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On a lifting problem of Fourier-Stieltjes transforms of measures

By Jyunji INOUE

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Let G and $\hat{G}$ be a LCA group and its dual group, respectively. $M(G)$ denotes the measure algebra on G, the Banach algebra of bounded regular complex Borel measures on G with convolution multiplication and total variation norm $\|\cdot\|$. $M_a(G)$ and $M_s(G)$ express the space of absolutely continuous measures and the space of singular measures on G with respect to the Haar measure of G, respectively. For $\mu\in M(G)$, $\hat{\mu}$ denotes the Fourier-Stieltjes transform of μ, and we put $B(\hat{G})=\{\hat{\mu}|\mu\in M(G)\}$, $A(\hat{G})=\{\hat{\mu}|\mu\in M_a(G)\}$, $B_s(\hat{G})=\{\hat{\mu}|\mu\in M_s(G)\}$. $B(\hat{G})$ is a Banach algebra with respect to the pointwise multiplication and the norm $\|\hat{\mu}\|=\|\mu\|$.

Let Λ be a closed subgroup of $\hat{G}$. The following theorem is well-known ([6]).

THEOREM1. $B(\hat{G})|_{\Lambda}=B(\Lambda)$, $A(\hat{G})|_{\Lambda}=A(\Lambda)$.

It follows from theorem 1 that each member of $B(\Lambda)$ (resp. $A(\Lambda)$) can be lifted to a member of $B(\hat{G})$ (resp. $A(\hat{G})$), but it is not clear whether there exist any liftings which are linear maps from $B(\Lambda)$ to $B(\hat{G})$ (resp. from $A(\Lambda)$ to $A(\hat{G})$).

On the other hand, in the recent papers [4] and [5], we can find partial answers to this lifting problem.

THEOREM 2 (cf. [4] and [5]). *Let Λ be a discrete subgroup of $\hat{G}$, let H be the annihilator of Λ in G, and let W be a neighborhood of $0\in\hat{G}$. Choose a neighborhood U of $0\in\hat{G}$ and a probability measure $\rho\in M_a(G)$ such that* $\operatorname{supp}\hat{\rho}\subset U\subset W$ *and $(U-U)\cap\Lambda=\{0\}$, and put*

$$\hat{J}\hat{\mu}(\gamma)=\sum_{\alpha\in\Lambda}\hat{\mu}(\alpha)\,\hat{\rho}(\gamma-\alpha)\qquad(\hat{\mu}\in B(\Lambda),\ \gamma\in\hat{G}).$$

Then we have $\hat{J}\hat{\mu}\in B(\hat{G})$ with the following additional properties.

i) $\hat{J}\hat{\mu}|_{\Lambda}=\hat{\mu}$,

ii) $\|\hat{J}\hat{\mu}\|=\|\hat{\mu}\|$,

(*) iii) *$\hat{J}\hat{\mu}$ is positive definite if $\hat{\mu}$ is positive definite,*

iv) $\hat{J}\hat{\mu}\in A(\hat{G})$ *if* $\hat{\mu}\in A(\Lambda)$,

v) $\hat{J}\hat{\mu}\in B_s(\hat{G})$ *if* $\hat{\mu}\in B_s(\Lambda)$,

vi) $\mathrm{supp}\,(\hat{J}\hat{\mu})\subset \mathrm{supp}\,(\hat{\mu})+W$.

REMARK 1. In theorem 2, we can consider $\hat{J}\hat{\mu}$ the Fourier-Stieltjes transform of a uniquely determined measure $J\mu\in M(G)$, and then J becomes a linear map of $M(G/H)$ into $M(G)$ with the following properties.

$$
(**)\quad
\begin{array}{rl}
\text{i)} & \widehat{(J\mu)}|_\Lambda=\hat{\mu}, \\
\text{ii)} & \|J\mu\|=\|\mu\|, \\
\text{iii)} & J\mu\geqq 0 \text{ if } \mu\geqq 0, \\
\text{iv)} & J\mu\in M_a(G) \text{ if } \mu\in M_a(G/H), \\
\text{v)} & J\mu\in M_s(G) \text{ if } \mu\in M_s(G/H), \\
\text{vi)} & \mathrm{supp}\,\widehat{(J\mu)}\subset \mathrm{supp}\,(\hat{\mu})+W.
\end{array}
$$

Obviously, the existence of $\hat{J}$ with (*) and the existence of J with (**) are equivalent each other.

The purpose of this paper is to prove the following theorem 3 which gives an answer to the lifting problem stated above.

THEOREM 3. *If Λ is a closed subgroup of $\hat{G}$, and if W is a neighborhood of $0\in\hat{G}$, there exists a linear map $\hat{J}$ of $B(\Lambda)$ into $B(\hat{G})$ which satisfies (*) of theorem 2.*

To prove theorem 3, we provide two lemmas. $\boldsymbol{R}^n$ and $\boldsymbol{T}^n$ denote the n-fold products of the real groups and the circle groups, respectively.

LEMMA 1. *For each neighborhood U of $0\in\hat{G}$, there exists a compact subgroup K of $\hat{G}$ contained in $U\cap\Lambda$ such that $\hat{G}/K$ and Λ/K split into direct sums of the forms*

$$\hat{G}/K=L\times F,\ \Lambda/K=L\times D,$$

where L is an open subgroup of Λ/K and F is a closed subgroup of $\hat{G}/K$.

PROOF. By the structure theorem of LCA groups, Λ contains an open subgroup of the form $\boldsymbol{R}^m\times K'$ with a compact subgroup K' of Λ. By (24.7) of [3], there exists a compact subgroup K contained in $K'\cap U$ such that $K'/K=\boldsymbol{T}^n\times F'$ with a finite subgroup F' of K'/K. Hence Λ/K contains an open subgroup L isomorphic to $\boldsymbol{R}^m\times\boldsymbol{T}^n$, and by theorem 6.16 of [1], there exists a discrete subfroup D of Λ/K such that $\Lambda/K=L\times D$. Likewise, since L is a closed subgroup of $\hat{G}/K$, there exists a closed subgroup F of $\hat{G}/K$

such that $\hat{G}/K = L\times F$.

LEMMA 2. *Let K be a compact subgroup of Λ and let H and G_0 be the annihilator of Λ and K in G, respectively. For each $\mu\in M(G/H)$, we denote by $\mu|_{(G_0/H)+x}$ the restriction of μ to a coset $(G_0/H)+x\in(G/H)/(G_0/H)$. Then the Fourier-Stieltjes transform of $(\mu|_{(G_0/H)+x})*\delta_{-x}$ is given by*

$$\left(\left(\mu|_{(G_0/H)+x}\right)*\delta_{-x}\right)^{\wedge}(\gamma) = \left(\left((x,\cdot)\hat{\mu}\right)*m_K\right)(\gamma) \qquad (\gamma\in\Lambda), \tag{1}$$

where δ_{-x} denotes the dirac measure at $-x\in G/H$, and $m_K(\in M(\Lambda))$ denotes the normalized Haar measure of K.

PROOF. First, suppose that $\mathrm{supp}(\hat{\mu})$ is compact. If $^{\vee}$ denotes the inverse Fourier transform, we have by the inversion theorem

$$\left(\left((x,\cdot)\,\hat{\mu}\right)*m_K\right)^{\vee} = (\mu*\delta_{-x})\cdot\chi_{G_0/H} = \left(\mu|_{(G_0/H)+x}\right)*\delta_{-x}\left(\in M_a(G/H)\right).$$

where $\chi_{G_0/H}$ is the characteristic function of G_0/H. Hence (1) holds.

Next, we consider the general case. For each $\gamma_0\in\Lambda$, there exists $\nu\in M_a(G/H)$ such that $\mathrm{supp}(\hat{\nu})$ is comact, $\hat{\nu}|_{K+\gamma_0}=1$ and $\mathrm{supp}(\nu)\subset G_0/H$, then we have

$$\begin{aligned}&\left(\left((\mu*\nu)|_{(G_0/H)+x}\right)*\delta_{-x}\right)^{\wedge}(\gamma_0) = \left(\left(\mu|_{(G_0/H)+x}\right)*\delta_{-x}\right)^{\wedge}(\gamma_0),\\ &\left(\left((x,\gamma')\,\hat{\mu}(\gamma')\,\hat{\nu}(\gamma')\right)*m_K\right)(\gamma_0) = \left(\left((x,\gamma')\,\hat{\mu}(\gamma')\right)*m_K\right)(\gamma_0).\end{aligned} \tag{2}$$

From the first paragraph we have

$$\left(\left((\mu*\nu)|_{(G_0/H)+x}\right)*\delta_{-x}\right)^{\wedge}(\gamma_0) = \left(\left((x,\gamma')\,\hat{\mu}(\gamma')\,\hat{\nu}(\gamma')\right)*m_K\right)(\gamma_0). \tag{3}$$

we get (1) from (2) and (3).

PROOF OF THEOREM 3. Let U be a compact neighborhood of $0\in\hat{G}$ such that $U+U\subset W$. By lemma 1 there exists a compact subgroup K of $\hat{G}$ contained in $\Lambda\cap U$ such that $\hat{G}/K$ and Λ/K split into direct sums of the form $\hat{G}/K=L\times F$ and $\Lambda/K=L\times D$ with a closed subgroup F of $\hat{G}/K$ and a discrete subgroup D of Λ/K, respectively.

(I). First, we consider the case $K=\{0\}$. Let τ be the group topology of $\hat{G}$ such that F with the subspace topology inherit from $\hat{G}$ forms an open subgroup of $\hat{G}$ with respect to τ. Obviously, τ is stronger than the original topology of $\hat{G}$. Since $L\cap F=\{0\}$ Λ is a discrete subgroup in the new topology τ. The group $\hat{G}$ with the topology τ forms a LCA group, which will be denoted by $\hat{G}_\tau$.

Let V be a compact neighborhood of $0\in\hat{G}_\tau$ such that $(V-V)\cap\Lambda=\{0\}$,

$V \subset U \cap F$, and let ρ be a probability measure in $M(\hat{F})$ ($\hat{F}$ is the dual group of F) such that supp $\hat{\rho} \subset V$. Then, we define the functions $\hat{\hat{\rho}}$ and $\hat{J}\hat{\mu}$ on $\hat{G}$ by

$$\hat{\hat{\rho}}(\gamma) = \begin{cases} \hat{\rho}(\gamma)\,; & \gamma \in F \\ 0\,; & \gamma \notin F \end{cases}$$
$$(\hat{J}\hat{\mu})(\gamma) = \sum_{\alpha \in \Lambda} \hat{\mu}(\alpha)\, \hat{\hat{\rho}}(\gamma - \alpha) \qquad (\gamma \in \hat{G}, \hat{\mu} \in B(\Lambda))\,. \tag{4}$$

By theorem 2, $\hat{J}\hat{\mu}$ is an element of $B(\hat{G}_\tau)$ which has the properties (*). If we can show that $\hat{J}\hat{\mu}$ is continuous in the original topology of $\hat{G}$, we have $\hat{J}\hat{\mu} \in B(\hat{G})$ with the properties i), ii), iii) and vi) of (*) in theorem 2.

Let $\gamma_0 \in \hat{G}$, and let $(\gamma_\beta)_{\beta \in B} \subset \hat{G}$ be a net which converges to γ_0 in the topology of $\hat{G}$. In the first, we consider the case that $\gamma_0 \in (\Lambda + \text{supp}\,(\hat{\hat{\rho}}))^0$ (the set of the interior points of $\Lambda + \text{supp}\,(\hat{\hat{\rho}})$ in $\hat{G}$). Then, we may assume $\gamma_\beta \in \Lambda + \text{supp}\,(\hat{\hat{\rho}})$ $(\beta \in B)$. Write $\gamma_0 = \alpha_0 + t_0$, $\gamma_\beta = \alpha_\beta + t_\beta$ $(\beta \in B)$, where α_0, $\alpha_\beta \in \Lambda$ and t_0, $t_\beta \in \text{supp}\,(\hat{\hat{\rho}})$. These expressions are unique by the condition $\Lambda \cap (V - V) = \{0\}$. Since Λ contains L, $\Lambda + V_1$ contains a neighborhood of γ_0 for each neighborhood V_1 of t_0 in F. This shows that $\lim t_\beta = t_0$, $\lim \alpha_\beta = \alpha_0$, and we have

$$\lim \hat{J}\hat{\mu}(\gamma_\beta) = \lim \sum_{\alpha \in \Lambda} \hat{\mu}(\alpha)\, \hat{\hat{\rho}}(\gamma_\beta - \alpha) = \lim \hat{\mu}(\alpha_\beta)\, \hat{\hat{\rho}}(t_\beta)$$
$$= \hat{\mu}(\gamma_0)\, \hat{\hat{\rho}}(t_0) = \hat{J}\hat{\mu}(\gamma_0) \qquad \left(\hat{\mu} \in B(\Lambda)\right).$$

Next, if $\gamma_0 \in \partial_{\hat{G}}\,(\Lambda + \text{supp}\,(\hat{\hat{\rho}}))$ (the boundary of $\Lambda + \text{supp}\,(\hat{\hat{\rho}})$ in $\hat{G}$), we have $\gamma_0 = \alpha_0 + t_0$ with $t_0 \in \partial_F\,(\text{supp}\,(\hat{\hat{\rho}}))$ and $\alpha_0 \in \Lambda$, since $\Lambda + \text{supp}\,(\hat{\hat{\rho}})$ is closed in $\hat{G}$. Let $B' = \{\beta \in B \mid \gamma_\beta \in \Lambda + \text{supp}\,(\hat{\hat{\rho}})\}$, and put $\gamma_\beta = \alpha_\beta + t_\beta\,(\beta \in B')$ with $\alpha_\beta \in \Lambda$, $t_\beta \in \text{supp}\,(\hat{\hat{\rho}})$. Then we have

$$\hat{J}\hat{\mu}(\gamma_\beta) = \begin{cases} \hat{\mu}(\alpha_\beta)\, \hat{\hat{\rho}}(t_\beta)\,; & \beta \in B' \\ 0\,; & B \backslash B'. \end{cases}$$

If B' is a cofinal set of B, we have $\lim_{\beta \in B'} \alpha_\beta = \alpha_0$, $\lim_{\beta \in B'} t_\beta = t_0$ in $\hat{G}$ as above, and

$$\lim_{\beta \in B'} \hat{J}\hat{\mu}(\gamma_\beta) = \lim_{\beta \in B'} \hat{\mu}(\alpha_\beta)\, \hat{\rho}(t_\beta)$$
$$= \hat{\mu}(\alpha_0)\, \hat{\rho}(t_0) = \hat{J}\hat{\mu}(\gamma_0) = 0\,.$$

Since $\hat{J}\hat{\mu}(\gamma_\beta) = 0$ $(\beta \in B \backslash B')$, we have $\lim_{\beta \in B} \hat{J}\hat{\mu}(\gamma_\beta) = 0 = \hat{J}\hat{\mu}(\gamma_0)$ whether B' is a cofinal set of B or not.

In the last case, if $\gamma_0 \notin \Lambda + \text{supp}\,(\hat{\hat{\rho}})$, we have at once $\lim \hat{J}\hat{\mu}(\gamma_\beta) = 0 = \hat{J}\hat{\mu}(\gamma_0)$ since supp $\hat{J}\hat{\mu} \subset \Lambda + \text{supp}\,(\hat{\hat{\rho}})$.

To prove iv), let $\hat{\mu} \in A\,(\Lambda)$ be arbitrary and choose $\hat{\mu}_k \in A(\Lambda)$ $(k = 1, 2, \cdots)$

with compact support such that $\lim \|\hat{\mu}_k - \hat{\mu}\| = 0$. Since supp $\hat{J}\hat{\mu}_k$ is contained in the compact set supp $\hat{\mu}_k$+supp $\hat{\hat{\rho}}$, we have $\hat{J}\hat{\mu}_k \in A(\hat{G})$. Thus, by lim $\|\hat{J}\hat{\mu}_k - \hat{J}\hat{\mu}\| = \lim \|\hat{\mu}_k - \hat{\mu}\| = 0$, we get $\hat{J}\hat{\mu} \in A(\hat{G})$.

To prove v), we use the Doss's criterion on singular measures ([2]). Let $\hat{\mu} \in B_s(\Lambda)$, and let C be an arbitrary compact set of $\hat{G}$, and $\varepsilon > 0$. Since $\Lambda \cap C$ is compact, we have by [2] a trigonometric polynomial $\bar{P}(\dot{x}) = \sum_{i=1}^{s} c_i (-\dot{x}, \gamma_i)$ $(\gamma_i \in \Lambda \backslash C)$ on G/H such that $|\sum_{i=1}^{s} c_i \mu(\gamma_i)| > \|\mu\| - \varepsilon$, $\sup_{\dot{x} \in G/H} |P(\dot{x})| \leqq 1$. Then the trigonometric polynomial $P(x) = \sum_{i=1}^{s} c_i(-x, \gamma_i)$ on G satisfies

$$\left|\sum_{i=1}^{s} c_i \hat{J}\hat{\mu}(\gamma_i)\right| = \left|\sum_{i=1}^{s} c_i \hat{\mu}(\gamma_i)\right| > \|\hat{\mu}\| - \varepsilon = \|\hat{J}\hat{\mu}\| - \varepsilon, \quad \sup_{x \in G} \left|P(x)\right| \leqq 1;$$

and we have, by [2] again, $\hat{J}\hat{\mu} \in B_s(\hat{G})$.

(II). Next, we consider the case $K \neq \{0\}$. Here, we express by π the natural map of $\hat{G}$ onto $\hat{G}/K$. If G_0 and H are the annihilator of K and Λ respectively, we have $G_0 \supset H$ and G_0 is open in G. Since G_0 is the dual group of $\hat{G}/K = L \times F$ and H is the annihilator in G_0 of the closed subgroup $\Lambda/K = L \times D$ of $\hat{G}/K$, we have by (I) a linear map J_0 of $M(G_0/H)$ into $M(G_0)$ which satisfies (**) of remark 1 and

$$(J_0\mu)\hat{}(\gamma) = \sum_{\alpha \in \Lambda/K} \hat{\mu}(\alpha) \hat{\hat{\rho}}(\gamma - \alpha) \qquad \left(\mu \in M(G_0/H),\ \gamma \in \hat{G}/K\right) \tag{5}$$

where $\hat{\hat{\rho}}$ is a function on $\hat{G}/K$ defined by (4) for some probability measure $\rho \in M(\hat{F})$ with supp $\hat{\rho} \subset \pi(U) \cap F$.

We now define a map J of $M(G/H)$ into $M(G)$. Choose a subset $\{x_\xi\}_{\xi \in G/G_0} \subset G$ such that the set $\{\dot{x}_\xi = x_\xi + H\}_{\xi \in G/G_0}$ becomes a complete set of representatives of $(G/H)/(G_0/H)$ by G_0/H, and put

$$J\mu = \sum_{\xi \in G/G_0} \left[J_0\left((\mu|_{(G_0/H)+\dot{x}_\xi})(*\delta_{-\dot{x}_\xi})\right)\right] * \delta_{x_\xi} \qquad \left(\mu \in M(G/H)\right) \tag{6}$$

For each $\mu \in M(G/H)$, the set $\{\xi \in G/G_0 : \mu|_{(G_0/H)+\dot{x}_\xi} \neq 0\}$ is at most countable, and the map J is well defined. It is easy to see that J is linear and satisfies (**) of remark 1 except i) and vi).

Let $\mu \in M(G/H)$ and $\gamma \in \Lambda$ arbitrary. Then we have from (6) and the property (**) i) of J_0 that

$$\begin{aligned}(J\mu)\hat{}(\gamma) &= \sum_{\xi \in G/G_0} \left[J_0\left((\mu|_{(G_0/H)+\dot{x}_\xi}) * \delta_{-\dot{x}_\xi}\right)\right]\hat{}(\dot{\gamma})\,(-x_\xi, \gamma) \\ &= \sum_{\xi \in G/G_0} \left((\mu|_{(G_0/H)+\dot{x}_\xi}) * \delta_{-\dot{x}_\xi}\right)\hat{}(\dot{\gamma})\,(-x_\xi, \gamma)\end{aligned}$$

$$= \sum_{\xi\in G/G_0} \left((\mu|_{(G_0/H)+\dot{x}_\xi})*\delta_{-\dot{x}_\xi}\right)^{\wedge}(\gamma)\,(-x_\xi, \gamma)$$

$$= \sum_{\xi\in G/G_0} (\mu|_{(G_0/H)+\dot{x}_\xi})^{\wedge}(\gamma) = \hat{\mu}(\gamma)\,,$$

where $\dot{\gamma}=\gamma+K\in\Lambda/K$. Thus i) holds.

To prove vi), we use lemma 2. By (1) and (5) we have

$$\operatorname{supp}\left(J_0(\mu|_{(G_0/H)+\dot{x}_\xi})*\delta_{-\dot{x}_\xi})\right)^{\wedge}\subset\pi\left(\operatorname{supp}(\hat{\mu})\right)+\pi(U) \qquad (\xi\in G/G_0)\,. \quad (7)$$

Then we get from (6) and (7) that

$$\operatorname{supp}(J\mu)^{\wedge}\subset\operatorname{supp}(\hat{\mu})+K+U\subset\operatorname{supp}(\hat{\mu})+W\,,$$

and this proves vi), and the proof is complete.

REMARK 2. In [7], analogous lifting operators are constructed and used under the special setting of the paper.

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Department of Mathematics
Hokkaido University