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On the group of isometries of an affine homogeneous convex domain

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Introduction

Let Ω be a convex domain in the n-dimensional real number space $\boldsymbol{R}^n$, not containing any affine line, and let $G(\Omega)$ be the Lie group of all affine transformations on $\boldsymbol{R}^n$ leaving the domain Ω invariant. If the group $G(\Omega)$ acts transitively on Ω, then Ω is said to be *(affine) homogeneous*. By using the characteristic function φ of Ω, we can define a $G(\Omega)$-invariant Riemannian metric g_Ω on Ω as follows:

$$g_\Omega = \sum_{1\leqslant i,j\leqslant n} \frac{\partial^2 \log\varphi}{\partial x^i \partial x^j} dx^i dx^j ,$$

where $(x^1, x^2, \cdots, x^n)$ denotes a system of affine coordinates on $\boldsymbol{R}^n$. The Riemannian metric g_Ω is called the *canonical metric* of Ω (cf. [7], [8]). A homogeneous convex domain is said to be *reducible* if it is affinely equivalent to a direct product of homogeneous convex domains. A homogeneous convex domain is said to be *irreducible* if it is not reducible. We note that a homogeneous convex cone is a special case of a homogeneous convex domain.

For a homogeneous convex domain Ω, we denote by $I(\Omega)$ the group of all isometries of the homogeneous Riemannian manifold (Ω, g_Ω). Then, it has been proved that the groups $G(V)$ and $I(V)$ for an irreducible homogeneous convex cone V have the same connected component containing the identity element ([3], [6]).

The aim of the present paper is to extend the above result to homogeneous convex domains. Namely, we will prove the following statement: *If a homogeneous convex domain Ω is irreducible and not affinely equivalent to an elementary domain, then the groups $G(\Omega)$ and $I(\Omega)$ have the same connected component containing the identity element* (Theorem 6.1). The definition of an elementary domain will be given in § 3. In order to prove the above result, we will need the theory of T-algebras developed by Vinberg [8], [9], and also, we will make use of the results obtained in [5], [6] and [7].

§ 1. Preliminaries.

In this section, we recall some of fundamental definitions and results on homogeneous convex domains and T-algebras. The details for them may be found in Vinberg [8], [9].

1.1. From a homogeneous convex domain Ω in $\boldsymbol{R}^n$, a homogeneous convex cone $V=V(\Omega)$ in $\boldsymbol{R}^{n+1}$ can be constructed as follows:

$$V=\left\{(tx, t)\in\boldsymbol{R}^n\times\boldsymbol{R}\,;\ x\in\Omega,\ t>0\right\}.$$

The convex cone $V=V(\Omega)$ is called the *cone fitted onto* the convex domain Ω (cf. [8]). The natural imbedding σ from Ω into V defined by

$$\sigma:\ x\in\Omega\longrightarrow(x, 1)\in \mathrm{V} \tag{1.1}$$

is equivariant with respect to the groups $G(\Omega)$ and $G(V)$. Moreover, σ is an isometric imbedding with respect to the canonical metrics. Therefore, the homogeneous Riemannian manifold (Ω, g_Ω) can be regarded as a Riemannian submanifold of (V, g_V) (cf. [7]).

1.2. We now recall a relation between homogeneous convex domains and T-algebras. Let $\mathfrak{A}=\sum_{1\leqslant i,j\leqslant r}\mathfrak{A}_{ij}$ be a T-algebra of rank r $(r\geqslant 2)$ provided with an involution $*$. General elements of the subspace $\mathfrak{A}_{ij}$ will be denoted as a_{ij}, b_{ij}, c_{ij}, $\cdots$, and also, an element a of $\mathfrak{A}$ will be denoted by the matrix $a=(a_{ij})$, where a_{ij} is the $\mathfrak{A}_{ij}$-component of a.

Throughout this paper, we will use the following notation:

$$n_{ij}=\dim\mathfrak{A}_{ij}=\dim\mathfrak{A}_{ji},\ n_i=1+\frac{1}{2}\sum_{k\neq i}n_{ki}\qquad(1\leqslant i,\ j\leqslant r).$$

$$e=(e_{ij}),\ e_{ij}=\delta_{ij}\qquad\text{(Kronecker delta)}.$$

Let us define subsets $T=T(\mathfrak{A})$, $X=X(\mathfrak{A})$ and $V=V(\mathfrak{A})$ of $\mathfrak{A}$ by

$$\begin{aligned}T&=\left\{t=(t_{ij})\in\mathfrak{A}\,;\ t_{ii}>0\ (1\leqslant i\leqslant r),\ t_{ij}=0\ (1\leqslant j<i\leqslant r)\right\},\\ X&=\{x\in\mathfrak{A}\,;\ x^*=x\}\quad\text{and}\quad V=\{tt^*\,;\ t\in T\},\end{aligned}\tag{1.2}$$

respectively. Then the set V is a homogeneous convex cone in the real vector space X and the set T is a connected Lie group having the element e as the identity element. The Lie group T acts linearly and simply transitively on V in the following manner:

$$(t, ss^*)\in T\times V\longrightarrow(ts)(ts)^*\in V.$$

The Lie algebra $\mathfrak{t}$ of T can be identified with the subspace $\sum_{1\leqslant i\leqslant j\leqslant r}\mathfrak{A}_{ij}$ of $\mathfrak{A}$

provided with the bracket product $[a, b]=ab-ba$ (cf. [8]).

1.3. We next define subsets $T_0=T_0(\mathfrak{A})$, $X_0=X_0(\mathfrak{A})$ and $\Omega=\Omega(\mathfrak{A})$ of $\mathfrak{A}$ by

$$T_0=\left\{t=(t_{ij})\in T\,;\ t_{rr}=1\right\}, \qquad X_0=\left\{x=(x_{ij})\in X\,;\ x_{rr}=0\right\}$$

and

$$\Omega=V(\mathfrak{A})\cap(X_0+e)=\left\{x=(x_{ij})\in V(\mathfrak{A})\,;\ x_{rr}=1\right\}, \tag{1.3}$$

respectively. Then Ω is a homogeneous convex domain in the affine subspace X_0+e of X such that $V(\mathfrak{A})$ is the cone $V(\Omega)$ fitted onto Ω. The set T_0 is a closed (normal) subgroup of T acting affinely and simply transitively on Ω in the following manner:

$$(t, ss^*)\in T_0\times\Omega\longrightarrow(ts)\,(ts)^*\in\Omega\,. \tag{1.4}$$

Conversely, every homogeneous convex domain is affinely equivalent to a convex domain of the form $\Omega(\mathfrak{A})$ by means of a T-algebra $\mathfrak{A}$. The subspace $\mathfrak{t}_0$ of $\mathfrak{t}$ defined by

$$\mathfrak{t}_0=\left\{t=(t_{ij})\in\mathfrak{t}\,;\ t_{rr}=0\right\}$$

is the Lie subalgebra of $\mathfrak{t}$ corresponding to the subgroup T_0 of T (cf. [8]).

1.4. For a T-algebra $\mathfrak{A}=\sum_{1\leqslant i,j\leqslant r}\mathfrak{A}_{ij}$ of rank r $(r\geqslant 2)$, we define a set I of indices by $I=\{1, 2, \cdots, r\}$. A permutation ε of the set I is said to be *admissible to* $\mathfrak{A}$ if the condition $n_{\varepsilon(i)\varepsilon(j)}=0$ holds for every pair (i, j) of indices satisfying $i<j$ and $\varepsilon(j)<\varepsilon(i)$. Using a permutation ε admissible to $\mathfrak{A}$, we have a new T-algebra $\mathfrak{A}^\varepsilon=\sum_{1\leqslant i,j\leqslant r}\mathfrak{A}^\varepsilon_{ij}$ which is different from $\mathfrak{A}$ only in the grading as follows:

$$\mathfrak{A}^\varepsilon_{ij}=\mathfrak{A}_{\varepsilon(i)\varepsilon(j)}\,.$$

Then the cone $V(\mathfrak{A})$ is linearly equivalent to the new cone $V(\mathfrak{A}^\varepsilon)$ under the permutation of the coordinates on $X(\mathfrak{A})$ by ε. As for homogeneous convex domains, we can easily see that a convex domain $\Omega(\mathfrak{A})$ is linearly equivalent to the domain $\Omega(\mathfrak{A}^\varepsilon)$ if $\varepsilon(r)=r$.

We now state a necessary condition for a homogeneous convex domain $\Omega=\Omega(\mathfrak{A})$ to be irreducible in terms of a T-algebra $\mathfrak{A}$. The following proposition will be used in § 4.

PROPOSITION 1.1. *If a homogeneous convex domain* $\Omega(\mathfrak{A})$ *is irreducible, then for every pair* (i, j) *of indices* $1\leqslant i<j\leqslant r-1$, *there exists a series* $i_1, i_2, \cdots, i_m$ *of indices* $1\leqslant i_1, i_2, \cdots, i_m\leqslant r-1$ *satisfying the conditions* $i_1=i$, $i_m=j$ *and* $n_{i_{\lambda-1}i_\lambda}\neq 0$ $(2\leqslant\lambda\leqslant m)$.

PROOF. Let us suppose that there exist proper subsets J_1 and J_2 of $J=\{1, 2, \cdots, r-1\}$ satisfying the conditions

$$J=J_1\cup J_2 \quad \text{and} \quad n_{ij}=0 \ (i\in J_1,\ j\in J_2)\,.$$

Then, the domain $\Omega(\mathfrak{A})$ is reducible. In fact, if

$$J_1=\{i_1, i_2, \cdots, i_p\}\ (i_1<i_2<\cdots<i_p) \text{ and } J_2=\{j_1, j_2, \cdots, j_q\}\ (j_1<j_2<\cdots<j_q)\,,$$

then the permutation ε of the set I defined by

$$\varepsilon(k)=\begin{cases} i_k & (1\leqslant k\leqslant p) \\ j_{k-p} & (p+1\leqslant k\leqslant r-1) \\ r & (k=r) \end{cases}$$

is admissible to $\mathfrak{A}$. As was stated above, the convex domain $\Omega(\mathfrak{A})$ is linearly equivalent to $\Omega(\mathfrak{A}^{\varepsilon})$. So, without loss of generality, we may assume that $J_1=\{1, 2, \cdots, p\}$ and $J_2=\{p+1, p+2, \cdots, r-1\}$ $(q=r-p-1)$. We now define two subspaces $\mathfrak{A}^{(1)}=\sum_{1\leqslant i,j\leqslant p+1}\mathfrak{A}_{ij}^{(1)}$ and $\mathfrak{A}^{(2)}=\sum_{1\leqslant i,j\leqslant q+1}\mathfrak{A}_{ij}^{(2)}$ of $\mathfrak{A}$ by

$$\mathfrak{A}_{ij}^{(1)}=\begin{cases} \mathfrak{A}_{ij}\ (1\leqslant i,\ j\leqslant p) \\ \mathfrak{A}_{ir}\ (1\leqslant i\leqslant p,\ j=p+1) \\ \mathfrak{A}_{rj}\ (i=p+1,\ 1\leqslant j\leqslant p) \\ \mathfrak{A}_{rr}\ (i=j=p+1) \end{cases}$$

and

$$\mathfrak{A}_{ij}^{(2)}=\begin{cases} \mathfrak{A}_{kl}\ (1\leqslant i,\ j\leqslant q,\ k=i+p,\ l=j+p) \\ \mathfrak{A}_{kr}\ (1\leqslant i\leqslant q,\ j=q+1,\ k=i+p) \\ \mathfrak{A}_{rl}\ (1\leqslant j\leqslant q,\ i=q+1,\ l=j+p) \\ \mathfrak{A}_{rr}\ (i=j=q+1) \end{cases},$$

respectively.

Then $\mathfrak{A}^{(k)}$ is closed with the multiplication and invariant under the involution $*$ $(k=1, 2)$. We now introduce a new multiplication in $\mathfrak{A}^{(1)}$ as follows: For arbitrary elements a, $b\in\mathfrak{A}_{i\,p+1}^{(1)}$ $(1\leqslant i\leqslant p)$, we employ a multiplication $(n_r/m)a*b$ in $\mathfrak{A}^{(1)}$ instead of $a*b$ in $\mathfrak{A}$, where $m=1+\frac{1}{2}\sum_{1\leqslant k\leqslant p}n_{kr}$, and we do not change other relations of multiplication between elements in $\mathfrak{A}_{ij}^{(1)}$ $(1\leqslant i,\ j\leqslant p+1)$. Then it is easy to see that $\mathfrak{A}^{(1)}$ is a T-algebra of rank $p+1$ with this multiplication and the involution $*$. Similarly, $\mathfrak{A}^{(2)}$ becomes a T-algebra of rank $q+1$ (For the definition of a T-algebra, see p. 380 of [8].). Next, let us show that the domain $\Omega(\mathfrak{A})$ is affinely equivalent to the product domain $\Omega(\mathfrak{A}^{(1)})\times\Omega(\mathfrak{A}^{(2)})$. For an arbitrary element $x\in\mathfrak{A}$, we write

$$x=\begin{bmatrix} x_1 & 0 & x_2 \\ 0 & x_3 & x_4 \\ x_5 & x_6 & x_7 \end{bmatrix},$$

where $x_1 \in \sum_{1\leq i,j\leq p} \mathfrak{A}_{ij}$, $x_2, x_5^* \in \sum_{1\leq i\leq p} \mathfrak{A}_{ir}$, $x_3 \in \sum_{p+1\leq i,j\leq r-1} \mathfrak{A}_{ij}$, $x_4, x_6^* \in \sum_{p+1\leq i\leq r-1} \mathfrak{A}_{ir}$ and $x_7 \in \mathfrak{A}_{rr}$. Then the mapping:

$$\begin{bmatrix} t_1 & 0 & t_2 \\ 0 & t_3 & t_4 \\ 0 & 0 & 1 \end{bmatrix} \in T_0(\mathfrak{A}) \longrightarrow \left(\begin{bmatrix} t_1 & t_2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} t_3 & t_4 \\ 0 & 1 \end{bmatrix} \right) \in T_0(\mathfrak{A}^{(1)}) \times T_0(\mathfrak{A}^{(2)})$$

is a Lie group isomorphism, and the following affine isomorphism:

$$\begin{bmatrix} x_1 & 0 & x_2 \\ 0 & x_3 & x_4 \\ x_2^* & x_4^* & 1 \end{bmatrix} \in Y \longrightarrow \left(\begin{bmatrix} x_1 & x_2 \\ x_2^* & 1 \end{bmatrix}, \begin{bmatrix} x_3 & x_4 \\ x_4^* & 1 \end{bmatrix} \right) \in Y^{(1)} \times Y^{(2)}$$

maps the domain $\Omega(\mathfrak{A})$ onto the product domain $\Omega(\mathfrak{A}^{(1)}) \times \Omega(\mathfrak{A}^{(2)})$, where $Y = X_0(\mathfrak{A}) + e$, $Y^{(k)} = X_0(\mathfrak{A}^{(k)}) + e^{(k)}$ and $e^{(k)}$ is the unit element of $\mathfrak{A}^{(k)}$ $(k=1, 2)$ (cf. (1.2), (1.3) and (1.4)). q. e. d.

It should be noted that for homogeneous convex cones, the above proposition has been proved by [1].

§ 2. Canonical metric and curvature tensor.

In this section, by following [6], we define an R-derivation for the curvature tensor R on a Riemannian manifold. Furthermore, for a homogeneous convex domain, we recall fundamental formulas on the curvature tensor in terms of a T-algebra.

2.1. Let M be a connected homogeneous Riemannian manifold. Let us take an arbitrary point $e \in M$ and denote by $\mathfrak{m}$ the tangent space of M at e. Then, by using the Riemannian metric at the point e, we have an inner product $\langle\ ,\ \rangle$ on $\mathfrak{m}$ and the norm $\|x\| = \langle x, x\rangle^{1/2}$ for $x \in \mathfrak{m}$. Let us consider the Riemannian curvature tensor R at the point e as a trilinear mapping

$$R: \mathfrak{m}\times\mathfrak{m}\times\mathfrak{m} \longrightarrow \mathfrak{m} \quad \text{by} \quad R(x, y, z) = R(x, y)\, z,$$

and define a quadrilinear function

$$(2.1) \qquad K: \mathfrak{m}\times\mathfrak{m}\times\mathfrak{m}\times\mathfrak{m} \longrightarrow \boldsymbol{R} \quad \text{by} \quad K(x, y, z, w) = \langle R(x, y, z), w\rangle .$$

Then the following identities are well-known (cf. e. g., p. 201 in vol. 1 of [2]):

$$K(x, y, z, w) = -K(y, x, z, w) = -K(x, y, w, z)\,,$$
$$K(x, y, z, w) + K(y, z, x, w) + K(z, x, y, w) = 0\,, \tag{2.2}$$
$$K(x, y, z, w) = K(z, w, x, y)$$

for all $x, y, z, w \in \mathfrak{m}$. Let D: $\mathfrak{m} \rightarrow \mathfrak{m}$ be a skew-symmetric linear mapping. Then, D is called an *R-derivation* if R is D-invariant, that is, D satisfies the following identity:

$$D\big(R(x, y, z)\big) = R(Dx, y, z) + R(x, Dy, z) + R(x, y, Dz)$$

for all $x, y, z \in \mathfrak{m}$. For an arbitrary linear mapping D: $\mathfrak{m} \rightarrow \mathfrak{m}$, we define a function DK: $\mathfrak{m} \times \mathfrak{m} \times \mathfrak{m} \times \mathfrak{m} \rightarrow \boldsymbol{R}$ by

$$\begin{aligned}(DK)(x, y, z, w) = {} & K(Dx, y, z, w) + K(x, Dy, z, w) \\ & + K(x, y, Dz, w) + K(x, y, z, Dw)\end{aligned} \tag{2.3}$$

for $x, y, z, w \in \mathfrak{m}$. Then, a skew-symmetric linear mapping D: $\mathfrak{m} \rightarrow \mathfrak{m}$ is an R-derivation if and only if D satisfies the following identity:

$$(DK)(x, y, z, w) = 0\,, \tag{2.4}$$

for all $x, y, z, w \in \mathfrak{m}$. From now on, we denote by $\mathfrak{D}(M)$ the Lie algebra of all R-derivations on $\mathfrak{m}$. It should be noted that the structure of the Lie algebra $\mathfrak{D}(M)$ is invariant under an isometric equivalence and independent of choosing the point e.

Let $\mathfrak{i} = \mathfrak{i}(M)$ be the Lie algebra of the group $I(M)$ of all isometries, and $\mathfrak{h} = \mathfrak{h}(M)$ the isotropy subalgebra of $\mathfrak{i}$ at the point e. Then the infinitesimal linear isotropy representation θ: $\mathfrak{h} \rightarrow \mathfrak{gl}(\mathfrak{m})$ at the point e is injective and the condition

$$\theta(\mathfrak{h}) \subset \mathfrak{D}(M) \tag{2.5}$$

is satisfied.

2.2. We now want to describe the curvature tensor for the canonical metric on a homogeneous convex domain in terms of a T-algebra. Let $\mathfrak{A} = \sum_{1 \leqslant i, j \leqslant r} \mathfrak{A}_{ij}$ be a T-algebra of rank r ($r \geqslant 2$). Then, the unit element e of the Lie group $T = T(\mathfrak{A})$ is contained in the homogeneous convex cone $V = V(\mathfrak{A})$. Hence, the tangent space $T_e(V)$ of V at the point e can be naturally identified with the ambient space $X = X(\mathfrak{A})$ and also with the Lie algebra $\mathfrak{t} = \mathfrak{t}(\mathfrak{A})$ by the following linear isomorphism:

$$\xi:\ t \in \mathfrak{t} \longrightarrow t + t^* \in X = T_e(V)\,.$$

By using the canonical metric g_V and the linear isomorphism ξ, we have an inner product $\langle\,,\rangle$ on $\mathfrak{t}$ as follows:

$$\langle x, y\rangle = g_V(e)\big(\xi(x), \xi(y)\big)$$

for $x, y\in\mathfrak{t}$. Then, the inner product $\langle\,,\rangle$ satisfies the following conditions (cf. [8], [6]):

$$\langle\mathfrak{A}_{ij}, \mathfrak{A}_{kl}\rangle = 0 \ \big((i,j)\neq(k,l),\ 1\leqslant i\leqslant j\leqslant r,\ 1\leqslant k\leqslant l\leqslant r\big), \tag{2.6}$$

$$||a_{ij}b_{jk}||^2 = \frac{1}{2n_j}||a_{ij}||^2||b_{jk}||^2 \tag{2.7}$$

and

$$\langle a_{ij}b_{jk}, c_{ik}\rangle = \langle b_{jk}, a_{ij}^*c_{ik}\rangle = \langle a_{ij}, c_{ik}b_{jk}^*\rangle \tag{2.8}$$

for $a_{ij}\in\mathfrak{A}_{ij}$, $b_{jk}\in\mathfrak{A}_{jk}$ and $c_{ik}\in\mathfrak{A}_{ik}$ $(1\leqslant i<j<k\leqslant r)$. Let us put

$$e_i = \frac{1}{2\sqrt{n_i}}e_{ii}\,,$$

where $e_{ii}=1$ is the unit element of the subalgebra $\mathfrak{A}_{ii}=\boldsymbol{R}$ $(1\leqslant i\leqslant r)$. Then $||e_i||=1$.

For a homogeneous convex domain $\Omega=\Omega(\mathfrak{A})$, similarly as in the above case, we can identify the Lie algebra $\mathfrak{t}_0=\mathfrak{t}_0(\mathfrak{A})$ with the tangent space $T_e(\Omega)$ of Ω at the point e and also with the real vector space $X_0=X_0(\mathfrak{A})$ by the following linear isomorphism:

$$\xi_0:\ t\in\mathfrak{t}_0 \longrightarrow t+t^*\in X_0 = T_e(\Omega)\,.$$

By using the canonical metric g_Ω and the linear isomorphism ξ_0, we have an inner product $\langle\,,\rangle_0$ on $\mathfrak{t}_0$ as follows:

$$\langle x, y\rangle_0 = g_\Omega(e)\big(\xi_0(x), \xi_0(y)\big)$$

for $x, y\in\mathfrak{t}_0$. Since the inclusion mapping from Ω into the cone $V(\mathfrak{A})$ coincides with the isometric imbedding σ defined by (1.1), we have the following relations:

$$\xi_0(x) = \xi(x) \quad\text{and}\quad \langle x, y\rangle_0 = \langle x, y\rangle$$

for all $x, y\in\mathfrak{t}_0$. So, we may omit the subscript zero in $\langle\,,\rangle_0$.

2.3. Let β (resp. R) be the connection function (resp. the curvature tensor) for the canonical metric on a homogeneous convex domain $\Omega=\Omega(\mathfrak{A})$. Then β and R are given by the following formulas (cf. [4]):

$$\begin{gathered}\beta:\ \mathfrak{t}_0\times\mathfrak{t}_0\longrightarrow\mathfrak{t}_0\,,\\ 2\big\langle\beta(x,y), z\big\rangle = \big\langle[z,x], y\big\rangle + \big\langle[z,y], x\big\rangle + \big\langle[x,y], z\big\rangle\end{gathered} \tag{2.9}$$

and

$$R: \mathfrak{t}_0\times\mathfrak{t}_0\times\mathfrak{t}_0\longrightarrow\mathfrak{t}_0,$$

(2.10) $$R(x, y, z)=R(x, y)\, z=\beta\big(x, \beta(y, z)\big)-\beta\big(y, \beta(x, z)\big)-\beta\big([x, y], z\big),$$

respectively. The connection function

$$\alpha: \mathfrak{t}\times\mathfrak{t}\longrightarrow\mathfrak{t}$$

and the curvature tensor

$$\tilde{R}: \mathfrak{t}\times\mathfrak{t}\times\mathfrak{t}\longrightarrow\mathfrak{t}$$

for the canonical metric g_V of a homogeneous convex cone $V=V(\mathfrak{A})$ are given by the same formulas as in the above (2.9) and (2.10), respectively. We note that the relations between the connection functions α and β have been clarified by Lemma 2.1 of [7].

Let $\gamma: \mathfrak{t}_0\times\mathfrak{t}_0\to\mathfrak{A}_{rr}$ be the second fundamental form. Then, by the formula (2.6) of [7], γ is given by

(2.11) $$\gamma(x, y)=\frac{-1}{2\sqrt{n_r}}\langle x_1, y_1\rangle\, e_r$$

for all $x, y\in\mathfrak{t}_0$, where x_1 and y_1 are the $\sum_{1\leqslant i\leqslant r-1}\mathfrak{A}_{ir}-$ components of x and y, respectively. Let

$$K: \mathfrak{t}_0\times\mathfrak{t}_0\times\mathfrak{t}_0\times\mathfrak{t}_0\longrightarrow\boldsymbol{R} \quad\text{and}\quad \tilde{K}: \mathfrak{t}\times\mathfrak{t}\times\mathfrak{t}\times\mathfrak{t}\longrightarrow\boldsymbol{R}$$

be the functions defined by (2.1) from the curvature tensors R and $\tilde{R}$, respectively. Then, from the equation of Gauss (cf. vol. 2 of [2]) and the formula (2.11), it follows that the identity

(2.12) $$\tilde{K}(x, y, z, w)=K(x, y, z, w)+\frac{1}{4n_r}\big(\langle x_1, z_1\rangle\langle y_1, w_1\rangle-\langle x_1, w_1\rangle\langle y_1, z_1\rangle\big)$$

holds for all $x, y, z, w\in\mathfrak{t}_0$.

§ 3. Elementary domains.

Let (,) be an inner product on the n-dimensional real number space $\boldsymbol{R}^n$ $(n\geqslant 2)$. Then the homogeneous convex domain $\Omega(n)$ in $\boldsymbol{R}^n$ defined by

$$\Omega(n)=\big\{(x, y)\in\boldsymbol{R}\times\boldsymbol{R}^{n-1};\ x-(y, y)>0\big\}$$

is called an *elementary domain*. It is known that the elementary domain $\Omega(n)$ is affinely equivalent to a domain of the form $\Omega(\mathfrak{A})$ given by means of a T-algebra $\mathfrak{A}$ of rank two satisfying the condition $n_{12}=n-1$ ([8]). In this section, we determine the Lie algebra $\mathfrak{D}(\Omega(n))$ of all R-derivations on $\Omega(n)$.

It is easy to see that the group $G(\Omega(n))$ is generated by the following three types of transformations:

$$(x, y) \longrightarrow \big(x+2(y, a)+(a, a), y+a\big) \quad (a \in \boldsymbol{R}^{n-1}),$$
$$(x, y) \longrightarrow (\lambda^2 x, \lambda y) \quad (\lambda \in \boldsymbol{R},\ \lambda > 0),$$
$$(x, y) \longrightarrow (x, Ay) \quad \big(A \in O(n-1)\big),$$

where $O(n-1)$ is the orthogonal group of degree $n-1$.

Let us put a point $e=(1, 0) \in \Omega(n)$. Then, an affine automorphism f of $\Omega(n)$ leaves the point e fixed if and only if f has the following form: $f(x, y) =(x, Ay)$ $(A \in O(n-1))$ for all $(x, y) \in \Omega(n)$.

On the other hand, it is known in Proposition 2. 4 of [7] that $\Omega(n)$ is a simply connected hyperbolic space form of the sectional curvature $-1/(2n+2)$. Therefore, $\Omega(n)$ is isometric to the Riemannian symmetric space $SO^0(1, n)/SO(n)$ (cf. p. 268 in vol. 2 of [2]). Since the covariant derivative of the curvature tensor vanishes, the Lie algebra $\mathfrak{D}(\Omega(n))$ coincides with the linear isotropy subalgebra $\theta(\mathfrak{h}(\Omega(n)))$ (cf. chap. VI in vol. 1 of [2]). Hence, summing up the results stated above, we have the following

PROPOSITION 3. 1. *For the elementary domain $\Omega=\Omega(n)$ of dimension n, the isotropy subalgebra $\mathfrak{k}(\Omega)$ of $\mathfrak{g}(\Omega)$ is isomorphic to the Lie algebra $\mathfrak{o}(n-1)$ of $O(n-1)$. Both the Lie algebra $\mathfrak{D}(\Omega)$ of all R-derivations and the isotropy subalgebra $\mathfrak{h}(\Omega)$ of $\mathfrak{i}(\Omega)$ are isomorphic to the Lie algebra $\mathfrak{so}(n)$ of $SO(n)$.*

§ 4. Some lemmas on *R*-derivations.

In this section, we study general properties of R-derivations on a homogeneous convex domain. Let Ω be a homogeneous convex domain. Then, according to the result of Vinberg recalled in § 1, we can assume that Ω is a domain of the form $\Omega(\mathfrak{A})$ given by means of a T-algebra $\mathfrak{A}=\sum_{1 \leq i, j \leq r} \mathfrak{A}_{ij}$ of rank r $(r \geq 2)$ (cf. (1. 3)). We denote by $\mathfrak{D}(\Omega)$ the Lie algebra of all R-derivations on Ω.

4. 1. We first remark that the value of the curvature tensor can be calculated explicitly by using Lemma 2.2 of [5], Lemma 2. 1 of [7] and the formula (2. 10). The following lemma can be proved by the same formulas on the curvature tensor as used in the proofs of Lemmas 3. 1, 3. 2 and 3. 3 of [6]. So, we may omit the proof.

LEMMA 4. 1. *Every $D \in \mathfrak{D}(\Omega)$ satisfies the following conditions:*

(1) $\langle D\mathfrak{A}_{ij}, \mathfrak{A}_{kl} \rangle = 0$ *for all indices* $1 \leq i \leq j \leq r-1$ *and* $1 \leq k < l \leq r$ *satisfying* $\{i, j\} \cap \{k, l\} = \phi$.

(2) $\langle D\mathfrak{A}_{ij}, \sqrt{n_i}\, e_i+\sqrt{n_j}\, e_j\rangle=0$ *for all indices* $1\leqslant i<j\leqslant r-1$.

(3) $\frac{1}{\sqrt{n_i}}\langle e_i, De_k\rangle=\frac{1}{\sqrt{n_j}}\langle e_j, De_k\rangle$ *for all indices* $1\leqslant k\leqslant r-1$ *and* $1\leqslant i<j\leqslant r-1$ *satisfying* $n_{ij}\neq 0$.

For each $D\in\mathfrak{D}(\Omega)$, we define a skew-symmetric element $a=(a_{ij})$ of $\mathfrak{A}$ (i. e. $a_{ij}^{*}=-a_{ji}$) by

$$a_{ij}=\begin{cases}-\sqrt{n_i}\, P_{ij}\big(D(e_i)\big) & (1\leqslant i<j\leqslant r)\\ 0 & (1\leqslant i=j\leqslant r)\end{cases}, \tag{4.1}$$

where P_{ij} is the projection from $\mathfrak{A}$ onto the subspace $\mathfrak{A}_{ij}$. By the condition (2) of Lemma 4.1, we have

$$a_{ij}=\sqrt{n_j}\, P_{ij}\big(D(e_j)\big) \qquad (1\leqslant i\leqslant j\leqslant r-1)\,.$$

We next show the following lemma which is quite similar to Lemma 6.1 of [6].

LEMMA 4.2. *Every* $D\in\mathfrak{D}(\Omega)$ *and the element* $a=(a_{ij})\in\mathfrak{A}$ *defined by* (4.1) *satisfy the following three identities*:

(1) $\langle Dx_{jk}, x_{ij}\rangle=\langle a_{ik}, x_{ij}x_{jk}\rangle$,

(2) $\langle Dx_{ik}, x_{ij}\rangle=\langle a_{jk}, x_{ij}^{*}x_{ik}\rangle$,

(3) $\langle Dx_{jk}, x_{ik}\rangle=\langle a_{ij}, x_{ik}x_{jk}^{*}\rangle$

for $x_{ij}\in\mathfrak{A}_{ij}$, $x_{jk}\in\mathfrak{A}_{jk}$ *and* $x_{ik}\in\mathfrak{A}_{ik}$ $(1\leqslant i<j<k\leqslant r)$.

PROOF. By using (2.2), (2.3) and the following formulas:

$$R(e_i, x_{jk})=0\,,\ R(e_i, x_{ij}, e_i)=\frac{1}{4n_i}x_{ij},\ R(e_i, x_{ij}, x_{jk})=\frac{-1}{4\sqrt{n_i}}x_{ij}x_{jk}\,,$$

we have

$$\begin{aligned}(DK)(e_i, x_{jk}, e_i, x_{ij})&=K(e_i, x_{ij}, e_i, Dx_{jk})-K(e_i, x_{ij}, x_{jk}, De_i)\\ &=\frac{1}{4n_i}\Big(\langle Dx_{jk}, x_{ij}\rangle-\langle a_{ik}, x_{ij}x_{jk}\rangle\Big)\,.\end{aligned}$$

Therefore, the identity (1) follows from the condition (2.4). For the proof of the second identity, we use the condition

$$(DK)(e_j, x_{ik}, e_j, x_{ij})=0\,.$$

From the formulas:

$$R(e_j, x_{ik})=0,\ R(e_j, x_{ij}, e_j)=\frac{1}{4n_j}x_{ij},\ R(e_j, x_{ij}, x_{ik})=\frac{-1}{4\sqrt{n_j}}x_{ij}^{*}x_{ik}\,,$$

it follows that

$$(DK)\ (e_j, x_{ik}, e_j, x_{ij}) = K(e_j, x_{ij}, e_j, Dx_{ik}) - K(e_j, x_{ij}, x_{ik}, De_j)$$
$$= \frac{1}{4n_j}\Big(\langle Dx_{ik}, x_{ij}\rangle - \langle a_{jk}, x_{ij}^* x_{ik}\rangle\Big),$$

which implies the identity (2). Similarly as in the above cases, the identity (3) follows from the condition $(DK)(e_i, x_{jk}, e_i, x_{ik})=0$. q. e. d.

From the above lemma, we have the following

LEMMA 4. 3. *For each* $D\in\mathfrak{D}(\Omega)$, *the element* $a=(a_{ij})\in\mathfrak{A}$ *defined by* (4. 1) *satisfies the following four identities*:

$$(1)\quad \langle x_{ij}^* a_{ik}, x_{ij}^* x_{ik}\rangle = \frac{1}{2n_i}\,||x_{ij}||^2\langle a_{ik}, x_{ik}\rangle,$$

$$(2)\quad \langle a_{ik} x_{jk}^*, x_{ik} x_{jk}^*\rangle = \frac{1}{2n_k}(1-\delta_{kr})\,||x_{jk}||^2\langle a_{ik}, x_{ik}\rangle,$$

$$(3)\quad \langle x_{ik} a_{jk}^*, x_{ik} x_{jk}^*\rangle = \frac{1}{2n_k}(1-\delta_{kr})\,||x_{ik}||^2\langle a_{jk}, x_{jk}\rangle,$$

$$(4)\quad \langle a_{ij}^* x_{ik}, x_{ij}^* x_{ik}\rangle = \frac{1}{2n_i}\,||x_{ik}||^2\langle a_{ij}, x_{ij}\rangle$$

for $x_{ij}\in\mathfrak{A}_{ij}$, $x_{jk}\in\mathfrak{A}_{jk}$ *and* $x_{ik}\in\mathfrak{A}_{ik}$ $(1\leqslant i<j<k\leqslant r)$.

PROOF. By Lemma 2. 1 of [7] and the formula (2. 10), we can see that the following identities hold:

$$R(x_{ij}, x_{ik}, e_j) = -R(e_j, x_{ij}, x_{ik}) = \frac{1}{4\sqrt{n_j}}\,x_{ij}^* x_{ik},$$

$$R(x_{ij}, e_j, x_{ij}) = \frac{1}{4n_j}\,||x_{ij}||^2 e_j - \frac{1}{4\sqrt{n_i n_j}}\,||x_{ij}||^2 e_i$$

and

$$R(x_{ij}, x_{ik}, x_{ij}) = \frac{1}{4n_i}\,||x_{ij}||^2 x_{ik} - \frac{1}{4}\,x_{ij}(x_{ij}^* x_{ik}).$$

From these formulas, (2. 2) and (1) of Lemma 4. 1, it follows that

$$(DK)(x_{ij}, x_{ik}, x_{ij}, e_j) = \frac{-1}{4\sqrt{n_j}}\left(2\langle x_{ij}^* x_{ik}, Dx_{ij}\rangle + \frac{1}{\sqrt{n_i}}\,||x_{ij}||^2\langle e_i, Dx_{ik}\rangle\right).$$

Hence, by (1) of Lemma 4. 2 and (2. 4), the identity (1) holds. Using the following formulas:

$$R(e_i, x_{jk}) = 0,\ \ R(x_{ik}, x_{jk}, e_i) = \frac{1}{4\sqrt{n_i}}\,x_{ik} x_{jk}^*,$$

$$R(x_{jk}, x_{ik}, x_{jk}) = \frac{1}{4n_k}(1-\delta_{kr})\,||x_{jk}||^2 x_{ik} - \frac{1}{4}(x_{ik} x_{jk}^*)\,x_{jk}$$

and the condition (1) of Lemma 4.2, we have

$$(DK)(e_i, x_{jk}, x_{jk}, x_{ik})$$
$$= \frac{-1}{4\sqrt{n_i}} \left\{ 2\left\langle a_{ik}, (x_{ik} x_{jk}^*) x_{jk} \right\rangle - \frac{1}{n_k}(1-\delta_{kr}) \|x_{jk}\|^2 \langle a_{ik}, x_{ik} \rangle \right\}.$$

Hence, by $(DK)(e_i, x_{jk}, x_{jk}, x_{ik})=0$ and (2.8), the identity (2) holds. The identity (3) can be proved similarly from the condition $(DK)(e_j, x_{ik}, x_{ik}, x_{jk})=0$ and the identity (2) of Lemma 4.2. Moreover, as in the above cases, the identity (4) follows from the conditions $(DK)(e_j, x_{ik}, x_{ij}, x_{ik})=0$ and (3) of Lemma 4.2. q.e.d.

By using the above lemma, we can prove the following

LEMMA 4.4. *The element $a=(a_{ij})\in\mathfrak{A}$ defined by (4.1) satisfies the condition $a_{ir}=a_{jr}=0$ for every indices $1\leqslant i<j\leqslant r-1$ with $n_{ij}\neq 0$.*

PROOF. Putting $k=r$ and $x_{ir}=a_{ir}$ in the identity (1) of Lemma 4.3 and using the condition (2.8), we have

$$\frac{1}{2n_i}\|x_{ij}\|^2\|a_{ir}\|^2 = \|x_{ij}^* a_{ir}\|^2 = \left\langle a_{ir}(a_{ir}^* x_{ij}), x_{ij} \right\rangle. \tag{4.2}$$

On the other hand, by putting $k=r$ and $x_{ir}=a_{ir}$ in the identity (2) of Lemma 4.3, we get

$$a_{ir} x_{jr}^* = 0$$

for every $x_{jr}\in\mathfrak{A}_{jr}$. Therefore, by putting $x_{jr}=x_{ij}^* a_{ir}$ in (4.2), we have $a_{ir}=0$. From the identity (3) of Lemma 4.3, it follows that

$$x_{ir} a_{jr}^* = 0$$

for every $x_{ir}\in\mathfrak{A}_{ir}$. By (2.7) and (2.8),

$$\frac{1}{2n_j}\|x_{ij}\|^2\|a_{jr}\|^2 = \|x_{ij} a_{jr}\|^2 = \left\langle x_{ij}, (x_{ij} a_{jr}) a_{jr}^* \right\rangle = 0$$

for every $x_{ij}\in\mathfrak{A}_{ij}$, which implies that $a_{jr}=0$. q.e.d.

Summing up the results obtained in the above lemmas and Proposition 1.1, we have the following

PROPOSITION 4.5. *Let $\mathfrak{A}$ be a T-algebra of rank r $(r\geqslant 3)$ and let the homogeneous convex domain $\Omega=\Omega(\mathfrak{A})$ be irreducible. Then the conditions $\langle De_i, e_j\rangle=0$ and $\langle D\mathfrak{A}_{ij}, \mathfrak{A}_{kr}\rangle=0$ are satisfied for every $D\in\mathfrak{D}(\Omega)$ and all indices $1\leqslant i, j, k\leqslant r-1$.*

PROOF. If $i=j$, then $\langle De_i, e_j\rangle=0$ since D is skew-symmetric. If $i\neq j$, then by Proposition 1. 1, there exists a series of different indices $1\leqslant i_1, i_2, \cdots, i_m\leqslant r-1$ satisfying the conditions $i_1=i$, $i_m=j$ and $n_{i_{\lambda-1}i_\lambda}\neq 0$ $(2\leqslant\lambda\leqslant m)$. Then, by (3) of Lemma 4. 1, the identity

$$\frac{1}{\sqrt{n_{i_{\lambda-1}}}}\langle e_{i_{\lambda-1}}, De_j\rangle=\frac{1}{\sqrt{n_{i_\lambda}}}\langle e_{i_\lambda}, De_j\rangle$$

holds for every index $2\leqslant\lambda\leqslant m$. Therefore, by $\langle De_j, e_j\rangle=0$, we have $\langle De_i, e_j\rangle=0$. Moreover, by Lemma 4. 4, $a_{i_{\lambda-1}r}=a_{i_\lambda r}=0$ $(2\leqslant\lambda\leqslant m)$. Thus, $a_{ir}=0$ $(1\leqslant i\leqslant r-1)$, and hence, the condition $\langle D\mathfrak{A}_{ij}, \mathfrak{A}_{kr}\rangle=0$ follows from (1) of Lemma 4. 1 and Lemma 4. 2. q. e. d.

4. 2. Finally, we prepare two lemmas which will be used in the next section. The following lemma can be proved by computation quite similar to the method in the proof of Proposition 5. 1 of [6].

LEMMA 4. 6. *Every $D\in\mathfrak{D}(\Omega)$ satisfies the following identity*:

$$P_{ik}\big(D(x_{ij}x_{jk})\big)=P_{ik}\big(D(x_{ij})\,x_{jk}+x_{ij}D(x_{jk})\big)$$

for $x_{ij}\in\mathfrak{A}_{ij}$ and $x_{jk}\in\mathfrak{A}_{jk}$ $(1\leqslant i<j<k\leqslant r)$.

PROOF. If $n_{ij}=0$, then the identity holds trivially. So, we may assume that $n_{ij}\neq 0$. Let us calculate the condition

$$(DK)(e_i, x_{ij}, x_{jk}, x_{ik})=0$$

by using the following formulas:

$$R(x_{jk}, x_{ik}, x_{ij})=\frac{1}{4}\langle x_{ij}x_{jk}, x_{ik}\rangle\left(\frac{1}{\sqrt{n_i}}e_i-\frac{1}{\sqrt{n_j}}e_j\right),$$

$$R(x_{jk}, x_{ik}, e_i)=\frac{-1}{4\sqrt{n_i}}x_{ik}x_{jk}^*,\quad R(e_i, x_{ij}, x_{ik})=\frac{1}{4\sqrt{n_i}}x_{ij}^*x_{ik},$$

$$R(e_i, x_{ij}, x_{jk})=\frac{-1}{4\sqrt{n_i}}x_{ij}x_{jk}.$$

Then, by (3) of Lemma 4. 1, we have

$$K(x_{jk}, x_{ik}, x_{ij}, De_i)=0.$$

Therefore, by (2. 2) and (2. 8),

$$\begin{aligned}(DK)&(e_i, x_{ij}, x_{jk}, x_{ik})\\&=K(x_{jk}, x_{ik}, e_i, Dx_{ij})-K(e_i, x_{ij}, x_{ik}, Dx_{jk})+K(e_i, x_{ij}, x_{jk}, Dx_{ik})\\&=\frac{-1}{4\sqrt{n_i}}\big(\langle x_{ik}x_{jk}^*, Dx_{ij}\rangle+\langle x_{ij}^*x_{ik}, Dx_{jk}\rangle+\langle x_{ij}x_{jk}, Dx_{ik}\rangle\big)\\&=\frac{-1}{4\sqrt{n_i}}\big\langle\big\{D(x_{ij})\,x_{jk}+x_{ij}D(x_{jk})-D(x_{ij}x_{jk})\big\}, x_{ik}\big\rangle.\end{aligned}$$

q. e. d.

LEMMA 4.7. *For each* $D \in \mathfrak{D}(\Omega)$, *the element* $a=(a_{ij}) \in \mathfrak{A}$ *defined by* (4.1) *satisfies the following four identities*:

(1) $(x_{ik}^* a_{ij})\, x_{jr} = x_{ik}^*(a_{ij} x_{jr})$,

(2) $(a_{ik}^* x_{ij})\, x_{jr} = a_{ik}^*(x_{ij} x_{jr})$,

(3) $(a_{ik} x_{jk}^*)\, x_{jr} = a_{ik}(x_{jk}^* x_{jr})$,

(4) $(x_{ik} a_{jk}^*)\, x_{jr} = x_{ik}(a_{jk}^* x_{jr})$

for $x_{ij} \in \mathfrak{A}_{ij}$, $x_{jk} \in \mathfrak{A}_{jk}$, $x_{ik} \in \mathfrak{A}_{ik}$ *and* $x_{jr} \in \mathfrak{A}_{jr}$ $(1 \leqslant i < j < k \leqslant r-1)$.

PROOF. By using the formula (VII′) in p. 380 of [8], we have $x_{ij}(x_{jr} x_{kr}^*) = (x_{ij} x_{jr})\, x_{kr}^*$. From this and the condition (2.8), it follows that the equality $(x_{ik}^* x_{ij})\, x_{jr} = x_{ik}^*(x_{ij} x_{jr})$ holds. Therefore, we have the identities (1) and (2). We next show the identity (4). In order to do this, we consider the condition $(DK)(e_k, x_{jr}, x_{ik}, x_{ir})=0$. Then, by the following formulas:

$$R(e_k, x_{jr}) = 0,\ R(x_{ik}, x_{ir}, x_{jr}) = \frac{-1}{4}(x_{jr} x_{ir}^*)\, x_{ik}\,,$$

$$R(x_{ik}, x_{ir}, e_k) = \frac{1}{4\sqrt{n_k}}\, x_{ik}^* x_{ir}\,,$$

we get

$$\begin{aligned}(DK)(e_k, x_{jr}, x_{ik}, x_{ir}) &= K(x_{ik}, x_{ir}, e_k, Dx_{jr}) - K(x_{ik}, x_{ir}, x_{jr}, De_k) \\ &= \frac{1}{4\sqrt{n_k}} \langle x_{ik}^* x_{ir}, Dx_{jr} \rangle + \frac{1}{4} \big\langle (x_{jr} x_{ir}^*)\, x_{ik}, De_k \big\rangle\,.\end{aligned}$$

Hence, by (3) of Lemma 4.2, we have the identity (4). By using the identity (1) of Lemma 4.2, we can similarly verify that the identity (3) follows from the condition $(DK)(e_i, x_{jk}, x_{ir}, x_{jr})=0$. q. e. d.

§ 5. Extension of an R-derivation.

In this section, we study an extension of an R-derivation on a homogeneous convex domain $\Omega=\Omega(\mathfrak{A})$ to an $\tilde{R}$-derivation on the homogeneous convex cone $V=V(\mathfrak{A})$ fitted onto Ω. We remark that the structure of the Lie algebra $\mathfrak{D}(V)$ of all $\tilde{R}$-derivations on V has been clarified by [6].

5.1. For a T-algebra $\mathfrak{A} = \sum_{1 \leqslant i,j \leqslant r} \mathfrak{A}_{ij}$ of rank r $(r \geqslant 2)$, we define subspaces $\mathfrak{A}_k$ $(k=0, 1, 2)$ of $\mathfrak{A}$ by

$$\mathfrak{A}_0 = \sum_{1 \leqslant i \leqslant j \leqslant r-1} \mathfrak{A}_{ij},\ \mathfrak{A}_1 = \sum_{1 \leqslant i \leqslant r-1} \mathfrak{A}_{ir} \quad \text{and} \quad \mathfrak{A}_2 = \mathfrak{A}_{rr}\,.$$

Let us assume that the domain $\Omega=\Omega(\mathfrak{A})$ is irreducible and $r \geqslant 3$. Then, by (2.6) and Proposition 4.5, the conditions

$$D(\mathfrak{A}_0) \subset \mathfrak{A}_0 \quad \text{and} \quad D(\mathfrak{A}_1) \subset \mathfrak{A}_1 \tag{5.1}$$

are satisfied for every $D \in \mathfrak{D}(\Omega)$. By (2.8) and (3) of Lemma 4.2, the identities

$$\langle Dx_{ir}, x_{kr}\rangle = \langle a_{ki}x_{ir}, x_{kr}\rangle \ (k<i) \text{ and } \langle Dx_{ir}, x_{kr}\rangle = -\langle a_{ik}^{*}x_{ir}, x_{kr}\rangle \ (i<k)$$

hold for all $x_{ir} \in \mathfrak{A}_{ir}$ and $x_{kr} \in \mathfrak{A}_{kr}$. We now denote by D_1 the grade-preserving part of D, that is,

$$D_1 = \sum_{1 \leqslant i<j \leqslant r} P_{ij} \circ D \circ P_{ij}\,.$$

Then, by (5.1), we have

$$Dx = D_1(x) + \sum_{k<i} a_{ki}x_{ir} - \sum_{i<k} a_{ik}^{*}x_{ir} \tag{5.2}$$

for every $x = \sum_{1 \leqslant i \leqslant r-1} x_{ir} \in \mathfrak{A}_1$. By (4.1), (5.1) and (1) of Lemma 4.1, it is easy to see that the identity

$$De_{ii} = 2\sum_{k<i} a_{ki} - 2\sum_{i<k} a_{ik} \tag{5.3}$$

holds $(1 \leqslant i \leqslant r-1)$. By using (5.1) and Lemma 4.2, we can similarly verify that the identity

$$\begin{aligned} Dz = {} & D_1(z_{ij}) + \frac{1}{\sqrt{n_i}}\langle a_{ij}, z_{ij}\rangle e_i - \frac{1}{\sqrt{n_j}}\langle a_{ij}, z_{ij}\rangle e_j \\ & + \sum_{k<i}(a_{kj}z_{ij}^{*} + a_{ki}z_{ij}) + \sum_{i<k<j}(z_{ij}a_{kj}^{*} - a_{ik}^{*}z_{ij}) \\ & - \sum_{j<k}(z_{ij}a_{jk} + z_{ij}^{*}a_{ik}) \end{aligned} \tag{5.4}$$

holds for every $z = z_{ij} \in \mathfrak{A}_{ij}$ $(1 \leqslant i<j \leqslant r-1)$.

Now, we prove

LEMMA 5.1. *Let $\Omega = \Omega(\mathfrak{A})$ be an irreducible homogeneous convex domain and* rank $\mathfrak{A} = r \geqslant 3$. *Then every $D \in \mathfrak{D}(\Omega)$ satisfies the following condition:*

$$\big\langle D(zx), x\big\rangle = \big\langle D(z)x, x\big\rangle + \big\langle zD(x), x\big\rangle$$

for $z \in \mathfrak{A}_0$ and $x \in \mathfrak{A}_1$.

PROOF. We first consider the case $z = e_{ii} \in \mathfrak{A}_{ii}$ $(1 \leqslant i \leqslant r-1)$. Putting $x = \sum_{1 \leqslant k \leqslant r-1} x_{kr} \in \mathfrak{A}_1$, we have $zx = x_{ir}$. Therefore, by (5.2) and $\langle D_1(x_{ir}), x\rangle = \langle Dx_{ir}, x_{ir}\rangle = 0$, it follows that the identity

$$\big\langle D(zx), x\big\rangle = \sum_{k<i}\langle a_{ki}x_{ir}, x_{kr}\rangle - \sum_{i<k}\langle a_{ik}^{*}x_{ir}, x_{kr}\rangle$$

holds. By (5.3), we have

$$\left\langle D(z)x,\,x\right\rangle=2\sum_{k<i}\langle a_{ki}x_{ir},\,x_{kr}\rangle-2\sum_{i<k}\langle a_{ik}x_{kr},\,x_{ir}\rangle\,.$$

Again, by (5.2), the identity

$$\left\langle zD(x),\,x\right\rangle=\sum_{i<k}\langle a_{ik}x_{kr},\,x_{ir}\rangle-\sum_{k<i}\langle a_{ki}^{*}x_{kr},\,x_{ir}\rangle$$

is satisfied. Therefore, from these identities, we have

$$\left\langle D(e_{ii}x),\,x\right\rangle=\left\langle D(e_{ii})x,\,x\right\rangle+\left\langle e_{ii}D(x),\,x\right\rangle\,.$$

We next consider the case $z=z_{ij}\in\mathfrak{A}_{ij}$ $(1\leqslant i<j\leqslant r-1)$. Then, $zx=z_{ij}x_{jr}$. Therefore, by (5.2),

$$\begin{aligned}\left\langle D(zx),\,x\right\rangle=&\left\langle D_1(z_{ij}x_{jr}),\,x_{ir}\right\rangle-\left\langle a_{ij}^{*}(z_{ij}x_{jr}),\,x_{jr}\right\rangle\\&+\sum_{k<i}\left\langle a_{ki}(z_{ij}x_{jr}),\,x_{kr}\right\rangle-\sum_{i<k<j}\left\langle a_{ik}^{*}(z_{ij}x_{jr}),\,x_{kr}\right\rangle\\&-\sum_{j<k}\left\langle a_{ik}^{*}(z_{ij}x_{jr}),\,x_{kr}\right\rangle\,.\end{aligned}\tag{5.5}$$

By (5.4), we can similarly verify that the identity

$$\begin{aligned}\left\langle D(z)x,\,x\right\rangle=&\left\langle D_1(z_{ij})x_{jr},\,x_{ir}\right\rangle+\frac{1}{2n_i}\|x_{ir}\|^2\langle a_{ij},\,z_{ij}\rangle\\&-\frac{1}{2n_j}\|x_{jr}\|^2\langle a_{ij},\,z_{ij}\rangle\\&+\sum_{k<i}\left(\left\langle(a_{kj}z_{ij}^{*})\,x_{ir},\,x_{kr}\right\rangle+\left\langle(a_{ki}z_{ij})\,x_{jr},\,x_{kr}\right\rangle\right)\\&+\sum_{i<k<j}\left(\left\langle(z_{ij}a_{kj}^{*})\,x_{kr},\,x_{ir}\right\rangle-\left\langle(a_{ik}^{*}z_{ij})\,x_{jr},\,x_{kr}\right\rangle\right)\\&-\sum_{j<k}\left(\left\langle(z_{ij}a_{jk})\,x_{kr},\,x_{ir}\right\rangle+\left\langle(z_{ij}^{*}a_{ik})\,x_{kr},\,x_{jr}\right\rangle\right)\end{aligned}\tag{5.6}$$

holds. By using (5.2), we have

$$zD(x)=z_{ij}D_1(x_{jr})+\sum_{j<k}z_{ij}(a_{jk}x_{kr})-\sum_{k<j}z_{ij}(a_{kj}^{*}x_{kr})$$

and hence,

$$\begin{aligned}\left\langle zD(x),\,x\right\rangle=&\left\langle z_{ij}D_1(x_{jr}),\,x_{ir}\right\rangle-\left\langle z_{ij}(a_{ij}^{*}x_{ir}),\,x_{ir}\right\rangle\\&-\sum_{k<i}\left\langle z_{ij}(a_{kj}^{*}x_{kr}),\,x_{ir}\right\rangle-\sum_{i<k<j}\left\langle z_{ij}(a_{kj}^{*}x_{kr}),\,x_{ir}\right\rangle\\&+\sum_{j<k}\left\langle z_{ij}(a_{jk}x_{kr}),\,x_{ir}\right\rangle\,.\end{aligned}\tag{5.7}$$

On the other hand, by Lemma 4.6, the identity

$$\left\langle D_1(z_{ij}x_{jr}),\,x_{ir}\right\rangle=\left\langle D_1(z_{ij})x_{jr},\,x_{ir}\right\rangle+\left\langle z_{ij}D_1(x_{jr}),\,x_{ir}\right\rangle\tag{5.8}$$

holds. Since the subalgebra $\mathfrak{t}$ of a T-algebra $\mathfrak{A}$ is associative, the following identities are satisfied:

(5. 9) $$a_{ki}(z_{ij}x_{jr})=(a_{ki}z_{ij})\,x_{jr}\ (k<i)\ \text{and}\ z_{ij}(a_{jk}x_{kr})=(z_{ij}a_{jk})\,x_{kr}\ (j<k).$$

Furthermore, by (2. 7), (2. 8) and (4) of Lemma 4. 3, we have

(5. 10) $$\left\langle a_{ij}^{*}(z_{ij}x_{jr}),\, x_{jr}\right\rangle=\frac{1}{2n_j}\|x_{jr}\|^2\langle a_{ij}, z_{ij}\rangle$$

and

(5. 11) $$\left\langle z_{ij}(a_{ij}^{*}x_{ir}),\, x_{ir}\right\rangle=\frac{1}{2n_i}\|x_{ir}\|^2\langle a_{ij}, z_{ij}\rangle\,,$$

respectively. Therefore, using (2. 8), Lemma 4. 7 and the identities (5. 5)-(5. 11), we can verify that the condition

$$\left\langle D(z_{ij}x),\, x\right\rangle=\left\langle D(z_{ij})\,x,\, x\right\rangle+\left\langle z_{ij}D(x),\, x\right\rangle$$

is satisfied. q. e. d.

5. 2. For an arbitrary R-derivation $D\in\mathfrak{D}(\Omega)$, we define an extension $\tilde{D}$ of D to $\mathfrak{t}=\mathfrak{t}_0+\mathfrak{A}_{rr}$ by

(5. 12) $$\tilde{D}\colon \mathfrak{t}\longrightarrow\mathfrak{t},\ \tilde{D}=\begin{cases} D \text{ on } \mathfrak{t}_0\\ 0 \text{ on } \mathfrak{A}_{rr}\end{cases}.$$

We next prove the following

LEMMA 5. 2. *Let $\Omega=\Omega(\mathfrak{A})$ be an irreducible homogeneous convex domain and* rank $\mathfrak{A}=r\geqq 3$. *Then the extension $\tilde{D}$ of an arbitrary $D\in\mathfrak{D}(\Omega)$ defined by* (5. 12) *satisfies the following three conditions*:

(1) $(\tilde{D}\tilde{K})(x, y, z, w)=0$ *for all* $x, y, z, w\in\mathfrak{t}_0$.

(2) $(\tilde{D}\tilde{K})(e_r, e_r, x, y)=(\tilde{D}\tilde{K})(e_r, x, e_r, y)=0$ *for all* $x, y\in\mathfrak{t}$.

(3) $(\tilde{D}\tilde{K})(e_r, x, y, z)=0$ *for all* $x, y, z\in\mathfrak{t}_0$.

PROOF. For an arbitrary element $x\in\mathfrak{t}$, we write

$$x=x_0+x_1+x_2\,,$$

where $x_k\in\mathfrak{A}_k$ $(k=0, 1, 2)$. Then, by (5. 1) and (5. 12), we have

(5. 13) $$(\tilde{D}x)_0=Dx_0,\ (\tilde{D}x)_1=Dx_1\ \text{ and }\ (\tilde{D}x)_2=0\,.$$

Hence, by using (5. 13) and the equation of Gauss stated in (2. 12), we can verify that $(\tilde{D}\tilde{K})(x, y, z, w)=0$ for all $x, y, z, w\in\mathfrak{t}_0$. From the condition $\tilde{R}(e_r, e_r)=0$ and (5. 13), it follows that

$$(\tilde{D}\tilde{K})(e_r, e_r, x, y)=0$$

for all $x, y\in\mathfrak{t}$. Moreover, using (5. 13) and the formula

$$\tilde{R}(e_r, x, e_r)=\frac{1}{4n_r}x_1,$$

we get

$$(\tilde{D}\tilde{K})(e_r, x, e_r, y)=\tilde{K}(e_r, \tilde{D}x, e_r, y)+\tilde{K}(e_r, x, e_r, \tilde{D}y)$$
$$=\frac{1}{4n_r}\left(\langle Dx_1, y_1\rangle+\langle x_1, Dy_1\rangle\right)=0$$

for all $x, y\in\mathfrak{t}$. Finally we use the following formulas:

$$\tilde{R}(e_r, x, y)=\frac{1}{2\sqrt{n_r}}\alpha(x_1, y)$$

and

$$\tilde{K}(e_r, x, y, z)=\frac{1}{2\sqrt{n_r}}\left\langle\alpha(x_1, y), z\right\rangle$$
$$=\frac{1}{4\sqrt{n_r}}\left(\left\langle[z, x_1], y\right\rangle+\left\langle[z, y], x_1\right\rangle+\left\langle[x_1, y], z\right\rangle\right)$$
$$=\frac{1}{4\sqrt{n_r}}\left(\langle z_0x_1, y_1\rangle+\langle z_0y_1, x_1\rangle-\langle y_0z_1, x_1\rangle-\langle y_0x_1, z_1\rangle\right)$$

for all $x, y, z\in\mathfrak{t}_0$ (cf. (2. 9) and (2. 10)). Now, by linearizing the variable x in the identity obtained by Lemma 5. 1, we have

$$\left\langle D(z_0x_1), y_1\right\rangle+\left\langle D(z_0y_1), x_1\right\rangle=\left\langle\left(D(z_0)\,x_1+z_0D(x_1)\right), y_1\right\rangle$$
$$+\left\langle\left(D(z_0)\,y_1+z_0D(y_1)\right), x_1\right\rangle$$

for all $x, y, z\in\mathfrak{t}_0$. Hence, from this identity and (5. 13), the identity (3) holds. q. e. d.

By using the above lemma, we have the following

PROPOSITION 5. 3. *Let* $\Omega=\Omega(\mathfrak{A})$ *be an irreducible homogeneous convex domain and* rank $\mathfrak{A}=r\geqslant 3$. *Then the extension* $\tilde{D}$ *of every* $D\in\mathfrak{D}(\Omega)$ *defined by* (5. 12) *is an* $\tilde{R}$*-derivation on the cone* $V(\Omega)$ *fitted onto* Ω.

PROOF. By the Bianchi's identities (2. 2) and the fact $\mathfrak{t}=\mathfrak{t}_0+\mathfrak{A}_{rr}$ (direct sum of subspaces, $r\geqslant 3$), we can see that the condition $\tilde{D}\tilde{K}=0$ is equivalent to the conditions (1), (2) and (3) in Lemma 5. 2. q. e. d.

§ 6. Isotropy subalgebras and the Lie algebra $\mathfrak{D}(\Omega)$.

In this section, we prove the main theorem of this paper. Let Ω be a homogeneous convex domain. Then, as was stated in the previous sections, it can be assumed that Ω is a domain of the form $\Omega(\mathfrak{A})$ by means of a T-algebra $\mathfrak{A}$ of rank r $(r\geqslant 2)$.

We now show the following main theorem.

THEOREM 6.1. *If a homogeneous convex domain Ω is irreducible and not affinely equivalent to an elementary domain, then the Lie algebra $\mathfrak{g}(\Omega)$ of the affine automorphism group $G(\Omega)$ is identical with the Lie algebra $\mathfrak{i}(\Omega)$ of the isometry group $I(\Omega)$.*

PROOF. It suffices to prove that the isotropy subalgebra $\mathfrak{k}=\mathfrak{k}(\Omega)$ in $\mathfrak{g}(\Omega)$ is identical with the isotropy subalgebra $\mathfrak{h}=\mathfrak{h}(\Omega)$ in $\mathfrak{i}(\Omega)$. We first consider the case $r=2$. If $n_{12}\neq 0$, then Ω is affinely equivalent to an elementary domain (cf. § 3). So, by the assumption, n_{12} must be zero and $\mathfrak{A}=\mathfrak{A}_{11}+\mathfrak{A}_{22}$. Therefore, by (1.2) and (1.3), Ω is the cone of all positive real numbers and $\mathfrak{k}=\mathfrak{h}=(0)$. We next consider the case $r\geqslant 3$. Then, by Proposition 5.3, the extension $\tilde{D}$ of every $D\in\mathfrak{D}(\Omega)$ is an $\tilde{R}$-derivation of the homogeneous convex cone V fitted onto Ω. Moreover, by Proposition 4.5 and (5.12), we have $\langle\tilde{D}e_i, e_j\rangle=0$ $(1\leqslant i, j\leqslant r)$. Therefore, the $\mathfrak{D}_0$-part of $\tilde{D}$ is zero (For the definition of the $\mathfrak{D}_0$-part of $\tilde{D}$, see § 3 of [6].). Hence, from Propositions 5.1 and 6.6 of [6], it follows that there exists a derivation $\tau:\mathfrak{A}\to\mathfrak{A}$ of a T-algebra such that

$$\tilde{D}=\xi^{-1}\circ(\tau_{|X}+k_a)\circ\xi\in\theta\big(\mathfrak{k}(V)\big),$$

where a is the element of $\mathfrak{A}$ given by (4.1) and $k_a: X\to X$ is the linear operator defined by $k_a(x)=ax-xa$. Since the derivation τ preserves the grading and commutes with the involution of $\mathfrak{A}$, τ keeps the subspace X_0 invariant and vanishes at e. Furthermore, by the condition $a_{ir}=0$ $(1\leqslant i\leqslant r-1)$ (cf. Proposition 4.5), we can see that $k_a(X_0)\subset X_0$ and $k_a(e)=0$. Therefore, by (1.2) and (1.3), the restrictions of τ and k_a to the subspace X_0 are contained in the subalgebra $\mathfrak{k}$. Hence,

$$D=\tilde{D}_{|\mathfrak{t}_0}=\xi_0^{-1}\circ(\tau_{|X_0}+k_{a|X_0})\circ\xi_0$$

is an element of $\theta(\mathfrak{k})$. From this, it follows that $\mathfrak{D}(\Omega)\subset\theta(\mathfrak{k})$. Combining this and the condition $\theta(\mathfrak{k})\subset\theta(\mathfrak{h})$ with (2.5), we get

$$\theta(\mathfrak{k})=\theta(\mathfrak{h})=\mathfrak{D}(\Omega)\,. \tag{6.1}$$

Since the infinitesimal linear isotropy representation θ is injective, $\mathfrak{k}$ is identical with $\mathfrak{h}$. q. e. d.

By Proposition 3.1 and the condition (6.1), we have the following

THEOREM 6.2. *Let Ω be an irreducible homogeneous convex domain. Then the isotropy subalgebra $\mathfrak{h}$ of $\mathfrak{i}(\Omega)$ is isomorphic to the Lie algebra $\mathfrak{D}(\Omega)$ of all R-derivations on Ω.*

Finally we remark that the above theorem holds for an arbitrary homogeneous convex cone (cf. Theorem 7.3 of [6]).

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