBr}(\bar{N})$.*

3) *Each p'-factor $\bar{H}/\bar{K}$ appeared in (*) is abelian, and every $\phi\in\mathrm{IBr}(\bar{K})$ is extendible to $\bar{H}$.*

PROOF. We may assume that $O_{p'}(G)=1$, and hence for any subnormal subgroup L of G, $O_{p'}(L)=1$ and L has only the principal block.

1)⇨2). Let H/K be a p'-factor appeared in (*). Let $\theta\in\mathrm{IBr}(H/K)$, then θ has p'-degree. On the other hand, the theorem of Clifford implies that IBr(H) satisfies 1). This follows that θ is linear, and H/K is abelian.

Let L/N be a p'-composition factor satisfying the condition in 2), then since L is subnormal in G, IBr(L) satisfies 1). Then, again, the theorem of Clifford means that L acts trivially on IBr(N).

2)$\Rightarrow$3). It is known the following lemma by the same way of **C**-characters (for details, refer to sections 51, 53 in [3] and section 11 in [9]).

LEMMA 5. *Let F be an algebraically closed field of any characteristic, $H \lhd G$ and G/H be cyclic. Suppose ψ is a G-invariant irreducible F-character (Brauer character) of H, then ψ is extendible to G.*

Let H/K be a p'-factor appeared in (*). Then 2) implies that every composition factor L/N of H/K is cyclic, and every $\psi \in$ IBr(N) is L-invariant. Hence it follows from Lemma 5 that ψ is extendible to L. Repeating this process, we have that every irreducible Brauer character of K is extendible to H.

3)$\Rightarrow$1). Let H be the maximal subgroup appeared in (*). Then H satisfies the condition 3), and hence IBr(H) satisfies 1) by induction on $|G|$.

If $|G:H|$ is a power of p, then U^G is indecomposable for every indecomposable FH-module U by Green's theorem (p. 337 in [4]). Then it follows from Nakayama's relation that $\phi_H = \psi \in$ IBr(H) or $\phi_H = \psi_1 + \cdots + \psi_r$, where ψ_i are distinct G-conjugate irreducible Brauer characters of H and $r = |G: I_G(\psi_1)|$ which devides $|G:H|$ (= a power of p). This implies that $\phi(1)$ is a power of p for every $\phi \in$ IBr(G).

If $|G:H|$ is prime to p, then 3) implies that every $\psi \in$ IBr(H) is extendible to G. Hence IBr(G) satisfies 1). This completes the proof of Theorem 4.

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Tokyo University of
Agriculture and Technology