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Principal functions and invariant subspaces of hyponormal operators

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1. Introduction and Theorems

A bounded linear operator T on a Hilbert space is said to be *hyponormal* if its self-commutator $[T^{*}, T] \equiv T^{*}T-TT^{*}$ is positive semi-definite, and *pure hyponormal* if, in addition, T has no nontrivial reducing subspace on which it is normal.

It is not known at present whether every hyponormal operator has a non-trivial invariant subspace. Putnam [7] and Apostol and Clancey [1] presented some conditions for a hyponormal operator to have invariant subspaces. In this paper, by using the principal function invented by Pincus [6], we shall improve the results of Putnam, and Apostol and Clancey.

Let $T=X+iY$ be a pure hyponormal operator, where X and Y are self-adjoint. Then it is known that X and Y are absolutely continuous (see [4, Chap. 2, Th. 3.2]). Let $X=\int tdG(t)$ be the spectral resolution of X. Then the *absolutely continuous support* E_X of X is defined as a Borel subset of the real line (determined uniquely up to a null set) having the least Lebesgue measure and satisfying $G(E_X)=I$. Analogously E_Y is defined for Y.

The main results in this paper are the following:

THEOREM 1. *Let $T=X+iY$ be a pure hyponormal operator. Suppose that, for some real μ_0, the spectrum of T, $\sigma(T)$, has non-empty intersection with each of the open half-planes $\{z: \operatorname{Re} z<\mu_0\}$ and $\{z: \operatorname{Re} z>\mu_0\}$. If*

$$\int_{E_X} \frac{F(x)}{(x-\mu_0)^2} dx<\infty$$

where $F(x)$ is the linear measure of the vertical cross section $\sigma(T) \cap \{z: \operatorname{Re} z=x\}$, then T has a non-trivial invariant subspace.

THEOREM 2. *In Theorem 1 the existence of a non-trivial invariant subspace is also guaranteed if the integrability condition is replaced by*

$$\int_{E_X} \frac{1}{|x-\mu_0|} dx<\infty .$$

Putnam [7] proved Theorem 1 under a more restrictive condition: $\int_{E_X} \frac{F(x)}{(x-\mu_0)^2} dx < 2\pi$, while Apostol and Clancey [1] established Theorem 2 for T with rank one self-commutator.

2. Principal functions

Let $T = X + iY$ be a pure hyponormal operator with trace class self-commutator. For each complex λ, T_λ will be used for $T - \lambda I$. Pincus [6] provided a useful unitary invariant for T, called the principal function of T. There is a non-negative summable function g on $\boldsymbol{C}$, which satisfies

$$\det\left[(X-z)(Y-w)(X-z)^{-1}(Y-w)^{-1}\right] = \exp\left\{\frac{1}{2\pi i}\iint_{\boldsymbol{R}^2} \frac{g(x+iy)}{(x-z)(y-w)} dxdy\right\}$$

for any (z, w) in $\boldsymbol{C}^2 \backslash \sigma(X) \times \sigma(Y)$. The principal function is known to yield much information about the structure of T (see, e. g. [2]). In this section we shall also present some such information.

LEMMA 1. *Let T be a pure hyponormal operator with trace class self-commutator $D \equiv [T^*, T]$. If the principal function g of T satisfies, for some $\lambda = \mu + i\nu \in \boldsymbol{C}$,*

$$\frac{1}{\pi}\iint_{\boldsymbol{R}^2} \frac{g(x+iy)}{(x-\mu)^2 + (y-\nu)^2} dxdy \leq M < \infty ,$$

then $cD \leq T_\lambda T_\lambda^*$ with $c = (\exp(M) - 1)^{-1}$.

PROOF. Since $[T_\lambda^*, T_\lambda] = [T^*, T]$ and the principal function of T_λ coincides with $g(z+\lambda)$, it suffices to consider the case $\lambda = 0$. It is a well-known result of M. G. Krein (see [2, § 3]) that there is a summable function $\delta(t)$ on $\boldsymbol{R}_+$, called the *phase shift* corresponding to the perturbation $TT^* \to T^*T = TT^* + D$, such that

$$\det\left(I + D(TT^* - z)^{-1}\right) = \exp\left(\int_0^\infty \frac{\delta(t)}{t-z} dt\right) \qquad \text{for} \quad z \in \boldsymbol{C} \backslash \sigma(TT^*) .$$

Carey and Pincus established the connection between the principal function g and the phase shift δ:

$$\delta(t) = \frac{1}{2\pi}\int_0^{2\pi} g(\sqrt{t}\, e^{i\theta})\, d\theta , \qquad t \geq 0$$

(see [2, § 7]). Then for any $\varepsilon > 0$

$$\det\left(I+D(TT^*+\varepsilon)^{-1}\right)=\exp\left(\int_0^\infty \frac{\delta(t)}{t+\varepsilon}dt\right)$$
$$=\exp\left(\frac{1}{\pi}\iint_{R^2}\frac{g(x+iy)}{x^2+y^2+\varepsilon}dxdy\right)$$
$$\leq \exp(M)<\infty .$$

Since

$$1+\operatorname{tr}\left(D^{1/2}(TT^*+\varepsilon)^{-1}D^{1/2}\right)\leq \det\left(I+D^{1/2}(TT^*+\varepsilon)^{-1}D^{1/2}\right)$$
$$=\det\left(I+D(TT^*+\varepsilon)^{-1}\right)$$
$$\leq \exp(M),$$

it follows that

$$D^{1/2}(TT^*+\varepsilon)^{-1}D^{1/2}\leq \exp(M)-1$$

or equivalently

$$cD\leq TT^*+\varepsilon$$

with $c=(\exp(M)-1)^{-1}$. Letting $\varepsilon\to 0$, the assertion follows.

REMARK 1. If D is of finite rank, the converse of Lemma 1 is also true in the following sense: if $cD\leq TT^*$ for some $c>0$, then

$$\frac{1}{\pi}\iint_{R^2}\frac{g(x+iy)}{x^2+y^2}dxdy<\infty .$$

In fact, the assumption $cD\leq TT^*$ implies for any $\varepsilon>0$

$$\operatorname{tr}\left(D^{1/2}(TT^*+\varepsilon)^{-1}D^{1/2}\right)\leq c^{-1}\operatorname{rank}(D).$$

Since

$$\det\left(I+D^{1/2}(TT^*+\varepsilon)^{-1}D^{1/2}\right)\leq \exp\left\{\operatorname{tr}\left(D^{1/2}(TT^*+\varepsilon)^{-1}D^{1/2}\right)\right\},$$

it is seen from the proof of Lemma 1 that

$$\frac{1}{\pi}\iint_{R^2}\frac{g(x+iy)}{x^2+y^2+\varepsilon}dxdy\leq \exp\left(c^{-1}\operatorname{rank}(D)\right).$$

Since ε is arbitrary and D is of finite rank, it follows that

$$\frac{1}{\pi}\iint_{R^2}\frac{g(x+iy)}{x^2+y^2}dxdy<\infty .$$

But above converse is not true in general. In fact, if T is a bilateral weighted shift with weights $\{a_n\}_{n=-\infty}^{\infty}$ where $a_n=\min(2^{n/2},1)$, $n=0,\pm 1,\pm 2,$

$\cdots$, then T becomes a pure hyponormal operator for which $[T^*, T] \leq TT^*$. But a simple calculation will show that the principal function g of T is the characteristic function of the unit disc (the index result for the principal function [2, § 8] will also show this fact), hence

$$\frac{1}{\pi}\iint_{R^2} \frac{g(x+iy)}{x^2+y^2}\, dxdy = \infty\,.$$

COROLLARY 1. *Let T be a pure hyponormal operator with trace class self-commutator D and the principal function g. If $\bar{\lambda}$ is an eigenvalue of T^*, then*

$$\iint_{R^2} \frac{g(x+iy)}{|x+iy-\lambda|^2}\, dxdy = \infty\,.$$

PROOF. Suppose, for $\lambda = \mu + i\nu \in \boldsymbol{C}$,

$$\iint_{R^2} \frac{g(x+iy)}{|x+iy-\lambda|^2}\, dxdy < \infty\,.$$

From Lemma 1, there exists a constant $c>0$ such that $cD \leq T_\lambda T_\lambda^*$. Since $D = T_\lambda^* T_\lambda - T_\lambda T_\lambda^*$, $cT_\lambda^* T_\lambda \leq (1-c)\, T_\lambda T_\lambda^*$. This implies ran $T_\lambda^* \subset$ ran T_λ. Because of ker $T_\lambda = \{0\}$, it follows that ker $T_\lambda^* = \{0\}$, contradicting that $\bar{\lambda}$ is an eigenvalue of T^*.

The assertion of Corollary 1 is similar to the following proposition that is proved by Carey and Pincus [3] in the case $0 \leq g \leq 1$.

PROPOSITION 1. *Let T be a pure hyponormal operator with trace class self-commutator D and the principal function g. Suppose $0 \leq g \leq n$ for an integer n. If $\bar{\lambda}$ is an eigenvalue for T^* and the dimension of* $\ker(T-\lambda)^*$ *is n, then for some $r>0$*

$$\iint_{B_r(\lambda)} \frac{n-g(x+iy)}{|x+iy-\lambda|^2}\, dxdy < \infty\,,$$

where $B_r(\lambda) = \{z \in \boldsymbol{C} : |z-\lambda| < r\}$.

PROOF. As in the proof of Lemma 1, it suffices to consider the case $\lambda = 0$. Let δ be the phase shift corresponding to the perturbation $TT^* \rightarrow T^*T$ and let $TT^* = \int_0^\infty t\, dE(t)$ be the spectral resolution of TT^*. Since, for any $\varepsilon > 0$,

$$\begin{aligned}(TT^*+\varepsilon)^{-1} &= \int_0^\infty \frac{1}{t+\varepsilon}\, dE(t) \\ &\geq \frac{1}{\varepsilon} E\big(\{0\}\big)\,,\end{aligned}$$

it follows that

$$\begin{aligned}\exp\left(\int_0^\infty \frac{\delta(t)}{t+\varepsilon}\,dt\right) &= \det\left(I+D^{1/2}(TT^*+\varepsilon)^{-1}D^{1/2}\right)\\ &\geq \det\left(I+\varepsilon^{-1}D^{1/2}E\left(\{0\}\right)D^{1/2}\right)\\ &= \det\left(I+\varepsilon^{-1}E\left(\{0\}\right)DE\left(\{0\}\right)\right).\end{aligned}$$

But $E(\{0\})\,DE(\{0\})=E(\{0\})\,T^*TE(\{0\})$. Since $E(\{0\})$ is the orthogonal projection onto the finite dimensional subspace $\ker T^*$ and $\ker(T^*T)=\{0\}$, there is an $\alpha>0$ such that $E(\{0\})\,T^*TE(\{0\})\geq\alpha E(\{0\})$. Thus

$$\begin{aligned}\det\left(I+\varepsilon^{-1}E\left(\{0\}\right)DE\left(\{0\}\right)\right) &\geq \det\left(I+\varepsilon^{-1}\alpha E\left(\{0\}\right)\right)\\ &= (1+\varepsilon^{-1}\alpha)^n\\ &= \exp\left(\int_0^\alpha \frac{n}{t+\varepsilon}\,dt\right).\end{aligned}$$

Therefore

$$\int_0^\infty \frac{\delta(t)}{t+\varepsilon}\,dt \geq \int_0^\alpha \frac{n}{t+\varepsilon}\,dt$$

and

$$\int_0^\alpha \frac{n-\delta(t)}{t+\varepsilon}\,dt \leq \int_\alpha^\infty \frac{\delta(t)}{t+\varepsilon}\,dt\,.$$

The hypothesis $0\leq g\leq n$ implies $0\leq\delta\leq n$. Taking limits as $\varepsilon\to 0$, the monotone convergence yields

$$\int_0^\alpha \frac{n-\delta(t)}{t}\,dt \leq \int_\alpha^\infty \frac{\delta(t)}{t}\,dt<\infty\,.$$

The result follows with $r=\alpha^{1/2}$ on substituting $\delta(t)=\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} g(\sqrt{t}\,e^{i\theta})\,d\theta$ into the left-hand side.

The following proposition gives an estimate of the principal function g on the point spectrum of T^*.

PROPOSITION 2. *Let T be a pure hyponormal operator with trace class self-commutator and the principal function g. Then for $\lambda\in\boldsymbol{C}$*

$$\dim\ker(T-\lambda)^* \leq \lim_{r\downarrow 0}\left\{\operatorname*{ess\,sup}_{z\in B_r(\lambda)} g(z)\right\}.$$

PROOF. It can be assumed $\lambda=0$ as before. Let δ be the phase shift corresponding to the perturbation $TT^*\to T^*T$. Then, for $n=1, 2, \cdots,$

$$\operatorname{tr}\left\{-(I+nT^*T)^{-1}+(I+nTT^*)^{-1}\right\}=\int_0^\infty \frac{n}{(1+nt)^2}\,dt$$

(see [2, § 3]). As $n\to\infty$, $(I+nT^*T)^{-1}$ converges strongly to 0 and $(I+nTT^*)^{-1}$ to the orthogonal projection onto $\ker T^*$, say P. Since $(I+nTT^*)^{-1}\geq(I+nT^*T)^{-1}$ for $n\geq 1$, by Fatou's lemma

$$\begin{aligned}\varliminf_{n\to\infty}\operatorname{tr}\left\{-(I+nT^*T)^{-1}+(I+nTT^*)^{-1}\right\}&\geq \operatorname{tr} P\\ &=\dim\ker T^*.\end{aligned}$$

Hence

$$\dim\ker T^*\leq\varliminf_{n\to\infty}\int_0^\infty\frac{n}{(1+nt)^2}\,\delta(t)\,dt\,.$$

For $r>0$, define

$$M(r)\equiv\operatorname*{ess\,sup}_{z\in B_r(0)} g(z)\,.$$

$M(z)$ is a positive, monotone non-decreasing function. Since, for any $a>0$,

$$\begin{aligned}\operatorname*{ess\,sup}_{0<t<a}\delta(t)&=\operatorname*{ess\,sup}_{0<t<a}\left\{\frac{1}{2\pi}\int_0^{2\pi} g(\sqrt{t}\,e^{i\theta})\,d\theta\right\}\\ &\leq M(a^{1/2})\,,\end{aligned}$$

it follows that

$$\begin{aligned}&\varliminf_{n\to\infty}\int_0^\infty\frac{n}{(1+nt)^2}\,\delta(t)\,dt\\ &=\varliminf_{n\to\infty}\left\{\int_0^a\frac{n}{(1+nt)^2}\,\delta(t)\,dt+\int_a^\infty\frac{n}{(1+nt)^2}\,\delta(t)\,dt\right\}\\ &\leq\varliminf_{n\to\infty}\left\{M(a^{1/2})\int_0^a\frac{n}{(1+nt)^2}\,dt+\frac{n}{(1+na)^2}\int_a^\infty\delta(t)\,dt\right\}\\ &\leq\varliminf_{n\to\infty}\left\{M(a^{1/2})+\frac{n}{(1+na)^2}\int_a^\infty\delta(t)\,dt\right\}\\ &=M(a^{1/2})\,.\end{aligned}$$

Therefore

$$\dim\ker T^*\leq\lim_{r\downarrow 0}M(r)\,.$$

3. Proofs of theorems

Both theorems are immediate consequences of the following.

THEOREM 3. *Let T be a pure hyponormal operator with trace class* *self-commutator* $D\equiv[T^*, T]$ *and the principal function* g. *If there exists a real* μ_0 *such that the spectrum of* T *has non-empty intersection with each of the open half-planes* $\{z:\ \mathrm{Re}\, z<\mu_0\}$ *and* $\{z:\ \mathrm{Re}\, z>\mu_0\}$ *and*

$$\iint_{\boldsymbol{R}^2} \frac{g(x+iy)}{(x-\mu_0)^2+(y-\nu)^2}\, dxdy \leq M<\infty \qquad \textit{for all real } \nu\,,$$

then T *has a non-trivial invariant subspace.*

PROOF. From Lemma 1, there exists a constant $c>0$ such that for any $\lambda=\mu_0+i\nu$ $(\nu\in\boldsymbol{R})$

$$cD\leq T_\lambda T_\lambda^*\,.$$

By [5], there exists a family $A(\lambda)$ $(\mathrm{Re}\,\lambda=\mu_0)$ of operators such that

$$T_\lambda A(\lambda)=D^{1/2} \quad \text{and} \quad \|A(\lambda)\|\leq c^{-1}\,.$$

Since each T_λ is one-to-one, it is easily seen that $A(\lambda)$ is weakly continuous. $A(\lambda)$ can be extended to $\{z:\ \mathrm{Re}\, z=\mu_0\}\cup\rho(T)$ by defining

$$A(z)\equiv(T-z)^{-1}D^{1/2} \qquad \text{for} \quad z\in\rho(T)\,.$$

Let Γ be a circle centered at μ_0 such that $\sigma(T)$ is interior of Γ and let Γ_1 and Γ_2 be the two semicircles determined by Γ and the line $\{\mathrm{Re}\, z=\mu_0\}$. Then

$$\begin{aligned} D^{1/2} &= -\frac{1}{2\pi i}\int_\Gamma A(z)\, dz \\ &= -\frac{1}{2\pi i}\int_{\Gamma_1} A(z)\, dz-\frac{1}{2\pi i}\int_{\Gamma_2} A(z)\, dz\,. \end{aligned}$$

Consequently, one of these later integrals is non-zero. Suppose, for definiteness, $B_1\equiv-\frac{1}{2\pi i}\int_{\Gamma_1} A(z)\, dz\neq 0$. The operator-valued function

$$A_1(z)\equiv\frac{1}{2\pi i}\int_{\Gamma_1}\frac{A(w)}{w-z}\, dw\,, \qquad z\notin\Gamma_1$$

is analytic off Γ_1 and satisfies $(T-z)\, A_1(z)=B_1$ in the interior of Γ_1.

Now let σ_1 be the part of $\sigma(T)$ contained in the closed semidisc with boundary Γ_1. Then

$$\bigcap_{z\notin\sigma_1} \mathrm{ran}\,(T-z)\supset \mathrm{ran}\, B_1\neq\{0\}\,.$$

By [4, Chap. 1, Th. 3.5], $\bigcap_{z\notin\sigma_1} \mathrm{ran}\,(T-z)$ is a closed invariant subspace for T and clearly not the whole space. This completes the proof.

PROOF OF THEOREM 1. T can be assumed to have a cyclic vector. For otherwise T has obviously a non-trivial invariant subspace. As pointed out by Berger and Shaw (see [4, Chap. 3, Th. 3.1]), the cyclicity implies that the self-commutator $[T^*, T]$ is of trace class. Furthermore, from the Berger's estimate (see [4, Chap. 5, Cor. 5.1]), the principal function g of T satisfies $0 \leq g \leq 1$. Since g vanishes a.e. off $\sigma(T)$ and off $\{x+iy: x \in E_X$ and $y \in E_Y\}$ (see [2, §5] and [4, Chap. 5, § 3]), for almost all x

$$\int_{\boldsymbol{R}} g(x+iy)\, dy \leq F(x)$$

and for all $\nu \in \boldsymbol{R}$

$$\begin{aligned}\iint_{\boldsymbol{R}^2} \frac{g(x+iy)}{(x-\mu_0)^2+(y-\nu)^2}\, dxdy &\leq \int_{E_X}\int_{\boldsymbol{R}} \frac{g(x+iy)}{(x-\mu_0)^2}\, dydx \\ &\leq \int_{E_X} \frac{F(x)}{(x-\mu_0)^2}\, dx \\ &< \infty .\end{aligned}$$

Thus, by Theorem 3, T has a non-trivial invariant subspace.

PROOF OF THEOREM 2. As in the proof of Theorem 1, it can be assumed that $0 \leq g \leq 1$. Then it follows

$$\begin{aligned}\iint_{\boldsymbol{R}^2} \frac{g(x+iy)}{(x-\mu_0)^2+(y-\nu)^2}\, dxdy &\leq \int_{E_X}\int_{\boldsymbol{R}} \frac{1}{(x-\mu_0)^2+(y-\nu)^2}\, dydx \\ &= \pi \int_{E_X} \frac{1}{|x-\mu_0|}\, dx \\ &< \infty .\end{aligned}$$

Now Theorem 3 can be applied.

REMARK 2. Define the measurable function G on $\boldsymbol{C}$ by

$$G(z) \equiv \iint_{\boldsymbol{R}^2} \frac{g(x+iy)}{|x+iy-z|^2}\, dxdy .$$

Then the assumption of Theorem 3 means that G is uniformly bounded on the vertical line $\{z: \operatorname{Re} z = \mu_0\}$. It is also seen from the proof of Theorem 3 that if G is uniformly bounded on some rectifiable closed curve Γ and the spectrum of T lies partly in both the exterior and the interior of Γ, then T has a non-trivial invariant subspace.

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