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Maximal ideals in regular rings

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Introduction. A von Neumann regular ring A may be characterised by any one of the following conditions: (a) every right (left) A-module is flat; (b) every right (left) A-module is p-injective. Note that if I is a p-injective right ideal of A, then A/I is a flat right A-module [21]. Von Neumann regular rings and associated rings are studied by many authors since several years (cf. for example, [1]-[5], [7]-[11]). For rings without identity, consult [13]. In this note, regular rings are considered essentially through maximal (right) ideals. We introduce a class of rings with particular maximal right ideals which generalise rings whose simple right modules are flat (cf. definition below). Conditions for von Neumann regularity and strong regularity are given. Commutative regular or self-injective regular rings with non-zero socle are characterised in terms of a special maximal ideal which acts as a "test module". Rings whose simple right modules are either p-injective or flat and biregular rings are also considered.

Throughout, A represents an associative ring with identity and A-modules are unitary. Z, Y, J will denote respectively the left singular ideal, the right singular ideal and the Jacobson radical of A. Recall that a right A-module M is p-injective if, for any principal right ideal P of A, any right A-homomorphism $g: P\rightarrow M$, there exists $y\in M$ such that $g(b)=yb$ for all $b\in P$. Note that A is a right p-injective ring if, and only if, every principal left ideal of A is a left annihilator [9, Theorem 1]. Following [3] and [13], A is called a right V-ring (resp. $p-V$-ring) if every simple right A-module is injective (resp. p-injective). It is now well-known that there is no inclusion between the classes of arbitrary von Neumann regular rings and V-rings, which has motivated the queries raised in [5] and many papers on connections between those rings. However, a well-known result of *I.* KAPLANSKY ensures that regular rings and V-rings coincide in the commutative case. As usual, an ideal of A means a two-sided ideal. A left (right) ideal of A is called rduced if it contains no non-zero nilpotent element. An element c of A is called a non-zero-divisor if $l(c)=r(c)=0$.

We introduce a generalisation of rings whose simple right modules are flat (cf. the proof of Lemma 1).

DEFINITION. *A is called a GSF-ring (generalised simple flat) if, for any maximal right ideal M of A, any* $b \in M$, bA/bM_A *is flat.*

LEMMA 1. *Let A be a GSF-ring. Then*

(1) *Any non-zero-divisor of A is right invertible*:

(2) $Z \cap Y \subseteq J$:

(3) *If I is a reduced right ideal of A, then I is a strongly regular ring. In particular, if* I_A *is cyclic, then it is a direct summand of* A_A.

PROOF. (1) Let k be a non-zero-divisor of A. If $kA \neq A$, let M be a maximal right ideal of A containing kA. Suppose that $kA = kM$. Then $k = kc$ for some $c \in M$ implies $1 - c \in r(k) = 0$, whence $1 = c \in M$, contradicting $M \neq A$. If $kA \neq kM$, the right A-homomorphism $g: A/M \rightarrow kA/kM$ defined by $g(a+M) = ka + kM$ for all $a \in A$ implies that $A/M \approx kA/kM$ is right A-flat (because A/M_A is simple). Then $k \in M$ implies $k = dk$ for some $d \in M$ which yields $1 = d \in M$, again a contradiction. This proves that $ku = 1$ for some $u \in A$.

(2) Let $y \in Z \cap Y$. For any $a \in A$, if $c \in r(1 - ya)$, then $c = yac$ and $cA \cap r(ya) = 0$ implies $c = 0$. Similarly, if $d \in l(1 - ya)$, since $ya \in Z$, then $d = 0$. By (1), $(1 - ya)w = 1$ for some $w \in A$, which proves that $y \in J$.

(3) Since I is reduced, for any $b \in I$, $l(b) \subseteq r(b)$. If $bA + r(b) \neq A$, let M be a maximal right ideal containing $bA + r(b)$. If $bA = bM$, then $b = bc$ for some $c \in M$ which implies $1 - c \in r(b) \subseteq M$, whence $1 \in M$, contradicting $M \neq A$. If $bM \neq bA$, then from (1), we have $A/M \approx bA/bM$ is right A-flat and $b = db$ for some $d \in M$. In that case, $1 - d \in l(b) \subseteq r(b) \subseteq M$ implies $1 \in M$, again a contradiction.
Thus $bA + r(b) = A$ for any $b \in I$ from which $b = b^2u$, $u \in A$. Now $b = b(b^2u)u = b^2v$, where $v = bu^2 \in I$, and $(b - bvb)^2 = 0$ implies $b = bvb$ which proves that I is a strongly regular ring. Since $bA = eA$, where $e = bv$ is idempotent, the proof of (3) is complete.

REMARK 1. In view of Lemma 1(3), [2, Corollary 6] holds for *GSF*-rings.

LEMMA 2. *Suppose that A has a maximal left ideal M with the following properties*: (a) ${}_AA/M$ *is flat and* (b) *for any* $u \in M$, uM *is a right ideal such that* A/uM_A *is flat. Then* $J = 0$.

PROOF. If $v \in J (\subseteq M)$, then A/vM_A is flat, and for any $a \in M$, $va \in vM$ implies that $va = vcva$ for some $c \in M$. Since $w(1 - vc) = 1$ for some $w \in A$ (because $vc \in J$), then $va = w(1 - vc)\, va = w(va - vcva) = 0$ which yields $vM = 0$. Now ${}_AA/M$ is flat which implies that $v \in vM = 0$, whence $J = 0$.

If A is a right p-injective ring whose complement right ideals are principal, then A/Y is a von Neumann regular ring. Applying [14, Lemma 4. 1] to Lemma 2, we get

COROLLARY 2. 1. *The following conditions are equivalent for a ring A satisfying the hypothesis of Lemma 2;*

(1) *A is right continuous regular;*

(2) *A is right continuous;*

(3) *A is right p-injective whose complement right ideals are principal.*

We now consider a generalisation of both right $p-V$-rings and rings whose simple right modules are flat.

DEFINITION. *A is called a SPF-ring if every simple right A-module is either p-injective or flat.*

LEMMA 3. *Let A be a SPF-ring. Then*

(1) *Any reduced right ideal of A is idempotent;*

(2) *$A=AcA$ for any non-zero-divisor c of A.*

PROOF. (1) Let R be a reduced right ideal of A. For any $b\in R$, $l(b)\subseteq r(b)$ and if $AbA+r(b)\neq A$, let M be a maximal right ideal containing $AbA+r(b)$. If A/M_A is flat, then $b=cb$ for some $c\in M$ and $1-c\in l(b)\subseteq r(b)\subseteq M$ yields $1\in M$, a contradiction. If A/M_A is p-injective, the right A-homomorphism $g: bA\rightarrow A/M$ defined by $g(ba)=a+M$ for all $a\in A$ yields $1+M=g(b)=db+M$ for some $d\in A$, whence $1\in M$, again a contradiction. Thus $AbA+r(b)=A$ for any $b\in R$, which proves that $R=R^2$.

(2) If $AcA\neq A$, let M be a maximal right ideal containing AcA. Since $l(c)=r(c)=0$, the proof of (1) shows that either A/M_A flat or A/M_A p-injective leads to a contradiction. This proves that $AcA=A$.

COROLLARY 3. 1. *If A is SPF, then any reduced ideal of A is a fully right idempotent ring.*

PROOF. Let T be a reduced ideal of A, I a right ideal of T. Then by Lemma 3(1), for any $b\in I$, $bA=(bA)^2=(bA)^4$ and since $(bA)^2\subseteq bT\subseteq I$, then $b\in I^2$ which proves that T is a fully right idempotent ring.

COROLLARY 3. 2. *If A is a prime left or right Goldie SPF-ring, then A is simple.* (*cf.* [3, p. 130].)

PROOF. Since any non-zero ideal T of A is both left and right essential, then whether A is left or right Goldie, T contains a non-zero-divisor, whence $T=A$ by Lemma 3(2).

Recall that (1) A is left duo if every left ideal of A is an ideal; (2)

A is *ELT* (resp. *MELT*) if every essential (resp. maximal essential, if it exists) left ideal of A is an ideal. Left duo rings and semi-simple Artinian rings are *ELT* and *MELT*. Similarly, *ERT* and *MERT* rings are defined on the right. Following [10], A is called a right *IF*-ring if every injective right A-module is flat. Note that if every factor ring of A is reduced (for example, a simple domain), then A is not necessarily regular. We now give a few characteristic properties of strongly regular rings.

THEOREM 4. *The following conditions are equivalent*:

(1) *A is strongly regular*;

(2) *A is a GSF-ring such that $l(a)\subseteq r(a)$ for every $a\in A$*;

(3) *A is a reduced GSF-ring*;

(4) *Every maximal left ideal M of A is an ideal and for every $b\in M$, both A/bM and bA/bM are right A-flat*;

(5) *A is a right IF-ring whose maximal left ideals are ideals such that there exists a maximal left ideal M with ${}_AA/M$ flat and A/bM_A flat for every $b\in M$*;

(6) *A is a semi-prime left duo SPF-ring*;

(7) *A is a MELT reduced SPF-ring*;

(8) *A is an ELT-ring all of whose factor rings are reduced*;

(9) *A is a fully idempotent ring whose simple left modules are flat such that any proper prime ideal is completely prime.*

PROOF. Obviously, (1) implies (2).

Assume (2). Suppose there exists $0\neq b\in A$ such that $b^2=0$. If $bA+r(b)\neq A$, let M be a maximal right ideal containing $bA+r(b)$. If $bA=bM$, we get $1\in M$, a contradiction. If $bA\neq bM$, following the proof of Lemma 1(1), since $l(b)\subseteq r(b)$, we get $1\in M$ again. Therefore $bA+r(b)=A$ which yields $b=0$. This proves that A is reduced and (2) implies (3).

(3) implies (4) by Lemma 1(3).

Assume (4). First suppose that there exists a maximal left ideal K such that $bK=bA$ for all $b\in K$. Then ${}_AA/K$ is flat and since A/bK_A is flat by hypothesis, then $J=0$ by Lemma 2. Since every maximal left ideal is an ideal, then A is reduced which implies that for any maximal left ideal M such that $cM=cA$ for all $c\in M$, then $c\in Mc$ (because A is reduced), which yields A/M_A flat. Now for any maximal left ideal L such that there exists $d\in L$ with $dA\neq dL$, since L is also a maximal right ideal, then $A/L\approx dA/dL$ is right A-flat. Now suppose that for every maximal left ideal T of A, there exists $u\in T$ such that $uA\neq uT$. Then $A/T\approx uA/uT$ is right A-flat again. Thus we see that in any case, A/M_A is flat for every maximal left ideal M of A, which proves A strongly regular [21, Theorem 1. 7(8)] and hence (4) implies (5).

Assume (5). Since any maximal left ideal of A is an ideal then A is reduced by Lemma 2. Since every principal right ideal of A is a right annihilator [10], then (5) implies (6) by [16, Theorem 1].

It is easy to see that (6) implies (7).

Since a MELT reduced fully right idempotent ring is fully left idempotent, then (7) implies (8) by [18, Proposition 9] and Lemma 3(1).

Assume (8). Since every factor ring of A is reduced, then A is fully idempotent which implies that each factor ring of A is fully idempotent. If B is a prime factor ring of A, then B is an integral domain by [20, Proposition 6] and since B is prime *ELT* fully idempotent, then B is fully left idempotent which yields B a simple domain. Now the only essential left ideal of B (being *ELT*) is B which proves B Artinian, whence B is a division ring. Therefore (8) implies (9) by [8, Corollary 1. 18].

Finally assume (9). If P is a proper prime ideal of A, then B=A/P is an integral domain. If M is a non-zero maximal left ideal of B, since ${}_BB/M$ is flat, for any $0\neq b\in M$, $b=bd$ for some $d\in M$ which implies $1=d\in M$, a contradiction. This proves that B is a division ring and (9) implies (1) by [8, Colollary 1. 18 and Theorem 3. 2].

Applying [1, Theorem 12] to Theorem 4, we get

COROLLARY 4. 1. *Let A be GSF-ring such that any reduced right ideal is essential in a principal right ideal. Then $A=B\oplus C$, where B is a (left and right) continuous strongly regular ring and C is the minimal direct summand of A_A containing the nilpotent elements.*

The next remark improves [16, Theorem 1(2)].

REMARK 2. If every principal left ideal of A generated by a non-nilpotent element is a left annihilator, then any reduced principal right ideal is a direct summand of A_A.

At this point, we raise the following question: In the non-reduced case, what connections are there between *GSF* and *SPF*-rings?

PROPOSITION 5. *If A is a MERT, GSF-ring, then A is SPF.*

PROOF. Let M be a maximal right ideal of A. If M_A is a direct summand of A_A, then A/M_A is projective which implies that it is flat. Now suppose that M is an essential right ideal (which is therefore an ideal of A). If $uA=uM$ for every $u\in M$, then ${}_AA/M$ is flat and for any $0\neq b\in A$, if g: $bA\to A/M$ is a non-zero right A-homomorphism, then $M+bA=A$

(because $g\neq 0$) which yields $b=cb+bdb$, where $c\in M$, $d\in A$ and since $b-bdb=cb=cbk$ for some $k\in M$, then $g(cb)=0$ implies $g(ba)=g(bd)ba$ for all $a\in A$, which proves that A/M_A is p-injective. If there exists $v\in M$ such that $vA\neq vM$, then $A/M\approx vA/vM$ is right A-flat. Thus A is *SPF*.

We know that right $p-V$-rings are fully right idempotent (cf. [13, Proposition 6]). Theorem 4 (2) and the proof of Proposition 5 yield

THEOREM 6. *The following conditions are equivalent for a commutative ring A:*

(1) *A is regular;*

(2) *Every simple A-module is either injective or flat;*

(3) *A is SPF;*

(4) *A is GSF.*

LEMMA 7. *Let A be a ring whose complement left ideals are ideals and containing a finitely generated p-injective maximal left ideal M such that for any $b\in M$, A/bM_A is flat. Then A is strongly regular with non-zero socle.*

PROOF. Since ${}_AM$ is finitely generated p-injective, then by [17, Lemma 1. 2], $A=M\oplus I$, where I is a minimal left ideal of A which implies A has non-zero socle. Since M is an ideal of A and ${}_AA/M$ is projective, then $J=0$ by Lemma 2, which implies that M is generated by a central idempotent e. Therefore $M=eA$ and A/M_A is projective which implies ${}_AA/M$ p-injective (cf. the proof of Proposition 5), whence I is a p-injective minimal left ideal. Thus $A=M\oplus I$ is left p-injective and since $J=0$, then $Z=0$ and by [16, Lemma 1], A is reduced. Then A is strongly regular by [9, Theorem 1] and [16, Theorem 1].

We are now in a position to give "test modules" for commutative rings to be regular and self-injective regular.

THEOREM 8. *The following conditions are equivalent for commutative ring A:*

(1) *A is regular with non-zero socle;*

(2) *A contains finitely generated p-injective maximal ideal M such that for any $a\in M$, aM is p-injective;*

(3) *A contains a finitely generated p-injective maximal ideal M such that for any $a\in M$, A/aM is flat.*

PROOF. Apply [21, Remark 1] to Lemma 7.

The maximal ideal M in Theorem 8 needs not be injective even in the case of continuous rings (cf. [14, Remark 7. 11]). Indeed, we may similarly have the next result for self-injective rings. Note that if A is commutative, then a minimal ideal of A is injective iff it is p-injective [19, Lemma 1.1].

THEOREM 9. *A commutative ring A is self-injective regular with non-zero socle if, and only if, A contains an injective maximal ideal M such that for any $b \in M$, A/bM is a flat A-module.*

We next consider various generalisations of strongly regular rings. ALD (almost left duo) rings, which generalise left duo rings and semi-simple Artinian rings, are studied in [19] and [21].

Applying [9, Theorem 1], [16, Theorem 1], [21, Lemma 1] to Lemmas 1 and 2, we get

PROPOSITION 10. (1) *If A is an ALD, GSF-ring containing a maximal left ideal M such that ${}_AA/M$ is flat and for any $u \in M$, A/uM_A is flat, then A is either semi-simple Artinian or strongly regular.*

(2) *The following conditions are equivalent*: (a) *A is either semi-simple Artinian or (left and right) self-injective strongly regular with non-zero socle*; (b) *A is a semi-prime ALD ring with an injective maximal left ideal.*

REMARK 3. A is simple Artinian if, and only if, A is a prime ALD ring with non-zero socle.

The next result is motivated by [18, Question (p. 128)] and [5, Query (b)].

PROPOSITION 11. *Let A be an ERT fully right idempotent ring such that for any maximal right ideal M of A, any $b \in M$, A/bM_A is flat. Then A is regular.*

PROOF. Suppose there exists $b \in A$ such that bA is not a direct summand of A_A. If K is a complement right ideal such that $R = bA \oplus K$ is an essential right ideal, let M be a maximal right ideal containing R. Since A is fully right idempotent and M is an ideal of A, then ${}_AA/M$ is flat which implies $b = bd$ for some $d \in M$. Now A/bM_A flat and $b \in bM$ together imply $b = bcb$ for some $c \in M$, which proves that bA is generated by the idempotent bc, contradicting our hypothesis. This proves the proposition.

Similarly, [18, Proposition 9], [21, Remark 1 and Theorem 1.7 (8)] and the proof of Proposition 5 yield

PROPOSITION 12. *The following conditions are equivalent*:

(1) *A is an ELT, ERT regular ring*;

(2) *A is a MELT, MERT ring such that AM is a p-injective left and right A-module for every maximal right ideal M of A.*

We here include some information on biregular rings. Recall that A is biregular if AaA is generated by a central idempotent for each $a\in A$.

PROPOSITION 13. *Suppose that for any $a\in A$, AaA is a direct summand of A_A. Then A is a biregular fully left and right idempotent ring. Consequently, A is a left (resp. right) V-ring iff every simple factor ring of A is a left (resp. right) V-ring.*

PROOF. If $u\in J$, then AuA is a right ideal generated by an idempotent e. Since J contains no non-zero idempotent, then $e=0$ which implies $u=0$, whence $J=0$. Then for any $a\in A$, AaA is generated by a central idempotent, whence ${}_AA/AaA$ is projective and therefore flat. This proves that $a\in aAaA$ which implies that A is fully right idempotent. Similarly, A is fully left idempotent. Since any non-zero ideal in a prime ring is both left and right essential, then any prime factor ring of A is simple and the last part of Proposition 13 follows from [5, Theorem 14].

Applying [8, Corollary 1.18 and Theorem 6.10] and [21, Theorem 1.7], we get

COROLLARY 13.1. *Let A satisfy the hypothesis of Proposition* 13.

(1) *If A is MELT, then A is a unit-regular left and right V-ring whose prime factor rings are Artinian*;

(2) *If every maximal left ideal of A is an ideal, then A is strongly regular.*

Applying [10, Theorem 3.3], [9, Theorem 1], [18, Proposition 9] to Proposition 13 and Corollary 13.1 (1), we get

COROLLARY 13.2. *If A is a semi-prime ERT right IF-ring such that for any $a\in A$, AaA is the right annihilator of an element, then A is a biregular, unit-regular left and right V-ring.*

COROLLARY 13.3. *If A is a GSF-ring such that for any $a\in A$, there exists $b\in A$ such that $AaA=r(b)=l(b)$, then A is biregular.*

PROOF. If $AaA=r(b)=l(b)$ and $bA+l(b)\neq A$, let M be a maximal right ideal containing $bA+l(b)$. Then, whether $bA=bM$ or not, we arrive at $1\in M$, a contradiction. Thus $bA+l(b)=A$ and $b=bcb$ for some $c\in A$. Then $AaA=r(e)=(1-e)A$, where $e=cb$ is idempotent, which proves A biregular by Proposition 13.

REMARK 4. If A is a fully right idempotent ring with maximun condition on left and right annihilators, then any ideal of A is generated by a central idempotent.

We now turn to semi-simple Artinian rings. [6, Theorem 3. 9], [10, Theorem 3. 3], [17, Lemma 1. 2], Lemma 1 (1) and Lemma 3 yield

THEOREM 14. *The following conditions are equivalent*:

(1) *A is semi-simple Artinian*;

(2) *A is a MELT ring such that ${}_AAM$ is injective for every maximal right ideal M of A*;

(3) *A is a semi-prime right Goldie GSF-ring*;

(4) *A is a semi-prime ELT right Goldie SPF-ring*;

(5) *A is a semi-prime ERT right Goldie SPF-ring*;

(6) *A is a right IF-ring with finitely generated projective essential left socle.*

We add a last result motivated by [12]. Write $Z_2(A)=\{a\in A/$there exists an essential left ideal L with $La\subseteq Z\}$. An element y of a left (or right) A-module is called left regular if $l(y)=0$ in A.

PROPOSITION 15. *If every non-zero non-singular left A-module is projective and contains a left regular element, then either Z is an essential left ideal of A or A is simple Artinian.*

PROOF. It is easy to see that Z is an essential left ideal of A if, and only if, $Z_2(A)=A$. Consequently, if we suppose that Z is not essential, then $A/Z_2(A)$ is a non-zero non-singular left A-module by [15, Lemma 1], which therefore contains a left regular element. By [15, Lemma 3], $Z=0$ and by [7, Theorem 5. 23], A is Artinian. Now every non-zero left ideal of A contains a left regular element which implies A semi-prime, and hence semi-simple Artinian. But then, every left A-module is non-singular and therefore every simple left A-module is faithful which implies A primitive and hence simple Artinian.

Applying [15, Proposition 8] to Proposition 15, we get

COROLLARY 15. 1. *The following conditions are equivalent*:

(1) *Either A is a division ring or Z is an essential left ideal of A*;

(2) *Any non-zero non-singular left A-module (if it exists) is projective and contains a left and right regular element.*

REMARK 5. In view of the above corollary, [12, Remark 3. 5] should be reformulated as follows: A is left non-singular if, and only if, there exists a non-singular left A-module containing a left regular element. It is easily seen that [12, Corollary 3. 7] cannot be true.

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